PROBLEM 7-1

GIVEN: the axial momentum equation presented in Equation 7-48:

$$\frac{1}{r}\frac{\partial}{\partial x}\left(r\rho V_x V_x\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho V_x V_r\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r \frac{\partial V_x}{\partial r}\right) + (\rho_\infty - \rho) g$$

and the overall continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r V_r\right) + \frac{\partial}{\partial x}\left(\rho V_x\right) = 0$$

FIND: the left-hand side of the momentum equation in a form involving the substantial derivative

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + V_r \frac{\partial()}{\partial r} + V_\theta \frac{\partial()}{\partial \theta} + V_x \frac{\partial()}{\partial x}$$

ASSUMPTIONS: Steady-State, axi-symmetric

APPROACH: expand the derivatives on the LHS of the momentum equation, group terms, and eliminate the terms that comprise the continuity equation, realizing that the sum of these terms is zero.

using assumptions $$\frac{D()}{Dt} = \overset{0,\ S.S.}{\cancel{\frac{\partial()}{\partial t}}} + V_r \frac{\partial()}{\partial r} + \overset{0\ \text{Axi-sym}}{\cancel{V_\theta \frac{\partial()}{\partial \theta}}} + V_x \frac{\partial()}{\partial x}$$

LHS of momentum equation $\longrightarrow$ expanding derivatives

$$\frac{1}{r}\frac{\partial}{\partial x}\left(r\rho V_x V_x\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho V_x V_r\right) = \frac{1}{r}\left[\rho V_x \frac{\partial}{\partial r}(rV_x) + rV_x \frac{\partial}{\partial x}(\rho V_x) + \rho r V_r \frac{\partial}{\partial r}(V_x) + V_x \frac{\partial}{\partial r}(\rho r V_r)\right]$$

since r is not a function of x, this becomes

$$\rho V_x \frac{\partial}{\partial r}(V_x) + \rho V_r \frac{\partial}{\partial r}(V_x) + \underbrace{V_x \frac{\partial}{\partial x}(\rho V_x) + \frac{V_x}{r}\frac{\partial}{\partial r}(\rho r V_r)}$$

This term is V_x times the continuity equation and is therefore zero since the continuity equation equals zero.

This leaves the following axial momentum equation:

$$\boxed{V_x \frac{\partial}{\partial r}(V_x) + V_r \frac{\partial}{\partial r}(V_x) = \frac{1}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r \frac{\partial V_x}{\partial r}\right) + (\rho_\infty - \rho) g\right]}$$

PROBLEM 7-2

GIVEN: the more general form of the energy equation for a 1-D cartesian reacting flow where no assumptions are made regarding the form of the species transport law or the relationship among properties (equation 7-55)

$$\sum \dot{m}_i'' \frac{dh_i}{dx} + \frac{d}{dx}\left(-k\frac{dT}{dx}\right) + \dot{m}'' V_x \frac{dV_x}{dx} = -\sum h_i \dot{m}_i'''$$

DERIVE: the Shvab-Zeldovich energy equation (equation 7-63)

ASSUMPTIONS: Fick's Law with binary diffusion, Le=1

APPROACH: mathmatical manipulation of the above equation usin $Le = \frac{\alpha}{\mathcal{D}} = 1$
$k = \rho\alpha c_p = \rho\mathcal{D}c_p$, and $c_p = \sum Y_i c_{p,i}$

recognizing that $\dot{m}_i''' = \frac{d\dot{m}_i''}{dx}$ and using product rule:

$$\sum \frac{d}{dx}(\dot{m}_i'' h_i) + \frac{d}{dx}\left(-k\frac{dT}{dx}\right) + \dot{m}'' V_x \frac{dV_x}{dx} = 0$$

substituting $h_i = h_{f,i}^\circ + \int c_p dT$

$$\sum \frac{d}{dx}(\dot{m}_i'' h_{f,i}^\circ) + \sum \frac{d}{dx}\left(\dot{m}_i'' \int c_p dT\right) + \frac{d}{dx}\left(-k\frac{dT}{dx}\right) + \dot{m}'' V_x \frac{dV_x}{dx} = 0$$

using the fact that $h_{f,i}^\circ$ is constant, first term becomes $\sum h_{f,i}\frac{d\dot{m}_i''}{dx}$ which can be moved to the RHS

