CHAPTER I

INTRODUCTION

Preliminaries. Perturbation theory is the study of the effects of small disturbances in the mathematical model of a physical system. The model could be expressed as an algebraic equation, integral equation, ordinary differential equation, partial differential equation, or systems of these. The list of types of differential equations which may be solved in closed form using elementary functions is short. These are first order exact, linear, homogeneous, the higher order linear equations with constant coefficients and the partial differential equations that are reducible to these by separation of variables. Beyond this one must use methods of advanced theory, numerical analysis, or approximation of solutions by means of formulas. The latter methods comprise the subject of perturbation theory.

The intent of this paper is to cover the main techniques of perturbation analysis as they have been applied to the solution of ordinary differential equations. Examples of each method will be given to show how it may be applied to the various different classifications of problems and to motivate the theoretical discussions. Because algebraic equations are easy to work with, the details of their solution do not obscure the perturbation methods employed to solve them. For this reason Chapter II will cover algebraic equations and introduce some fundamental concepts of perturbation theory. Chapter III will begin the study of perturbations of ordinary differential equations.

This paper will be organized according to the type of perturbation method being studied. Different types of problems will be solved using each method to illustrate the correct application of that method and to reveal its scope and limitations. Perturbation methods are broadly organized into *regular methods* and *singular methods*.

Problems are often referred to as being singular or regular because singular or regular perturbation methods are required to solve them. We may classify problems according to this convention but it is not an absolute distinction as some problems respond to both methods. Regular problems are usually less difficult as their solutions may be approximated by direct substitution of an asymptotic series for the independent variable. If the resulting series solution holds uniformly in the desired domain of the independent variable then the regular perturbation method is the preferred method.

SIAM Review

The first four examples introduce some terms and motivate the different problem classifications used in perturbation theory.

EXAMPLE 1.1: Regular problem.

$$z^2 - 2z + \epsilon = 0.$$

$\epsilon = 0 \Rightarrow$ roots of the *reduced equation* $z^2 - 2z = 0$ are $\{0, 2\}$,

so as $\epsilon \to 0$, $z_1(\epsilon) \to 0$ and $z_2(\epsilon) \to 2$.

A singular problem either cannot be solved by direct substitution of an asymptotic series, or the resulting series solution holds pointwise and not uniformly in the full domain or it is not uniform except for certain subsets of the domain.

EXAMPLE 1.2: Singular problem. Lin and Segel[5], pp. 278.

$$\epsilon z^2 - 2z + 1 = 0.$$

$\epsilon = 0 \Rightarrow$ root of the reduced equation $-2z + 1 = 0$ is $\left\{\frac{1}{2}\right\}$,

so as $\epsilon \to 0$, $z_1(\epsilon) \to \frac{1}{2}$. Typically, as $\epsilon \to 0$ in a singular problem:

i) the number of solutions change or

ii) the order of the reduced equation changes from the original equation.

The singularity may be caused by the domain or may be inherent in the model.

A singularity in the domain usually occurs as the independent variable $t \to \infty$. These *secular type* problems were first studied by Poincaré in celestial mechanics. The word *secular* is derived from the French word for century because in celestial mechanics the regular asymptotic expansions of equations describing planetary orbits develop terms which only become significant on the order of a century of time.

EXAMPLE 1.3: Secular Type Problem.

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0$$

$$u(0) = a$$
$$\frac{du}{dt}(0) = b.$$

We will see in Chapter III that the regular solution of this problem is not uniformly valid for all $t > 0$.

A singularity in the model typically arises when the phenomenon being modeled is not accurately predicted as the perturbation parameter $\epsilon \to 0$. These *layer type* problems were first studied by Prandtl in fluid flow boundary layer problems.

EXAMPLE 1.4: Layer Type Problem.

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + u = 0$$

$$y(0) = a$$
$$y(1) = b$$
$$0 < x < 1$$

As $\epsilon \to 0$ the order of the equation is changed and the solution cannot represent the system being modeled.

