Expressions in Cylindrical Coordinates

Velocity: $\mathbf{V} = V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\mathbf{k}$

Gravity: $\mathbf{g} = g_r\mathbf{e}_r + g_\theta\mathbf{e}_\theta + g_z\mathbf{k}$

Differential Operator: $\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{k}$

Gradient: $\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial\theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{k}$

Laplacian of a Vector: $\nabla^2\mathbf{V} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\mathbf{k})$

Divergence: $\nabla\cdot\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z} \equiv \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}$

Curl: $\nabla\times\mathbf{V} = \left(\frac{1}{r}\frac{\partial V_z}{\partial\theta} - \frac{\partial V_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right]\mathbf{k}$

Continuity: $\nabla\cdot\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z} \equiv \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z} = 0$

Incompressible NS: $\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} \equiv \frac{\partial\mathbf{V}}{\partial t} + \nabla\left(\frac{\mathbf{V}^2}{2}\right) - \mathbf{V}\times(\nabla\times\mathbf{V}) = -\frac{\nabla p}{\rho} + \mathbf{g} + \nu\nabla^2\mathbf{V}$

Using Vector Identity: $(\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \mathbf{V}\times(\nabla\times\mathbf{V})$ and $\nu = \frac{\mu}{\rho}$

NS *r*-momentum: $\rho\left(\frac{\mathrm{D}V_r}{\mathrm{D}t} - \frac{V_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 V_r - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta}\right) + \rho g_r$

NS *θ*-momentum: $\rho\left(\frac{\mathrm{D}V_\theta}{\mathrm{D}t} + \frac{V_r V_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left(\nabla^2 V_\theta - \frac{V_\theta}{r^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\right) + \rho g_\theta$

NS *z*-momentum: $\rho\frac{\mathrm{D}V_z}{\mathrm{D}t} = -\frac{\partial p}{\partial z} + \mu\nabla^2 V_z + \rho g_z$

Substantial Derivative: $\frac{\mathrm{D}A}{\mathrm{D}t} = \frac{\partial A}{\partial t} + V_r\frac{\partial A}{\partial r} + \frac{V_\theta}{r}\frac{\partial A}{\partial\theta} + V_z\frac{\partial A}{\partial z}$

Laplacian: $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$

Stream Function Definition: $V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\quad V_\theta = -\frac{\partial\psi}{\partial r}$ (in 2-D only)

Stream Function Equation: $\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$ from $\nabla\times\mathbf{V} = \mathbf{0}$

Potential Function Definition: $V_r = -\frac{\partial\phi}{\partial r}\quad V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\quad V_z = -\frac{\partial\phi}{\partial z}$ from $\mathbf{V} = -\nabla\phi$

Potential Function Equation: $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$ from $\nabla\cdot\mathbf{V} = 0$

Viscous Shear Stresses: $\tau_{rr} = 2\mu\frac{\partial V_r}{\partial r}\quad \tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r}\right)\quad \tau_{zz} = 2\mu\frac{\partial V_z}{\partial z}$

Off-Diagonal: $\tau_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial V_r}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right]\quad \tau_{\theta z} = \mu\left(\frac{1}{r}\frac{\partial V_z}{\partial\theta} + \frac{\partial V_\theta}{\partial z}\right)\quad \tau_{zr} = \mu\left(\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z}\right)$