$$\sum \frac{d}{dx}\left(\dot{m}_i'' \int c_p dT\right) + \frac{d}{dx}\left(-k\frac{dT}{dx}\right) + \dot{m}'' V_x \frac{dV_x}{dx} = -\sum h_{f,i}^\circ \dot{m}_i'''$$

now invoking Fick's Law, $\dot{m}_i'' = Y_i \dot{m}'' - \rho\mathcal{D}\frac{dY_i}{dx}$:

$$\sum \frac{d}{dx}\left(Y_i \dot{m}'' \int c_{p,i} dT\right) - \sum \frac{d}{dx}\left(\rho\mathcal{D}\frac{dY_i}{dx}\int c_{p,i}\, dT\right) - \frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{m}'' V_x \frac{dV_x}{dx} = -\sum h_{f,i}^\circ \dot{m}_i'''$$

$k = \rho\alpha c_p = \rho\alpha \sum Y_i c_{p,i}$ which for Le=1 becomes $k = \rho\mathcal{D}\sum Y_i c_{p,i}$

$$\sum \frac{d}{dx}\left(Y_i \dot{m}'' \int c_{p,i}\, dT\right) - \sum \frac{d}{dx}\left(\rho\mathcal{D}\frac{dY_i}{dx}\int c_{p,i}\, dT\right) - \frac{d}{dx}\left(\rho\mathcal{D}\sum Y_i c_{p,i}\frac{dT}{dx}\right) + \dot{m}'' V_x \frac{dV_x}{dx} = -\sum h_{f,i}^\circ \dot{m}_i'''$$

if diffusion is binary, 2^{nd} and 3^{rd} terms can be combined

$$\sum \frac{d}{dx}\left(Y_i \dot{m}'' \int c_{p,i}\, dT\right) - \frac{d}{dx}\left(\rho\mathcal{D}\sum\left(\frac{dY_i}{dx}\int c_{p,i}\, dT + Y_i c_{p,i}\frac{dT}{dx}\right)\right) + \dot{m}'' V_x \frac{dV_x}{dx} = -\sum h_{f,i}^\circ \dot{m}_i'''$$

PROBLEM 7-2 (continued)

the above expression can be simplified using the product rule:

$$\Sigma \frac{d}{dx}\left(Y_i \dot{m}'' \int C_{p,i}\, dT\right) - \frac{d}{dx}\left(\rho \mathcal{D} \frac{d}{dx}\left(\Sigma Y_i \int C_{p,i}\, dT\right)\right) + \dot{m}'' V_x \frac{dV_x}{dx} = -\Sigma h^{\circ}_{f,i}\, \dot{m}_i'''$$

$\dot{m}'' = \text{const}$ and $\Sigma Y_i \int C_{p,i}\, dT = h_{sensible} = \int C_p\, dT$ yielding

$$\dot{m}'' \frac{d}{dx}\left(\int C_p\, dT\right) - \frac{d}{dx}\left(\rho \mathcal{D} \frac{d}{dx}\left(\int C_p\, dT\right)\right) + \dot{m}'' V_x \frac{dV_x}{dx} = -\Sigma h^{\circ}_{f,i}\, \dot{m}_i'''$$

which is the same as equation 7-47 for negligible change in kinetic energy

$$\boxed{\dot{m}'' \frac{d}{dx}\left(\int C_p\, dT\right) - \frac{d}{dx}\left(\rho \mathcal{D} \frac{d}{dx}\left(\int C_p\, dT\right)\right) = -\Sigma h^{\circ}_{f,i}\, \dot{m}_i'''}$$

COMMENTS: Note that we still retain variable properties with the one one restriction being that the diffusion process is binary. Also note that the above expression can be expressed more simply as

$$\dot{m}'' C_p \frac{dT}{dx} - \frac{d}{dx}\left(\rho \mathcal{D} C_p \frac{dT}{dx}\right) = -\Sigma h^{\circ}_{f,i}\, \dot{m}_i'''$$

PROBLEM 7-3

GIVEN: the energy equation for one-dimensional spherical flow (Equation 7-65)