Terminology. Consider a differential equation with initial conditions

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = \epsilon f(x, y, y', \epsilon) \tag{1.1}$$

$$y(0) = \alpha,\ \frac{dy}{dx}(0) = \beta.$$

The variables ($a,\ b,\ c,\ \alpha, \beta, \epsilon$) are called *parameters*. Fixing their value as constants does not change the character of the problem (as long as $a \neq 0$). The variables x and y may not be fixed or the problem becomes meaningless. These two variables are called *coordinates.* The solution of (1.1) has the form

$$y = y(x; a, b, c, \alpha, \beta; \epsilon) = y(x; \boldsymbol{p}; \epsilon)$$

that is, the coordinate y is the *dependent* variable and the coordinate x with the parameters a, b, c, α, β, and ϵ are *independent* variables. Parameters will always be independent variables while coordinates may be dependent or independent. The parameter ϵ is singled out as the *perturbation parameter* because when $\epsilon = 0$ a *reduced problem* is obtained which belongs to a solvable class. The other parameters are *control parameters*.

Gauge functions. We will be interested in the limit of functions such as $f(\epsilon)$ as $\epsilon \to 0$. From Nayfeh[7], pp. 7, if the limit of $f(\epsilon)$ exists then there are three possibilities:

$f(\epsilon) \to 0$
$f(\epsilon) \to A$
$f(\epsilon) \to \infty$

as $\epsilon \to 0$, $0 < A < \infty$. In the first and last cases the rate at which $f(\epsilon) \to 0$ and $f(\epsilon) \to \infty$ is expressed by comparing $f(\epsilon)$ with known functions called *gauge functions*.

DEFINITION 1.1: A gauge function is a positive, monotone function, $\delta(\epsilon)$, defined in some interval, $0 < \epsilon < \epsilon_0$, of interest.

The simplest and most useful of these are the powers of ϵ.

$$1, \epsilon, \epsilon^2, \epsilon^3, \ldots$$

and the inverse powers of ϵ

$$\epsilon^{-1}, \epsilon^{-2}, \epsilon^{-3}, \ldots$$

For $0 < \epsilon < 1$ we know that

$$1 > \epsilon > \epsilon^2 > \epsilon^3 > \cdots$$

and

$$\epsilon^{-1} < \epsilon^{-2} < \epsilon^{-3} < \cdots$$

In some cases these gauge functions must be supplemented by

$$\log \epsilon^{-1}, \log\log \epsilon^{-1}, e^{\epsilon^{-2}}, \text{ etc.}$$

Other possible gauge functions are

$$\sin\epsilon, \cos\epsilon, \tan\epsilon, \sinh\epsilon, \cosh\epsilon, \text{etc.}$$

The behavior of a function $f(\epsilon)$ is compared with a gauge function $g(\epsilon)$ as $\epsilon \to 0$ by employing either of the *Landau symbols, O* or *o*.

Order Symbols.

DEFINITION 1.2: $f(\epsilon)$ is "big oh" of $g(\epsilon)$ written $f(\epsilon) = O(g(\epsilon))$ as $\epsilon \to \epsilon_0$ if a neighborhood of ϵ_0 exists and a constant, $k > 0$, exists such that $|f(\epsilon)| \leq k\,|g(\epsilon)|$. Thus, $f = O(g)$ as $\epsilon \to \epsilon_0$ if f/g is bounded. Cole[2], pp. 1.
For example, as $\epsilon \to 0$

$$\epsilon^n = O(\epsilon^n) \qquad \sin\epsilon = O(\epsilon)$$

$$\sin\epsilon^2 = O(\epsilon^2) \qquad \sin 2\epsilon - 2\epsilon = O(\epsilon^3)$$

$$\cos\epsilon = O(1) \qquad 1 - \cos\epsilon = O(\epsilon^2)$$

$$\cot\epsilon = O(\epsilon^{-1}) \qquad \coth\epsilon = O(\epsilon^{-1})$$

$$J_0(\epsilon) = O(1) \qquad [a_0 + O(\epsilon)]^2 = a_0^2 + O(\epsilon) \quad , a_0 \neq 0$$

For the purposes of this paper we need to know that as $\epsilon \to 0$

$$kO(\epsilon^n) = O(\epsilon^n)$$
$$O(\epsilon^n) + O(\epsilon^k) = O(\epsilon^n) \text{ for } n < k$$
$$O(\epsilon^n)O(\epsilon^k) = O\left(\epsilon^{n+k}\right)$$
$$[O(\epsilon^n)]^k = O(\epsilon^{nk})$$

It is worth noting that the O does not mean "the order of magnitude of" but more accurately means "the asymptotic order of". As an example we see that $10,000 = O(1)$ and $.0001 = O(1)$ in spite of the fact that these functions are not at all the order of magnitude of 1.

DEFINITION 1.3: $f(\epsilon)$ is "little oh" of $g(\epsilon)$ written $f(\epsilon) = o(g(\epsilon))$ as $\epsilon \to \epsilon_0$ if a neighborhood of ϵ_0 exists and a function $\delta(\epsilon) > 0$ exists where $\lim_{\epsilon \to \epsilon_0} \delta(\epsilon) = 0$ and $|f(\epsilon)| \leq \delta(\epsilon)|g(\epsilon)|$. Thus, $f = o(g)$ as $\epsilon \to \epsilon_0$ if $f/g \to 0$. Cole[2], pp. 1.
For example, as $\epsilon \to 0$

$$\epsilon^n = o(\epsilon^{n-1}) \qquad \sin\epsilon = o(1)$$
$$\sin\epsilon^2 = o(\epsilon) \qquad \cos\epsilon = o\left(\epsilon^{-\frac{1}{2}}\right)$$
$$1 - \cos 7\epsilon = o(\epsilon) \qquad J_0(\epsilon) = o(\epsilon^{-1})$$
$$\coth\epsilon = o\left(\epsilon^{-\frac{3}{2}}\right)$$

DEFINITION 1.4: Following Cole[2], pp. 1, any two functions, f and g, of ϵ belong to the same *equivalence class* in a neighborhood of ϵ_0, (ord f or ord g), if

$$0 < \lim_{\epsilon \to \epsilon_0} \frac{f(\epsilon)}{g(\epsilon)} < \infty.$$

A *partial ordering* of equivalence classes is given by

$$\text{ord}\, f(\epsilon) < \text{ord}\, g(\epsilon)$$

if

$$\lim_{\epsilon \to \epsilon_0} \frac{f(\epsilon)}{g(\epsilon)} \to 0$$

We notice that $f(\epsilon) = o(g(\epsilon)) \Rightarrow f(\epsilon) = O(g(\epsilon))$ but that the converse is not true. Order symbols are used to compare the relative "size" of gauge functions. If $f(\epsilon) = o(g(\epsilon))$ then $f(\epsilon) \ll g(\epsilon)$ which means that $f(\epsilon)$ is negligible compared to $g(\epsilon)$. If $f(\epsilon) = O(g(\epsilon))$ and $g(\epsilon) = O(f(\epsilon))$ then we can write $f(\epsilon) \approx g(\epsilon)$ where the symbol $\approx$ means *asymptotic to* and $f(\epsilon)$ and $g(\epsilon)$ are *asymptotically equivalent*.

Asymptotic Expansions. A measure of decreasing orders of magnitude is provided by an asymptotic sequence of functions.