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\rho V_r \int c_p dT - \rho \mathcal{D} \frac{d \int c_p dT}{dr}\right)\right] = -\Sigma h^\circ_{f,i}\, \dot{m}_i'''$$

DERIVE: the conserved scalar equation for enthalpy (Equation 7-82)

$$\frac{d}{dr}\left[r^2\left(\rho V_r h - \rho \mathcal{D} \frac{dh}{dr}\right)\right] = 0$$

ASSUMPTIONS: $Le = 1$, Fick's Law for diffusion, ΔKE negligible

APPROACH: Substitute for $\dot{m}_i'''$ using species mass conservation and simplify, realizing that $h^\circ_{f,i}$ is constant, $\Sigma Y_i h^\circ_{f,i} = h^\circ_f$ and $h = h^\circ_f + \int c_p dT$

species mass conservation:

$$\dot{m}_i''' = \frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\rho V_r Y_i - \rho \mathcal{D} \frac{dY_i}{dr}\right)\right]$$

substituting into given energy equation and rearranging:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\rho V_r \int c_p dT - \rho \mathcal{D} \frac{d \int c_p dT}{dr}\right)\right] + \Sigma \frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\rho V_r Y_i - \rho \mathcal{D} \frac{dY_i}{dr}\right)\right] h^\circ_{f,i} = 0$$

$h^\circ_{f,i}$ is constant and can be moved inside derivatives to yield

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\rho V_r \int c_p dT - \rho \mathcal{D} \frac{d \int c_p dT}{dr}\right)\right] + \frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\rho V_r \Sigma Y_i h^\circ_{f,i} - \rho \mathcal{D} \frac{d}{dr} \Sigma h^\circ_{f,i} Y_i\right)\right] = 0$$

recognizing that $\Sigma Y_i h^\circ_{f,i} = h^\circ_f$ and $h = h_{f,o} + \int c_p dT$ the above becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(\rho V_r h - \rho \mathcal{D} \frac{dh}{dr}\right)\right] = 0$$

$$\boxed{\frac{d}{dr}\left[r^2\left(\rho V_r h - \rho \mathcal{D} \frac{dh}{dr}\right)\right] = 0}$$

PROBLEM 7-4

GIVEN: the energy equation for one-dimensional axi-symmetric flow (Equation 7-66)

DERIVE: the conserved scalar equation for enthalpy (Equation 7-83)

ASSUMPTIONS: Le=1, Fick's Law for diffusion, δKE negligible, no axial diffusion

APPROACH: substitute for $\dot{m}_i'''$ using species mass conservation and simplify

energy equation:

$$\frac{1}{r}\frac{\partial}{\partial x}\left(r\rho v_x \int c_p dT\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho v_r \int c_p dT\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho \mathcal{D} \frac{\partial \int c_p dT}{\partial r}\right) = -\sum h^\circ_{f,i}\, \dot{m}_i'''$$

species mass conservation:

$$\dot{m}_i''' = \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r Y_i\right) + \frac{1}{r}\frac{\partial}{\partial x}\left(\rho r v_x Y_i\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r \mathcal{D}\frac{\partial Y_i}{\partial r}\right)$$

substituting for $\dot{m}_i'''$ in energy equation and rearranging

$$\frac{1}{r}\frac{\partial}{\partial x}\left(r\rho v_x \int c_p dT\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho v_r \int c_p dT\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho \mathcal{D} \frac{\partial \int c_p dT}{\partial r}\right) + \sum \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(\rho r v_r Y_i\right) + \frac{\partial}{\partial x}\left(\rho r v_x Y_i\right) - \frac{\partial}{\partial r}\left(\rho r \mathcal{D}\frac{\partial Y_i}{\partial r}\right)\right\} h^\circ_{f,i} = 0$$

using $h^\circ_{f,i}$ = constant, $\sum Y_i h^\circ_{f,i} = h^\circ_f$, and $h = h_{f,o} + \int c_p dT$ as was done in problem 7-3, the above becomes:

$$\frac{1}{r}\left\{\frac{\partial}{\partial x}\left(\rho r v_x h\right) + \frac{\partial}{\partial r}\left(\rho r v_r h\right) - \frac{\partial}{\partial r}\left(\rho r \mathcal{D}\frac{\partial h}{\partial r}\right)\right\} = 0$$

$$\boxed{\frac{\partial}{\partial x}\left(\rho r v_x h\right) + \frac{\partial}{\partial r}\left(\rho r v_r h\right) - \frac{\partial}{\partial r}\left(\rho r \mathcal{D}\frac{\partial h}{\partial r}\right) = 0}$$