DEFINITION 1.5: A sequence, $\delta_n(\epsilon)$, $n = 1, 2, \ldots$ (finite or infinite) is an *asymptotic sequence* if

$$\delta_{n+1}(\epsilon) = o(\delta_n(\epsilon)) \text{ as } \epsilon \to \epsilon_0$$

which is equivalent to

$$\lim_{\epsilon \to \epsilon_0} \left(\frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} \right) = 0. \tag{1.2}$$

Some examples of asymptotic sequences are

$$\epsilon^n, \ (\sin \epsilon)^n, \ (\log \epsilon)^{-n}.$$

DEFINITION 1.6: From Murdock[6], pp. 69, any approximation

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) \sim \boldsymbol{f}^*(\boldsymbol{x}; \boldsymbol{p}; \epsilon)$$

is called a *pointwise asymptotic approximation* of order $\delta(\epsilon)$ if

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) = \boldsymbol{f}^*(\boldsymbol{x}; \boldsymbol{p}; \epsilon) + o(\delta(\epsilon)) \quad \text{as } \epsilon \to 0.$$

The approximation is called a *uniform asymptotic approximation* for $\boldsymbol{x}$ and $\boldsymbol{p}$ in specified sets if the *o* symbol holds uniformly in those sets.

Most asymptotic approximations are built up sequentially using a set of gauge functions.

$$\delta_0(\epsilon) > \delta_1(\epsilon) > \cdots > \delta_k(\epsilon)$$

DEFINITION 1.7: Let $\delta_n(\epsilon)$ be a sequence of gauge functions. An approximation

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) \sim \boldsymbol{f}_0(\boldsymbol{x}; \boldsymbol{p})\delta_0(\epsilon) + \cdots + \boldsymbol{f}_k(\boldsymbol{x}; \boldsymbol{p})\delta_k(\epsilon) \tag{1.3}$$

is called an *asymptotic series*(pointwise or uniform) provided that each of the following

statements holds(pointwise or uniformly):

$$
\begin{aligned}
(1.4) \qquad \boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) &= \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p})\delta_0(\epsilon) + o(\delta_0(\epsilon)) \\
\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) &= \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p})\delta_0(\epsilon) + \boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p})\delta_1(\epsilon) + o(\delta_1(\epsilon)) \\
&\vdots \\
\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) &= \sum_{n=0}^{k} \boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) + o(\delta_k(\epsilon))
\end{aligned}
$$

This last sum is often called an *asymptotic expansion* of a function $\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon)$ to k terms as $\epsilon \to \epsilon_0$. Murdock[6], pp. 69. We will use the terms asymptotic expansion and asymptotic series interchangeably.

Each error term, or remainder, with fixed n has behavior as $\epsilon \to \epsilon_0$ specified by $o(\delta_n(\epsilon))$, but for fixed ϵ the remainders need not go to zero as $n \to \infty$. An asymptotic series need only satisfy a sequence of error estimates of the form (1.4). Hence, convergence of an asymptotic series is not guaranteed.

Asymptotic series need not be convergent to be useful in perturbation theory. A convergent series must have an infinite number of terms. An asymptotic series may have a finite number of terms, and even if it is infinite, one is never concerned with letting n become large. Instead, (1.4) indicates that the focus is on the error in approximating a function by a partial sum when the number of terms in the partial sum is held constant. This error should decrease at a specified rate as $\epsilon \to 0$ and not as $n \to \infty$.

It may happen that an asymptotic series solution is divergent. Even so, the underlying asymptotic sequence of gauge functions must still satisfy (1.2). This implies that the remainder term may be computed only to within some irreducible error for a given value of ϵ. If we compute terms beyond a certain point, the approximation becomes less and less accurate. In practice, however, one rarely determines more than one or two terms in the expansion because they are usually difficult to compute. Also, additional terms are not necessarily desirable to decrease error in the divergent case where estimates improve as $\epsilon \to 0$ and not as $n \to \infty$.

For instance, in an asymptotic power series, the error after the first two terms(constant and linear term) may be bounded by $10\epsilon^2$; including the quadratic term might give an error bounded by $1000\epsilon^3$. The asymptotic nature of the series specifies the powers in these error bounds, but not the coefficients. In this example, the linear approximation has error less than 0.1 when $\epsilon = 0.1$; the error of the quadratic approximation is only bounded by 1, so the linear approximation is probably best. What we do gain by increasing the number of terms in an asymptotic series is an

improvement in the *rate* at which the error goes to zero when ϵ is decreased. In this illustration the linear approximation is better for $\epsilon = 0.1$, but for $\epsilon = 0.01$ the quadratic is better. One always gains accuracy by increasing n *and decreasing* ϵ *sufficiently*. But for fixed ϵ in the divergent case there may be a certain number of terms beyond which the accuracy begins to decline. Even so, there are cases where a divergent asymptotic series develops more significant figures with fewer computed terms than the convergent power series representing the same function! Consider for example two different representations of Bessel's function of order zero.

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \tag{1.5}$$

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\,[u\cos(x - \tfrac{1}{4}\pi) + v\sin(x - \tfrac{1}{4}\pi)] \quad \text{as } x \to \infty \tag{1.6}$$

where

$$u(x) = 1 - \frac{1^2 \cdot 3^2}{4^2 \cdot 2^2 \cdot 2! x^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{4^4 \cdot 2^2 \cdot 4! x^4} + \cdots$$

$$v(x) = \frac{1}{4 \cdot 2x} - \frac{1^2 \cdot 3^2 \cdot 5^2}{4^3 \cdot 2^3 \cdot 3! x^3} + \cdots$$

The series (1.5) is uniformly and absolutely convergent for all values of x, whereas the series $u(x)$ and $v(x)$, and hence, (1.6) diverge for all values of x. However, the representation (1.6) is asymptotic because the error committed by truncating the series is of the order of the first neglected term. Nayfeh[7], pp. 16.

For small x, the first few terms in (1.5) give fairly accurate results. In fact, the first nine terms give a value of $J_0(2)$ correct to 11 significant figures. However, as x increases, the number of terms needed to yield the same accuracy increases rapidly. At $x = 4$, eight terms are needed to give an accuracy of three significant figures, whereas the first term of the asymptotic expansion (1.6) yields the same accuracy. As x increases further, an accurate result is obtained with far less labor by using the asymptotic divergent series (1.6). In fact, for very large values of x, the convergent series is useless from a computational point of view.

For any given function $\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon)$ the asymptotic expansion (1.4) is not unique because there exists an infinite number of asymptotic sequences that can be used in the representation (1.4). However, given an asymptotic sequence we have the following uniqueness theorem.

THEOREM 1.1: Given an asymptotic sequence, $\delta_n(\epsilon)$, the representation of any function $\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon)$ in terms of this sequence is unique.

Proof: Let

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) = \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p})\delta_0(\epsilon) + \boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p})\delta_1(\epsilon) + \boldsymbol{f}_2(\boldsymbol{x};\boldsymbol{p})\delta_2(\epsilon) + \cdots \tag{1.7}$$

dividing by $\delta_0(\epsilon)$ we have

$$\frac{\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon)}{\delta_0(\epsilon)} = \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p}) + \frac{\boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p})\delta_1(\epsilon)}{\delta_0(\epsilon)} + \frac{\boldsymbol{f}_2(\boldsymbol{x};\boldsymbol{p})\delta_2(\epsilon)}{\delta_0(\epsilon)} + \cdots$$

which upon letting $\epsilon \to \epsilon_0$ yields

$$\boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p}) = \lim_{\epsilon\to\epsilon_0} \frac{\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon)}{\delta_0(\epsilon)}.$$

Moving $\boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p})\delta_0(\epsilon)$ to the left side of (1.7) and dividing by $\delta_1(\epsilon)$ we have

$$\frac{[\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) - \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p})\delta_0(\epsilon)]}{\delta_1(\epsilon)} = \boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p}) + \frac{\boldsymbol{f}_2(\boldsymbol{x};\boldsymbol{p})\delta_2(\epsilon)}{\delta_1(\epsilon)} + \cdots$$