PROBLEM 7-5

GIVEN: the combustion of propane with air with products consisting of CO, CO_2, H_2O, H_2, O_2, and N_2

FIND: the mixture fraction, f, in terms of the various product species mole fractions, χ_i

APPROACH: using the definition of $f = \dfrac{\text{mass originating in fuel}}{\text{mass of mixture}}$, f can be defined in terms of the species mass fractions and then converted to a form involving the species mole fractions.

$$f = \frac{\text{mass originating in fuel}}{\text{mixture mass}} = \frac{[m_c + m_{H_2}]_{MIX}}{m_{mix}} = Y_c + Y_{H_2}$$

$$f = Y_{CO}\frac{MW_c}{MW_{CO}} + Y_{CO_2}\frac{MW_c}{MW_{CO_2}} + Y_{H_2O}\frac{MW_{H_2}}{MW_{H_2O}} + Y_{H_2}\frac{MW_{H_2}}{MW_{H_2}}$$

using the relationship between Y_i and χ_i : $Y_i = \chi_i \dfrac{MW_i}{MW_{mix}}$

$$f = \chi_{CO}\frac{MW_{CO}}{MW_{mix}}\frac{MW_c}{MW_{CO}} + \chi_{CO_2}\frac{MW_{CO_2}}{MW_{mix}}\frac{MW_c}{MW_{CO_2}} + \chi_{H_2O}\frac{MW_{H_2O}}{MW_{MIX}}\frac{MW_{H_2}}{MW_{H_2O}} + \chi_{H_2}\frac{MW_{H_2}}{MW_{MIX}}$$

$$f = \frac{(\chi_{CO} + \chi_{CO_2})MW_c + (\chi_{H_2O} + \chi_{H_2})MW_{H_2}}{MW_{mix}}$$

using the definition of MW_{mix} : $MW_{MIX} \equiv \Sigma \chi_i MW_i$

$$\boxed{f = \frac{(\chi_{CO} + \chi_{CO_2})MW_c + (\chi_{H_2O} + \chi_{H_2})MW_{H_2}}{\chi_{CO}MW_{CO} + \chi_{CO_2}MW_{CO_2} + \chi_{H_2O}MW_{H_2O} + \chi_{H_2}MW_{H_2} + \chi_{O_2}MW_{O_2} + \chi_{N_2}MW_{N_2}}}$$

PROBLEM 7-8

GIVEN: Conservation of energy, Eqn. 7.51

FIND: Derive Eqn. 7.67

ASSUMPTIONS: 1-D, $v_x \, dv_x/dx \approx 0$

SOLUTION: Starting with Eqn. 7.51

$$-\frac{d\dot{Q}''}{dx} = \dot{m}''\frac{dh}{dx}$$

Now $\dot{Q}_x'' = -k\,dT/dx + \sum \dot{m}_{i,diff}''\, h_i$ (Eqn. 7.52a)

where $\dot{m}_{i,diff}'' = \rho Y_i v_{diff,i}$ (Eqn. 7.14), so,

$$-\frac{d}{dx}\left[-k\frac{dT}{dx}\right] - \frac{d}{dx}\left[\sum \rho Y_i v_{d,i} h_i\right] = \dot{m}''\frac{dh}{dx}$$

where $d \equiv diff$.