Taking the limit as $\epsilon \to \epsilon_0$ yields

$$\boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p}) = \lim_{\epsilon\to\epsilon_0}\left[\frac{[\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) - \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p})\delta_0(\epsilon)]}{\delta_1(\epsilon)}\right].$$

Continuing in this manner we can see that the $\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})$ are all uniquely determined where

$$\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p}) = \lim_{\epsilon\to\epsilon_0}\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) - \sum_{m=0}^{n-1}\frac{\boldsymbol{f}_m(\boldsymbol{x};\boldsymbol{p})\,\delta_m(\epsilon)}{\delta_n(\epsilon)}. \tag{1.8}$$

□

An expansion obtained by formula (1.8) is called a *limit process expansion,* Murdock[6], pp. 74-75. If $\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon)$ is any function for which the limits (1.8) exist, the series $\sum \boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n$ defined from (1.8) will be asymptotic to $\boldsymbol{f}$. To see that this is more general than a Taylor expansion, even when $\delta_n = \epsilon^n$, consider the function

$$\boldsymbol{f}(\boldsymbol{x};\epsilon) = e^{\epsilon x} + e^{-x/\epsilon}$$

for $\epsilon > 0$ and $x > 0$. This has no Taylor series expansion in ϵ since it is not even defined for $\epsilon = 0$. But it is easy to calculate the limits using (1.8) with $\delta_n = \epsilon^n$ and find an asymptotic expansion, which turns out to be the same as the Taylor series for $e^{\epsilon x}$. A function such as $e^{-x/\epsilon}$, which approaches zero so rapidly as $\epsilon \to 0$ that its influence never shows up in an asymptotic power series is called *transcendentally small*. This example

shows that an asymptotic series does not determine a function uniquely but that the function does determine the series uniquely (provided the gauges are fixed).

Uniformity. Uniformity of an asymptotic series refers to uniform validity of the error estimates in (1.4). The problem is that in parameter perturbations, the quantities to be expanded can be functions of one or more variables besides the perturbation parameter.

DEFINITION 1.8: If we derive the asymptotic expansion of a function $\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon)$, where x is a variable independent of ϵ, we have, in terms of the asymptotic sequence $\delta_n(\epsilon)$

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) \sim \sum_{n=0}^{k} \boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) \qquad \text{as } \epsilon \to 0.$$

This expansion is said to be *uniformly valid* or *regular* if

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) \sim \sum_{n=0}^{k} \boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) + R_{k+1}(\boldsymbol{x};\boldsymbol{p};\epsilon) \tag{1.9a}$$

$$R_{k+1}(\boldsymbol{x};\boldsymbol{p};\epsilon) = o(\delta_k(\epsilon)) \tag{1.9b}$$

holds uniformly for all $\boldsymbol{x}$ and $\boldsymbol{p}$ in specified sets, and if the o symbol holds uniformly in those sets. Otherwise the expansion is said to be *nonuniformly valid* or *singular*. For the conditions (1.4) to hold uniformly, $\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon)$ must be small compared to $\boldsymbol{f}_{n-1}(\boldsymbol{x};\boldsymbol{p})\delta_{n-1}(\epsilon)$ for each n.

Since the sequence $\delta_n(\epsilon)$ is asymptotic, we require that $\boldsymbol{f}_{n-1}(\boldsymbol{x};\boldsymbol{p})/\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})$ be bounded. If this condition is true then the expansion is uniform. In other words, each term must be a small correction to the preceding term irrespective of the value of $\boldsymbol{x}$ and $\boldsymbol{p}$ in their domains of interest.

There is a closely related, but weaker, notion which is sometimes treated as if it were equivalent to uniformity. Each term of (1.3) is clearly of the order of its gauge for fixed $\boldsymbol{x}$ and $\boldsymbol{p}$. That is,

$$\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) = O(\delta_n(\epsilon)) \tag{1.10}$$

always holds pointwise.