Expanding 2nd term:

$$\underset{①}{\frac{d}{dx}\left[k\frac{dT}{dx}\right]} - \underset{②}{\sum \rho Y_i v_{d,i}\frac{dh_i}{dx}} - \underset{③}{\sum h_i \frac{d}{dx}(\rho Y_i v_{d,i})} = \underset{④}{\dot{m}''\frac{dh}{dx}}$$

Species conservation, Eqn. 7.16:

$$\frac{d}{dx}(\underbrace{\rho v_x}_{\dot{m}''} Y_i) + \frac{d}{dx}(\rho Y_i v_{d,i}) = \dot{m}_i''' = MW_i\dot{\omega}_i$$

or

$$\frac{d}{dx}(\rho Y_i v_{d,i}) = MW_i\dot{\omega}_i - \dot{m}''\frac{dY_i}{dx}$$

Substituting this result into term ③ above &

PROBLEM 7-8 (continued)

and expanding term ④ ($h = \sum Y_i h_i$) :

$$\frac{d}{dx}\left[k\frac{dT}{dx}\right] - \sum \rho Y_i v_{d,i} \frac{dh_i}{dx} - \sum h_i (MW_i \dot{\omega}_i)$$

$$+ \dot{m}'' \sum \cancel{\frac{dY_i}{dx}} h_i = \dot{m}'' \left[\sum Y_i \frac{dh_i}{dx} + \sum h_i \cancel{\frac{dY_i}{dx}} \right]$$

cancel

Since $h_i \equiv h^o_{f_i} + \int c_{p_i} dT$, $\frac{dh_i}{dx} = c_{p_i} \frac{dT}{dx}$:

$$\frac{d}{dx}\left[k\frac{dT}{dx}\right] - \sum \rho Y_i v_{d,i} c_{p_i} \frac{dT}{dx} - \sum h_i (MW_i \dot{\omega}_i)$$

$$= \dot{m}'' \underbrace{\sum Y_i c_{p_i}}_{\equiv c_p} \frac{dT}{dx} \qquad Q.E.D.$$

COMMENT: Note the importance of species conservation in this derivation

PROBLEM 7-9

GIVEN: $\dot{Q}''_x = \dot{Q}''_{cond} + \dot{Q}''_{\substack{species\\diff}}$

FIND: Apply constitutive relationships for rhs-fluxes & show that

$$\dot{Q}''_x = \rho \mathcal{D} c_p (1-Le)\, dT/dx - \rho \mathcal{D}\, dh/dx$$

ASSUMPTIONS: 1-D

Single $\mathcal{D}$ characterizes mixture

SOLUTION:

$$\dot{Q}''_x = -k\, dT/dx + \Sigma \dot{m}''_{i,diff}\, h_i$$

For Fickian diffusion, $\dot{m}''_{i,diff} = -\rho \mathcal{D} \frac{dY_i}{dx}$.

Thus

$$\dot{Q}''_x = -k\, dT/dx - \Sigma \rho \mathcal{D} \frac{dY_i}{dx} h_i \qquad (I)$$

From the product rule for differentiation:

$$\Sigma \frac{dY_i}{dx} h_i = \frac{d}{dx}\big(\underbrace{\Sigma Y_i h_i}_{h}\big) - \Sigma Y_i \frac{dh_i}{dx}$$

Substituting this result into Eqn I,

$$\dot{Q}''_x = -k\, dT/dx - \rho \mathcal{D} \frac{dh}{dx} - \rho \mathcal{D} \Sigma Y_i \frac{dh_i}{dx}$$

Applying the calorific E.O.S.,

$$h_i = h^o_{fi} + \int c_{pi}\, dT \quad \text{or} \quad \frac{dh_i}{dx} = c_{pi} \frac{dT}{dx}$$

PROBLEM 7-9 (continued)

$$\dot{Q}''_x = -k\, dT/dx - \rho \mathcal{D}\, dh/dx + \rho \mathcal{D} \underbrace{\sum Y_i c_{pi}}_{\equiv c_p} dT/dx$$

Noting that $c_p \equiv \sum Y_i c_{pi}$ & applying the definition of Lewis number,

$$Le = \frac{\alpha}{\mathcal{D}} = \frac{k}{\rho c_p \mathcal{D}},$$

or $k = \rho c_p \mathcal{D} Le$, we write

$$\dot{Q}''_x = -\rho \mathcal{D} c_p Le \frac{dT}{dx} - \rho \mathcal{D} \frac{dh}{dx} + \rho \mathcal{D} c_p\, dT/dx$$

$$= \rho \mathcal{D} c_p (1 - Le) \frac{dT}{dx} - \rho \mathcal{D} \frac{dh}{dx} \qquad \text{Q.E.D.}$$

COMMENT: This result could be used to derive an energy conservation equation in which the unity Lewis number assumption was <u>not</u> invoked.