DEFINITION 1.9: If for a given set of $\boldsymbol{x}$ and $\boldsymbol{p}$ equation (1.10) holds uniformly for each $n > 0$, the approximation (1.3) will be called *uniformly ordered* for $\boldsymbol{x}$ and $\boldsymbol{p}$ in their respective sets. Murdock[6], pp. 69. Note that no requirement is placed on the leading term $n = 0$.

Uniform ordering is often confused with uniformity. Since uniform ordering depends only upon the terms appearing in the approximation, it is much easier to check than to check for uniformity, which requires an estimate for the error. An important insight is that uniformity implies uniform ordering but the converse is not true. In practice, this means that we look for uniform ordering because it is easier to verify. If the approximation is not uniformly ordered then it can not be uniform. If the expansion is determined to be uniformly ordered then we may proceed to the more difficult process of finding error estimates or to proving uniformity.

THEOREM 1.2:

a. The series

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) \sim \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p})\delta_0(\epsilon) + \cdots + \boldsymbol{f}_k(\boldsymbol{x};\boldsymbol{p})\delta_k(\epsilon) \tag{1.3}$$

is uniformly ordered ($\boldsymbol{x}$ and $\boldsymbol{p}$ in specific sets) if and only if

$$\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p}) \text{ is bounded (for } \boldsymbol{x} \text{ and } \boldsymbol{p} \text{ in those sets)}, \quad n > 0.$$

b. If the series (1.3) is uniform then it is uniformly ordered.

c. If the series (1.3) is uniformly ordered and the last equation of (1.4) holds uniformly, then all of (1.3) holds uniformly and the series is uniform.

Proof:

a. By the definition of uniform ordering

$$\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) = O(\delta_n(\epsilon)) \quad \text{for } n > 0$$

$$\Rightarrow \qquad \left| \frac{\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon)}{\delta_n(\epsilon)} \right| \leq A$$

$$\Rightarrow \qquad \boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p}) \text{ is bounded.}$$

The ($\Leftarrow$) case may be proved by reversing the above steps.

b. If (1.3) is uniform then (1.4) holds uniformly. Comparing the first two equations we see that

$$\boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p})\delta_1(\epsilon) = o(\delta_0(\epsilon)) - o(\delta_1(\epsilon)) = o(\delta_0(\epsilon))$$

uniformly. This means that

$$\frac{\boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p})\delta_1(\epsilon)}{\delta_0(\epsilon)} \to 0 \quad \text{uniformly as } \epsilon \to 0$$

since

$$\frac{\delta_1(\epsilon)}{\delta_0(\epsilon)} \neq 0 \text{ when } \epsilon \neq 0$$

then $\boldsymbol{f}_1(\boldsymbol{x};\boldsymbol{p})$ is bounded. The same argument may be applied sequentially to each $\boldsymbol{f}_n$.
c. Suppose that (1.3) is uniformly ordered and the last equation of (1.4) holds uniformly. Then, since $\boldsymbol{f}_k(\boldsymbol{x};\boldsymbol{p}) = O(\delta_k(\epsilon))$,

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) &= \sum_{n=0}^{k-1}\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) + \{\boldsymbol{f}_k(\boldsymbol{x};\boldsymbol{p})\delta_k(\epsilon) + o(\delta_k(\epsilon))\} \\ &= \sum_{n=0}^{k-1}\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) + O(\delta_k(\epsilon))\end{aligned}$$

uniformly. And since $O(\delta_k(\epsilon)) \Rightarrow o(\delta_{k-1}(\epsilon))$ we have

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) = \sum_{n=0}^{k-1}\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p})\delta_n(\epsilon) + o(\delta_{k-1}(\epsilon))$$

uniformly which establishes uniformity of the next to last line of (1.4). Working up the list we may similarly establish the uniformity of the remaining lines.
Murdock[6], pp. 70. □

To summarize, if an approximation is uniformly ordered it does not follow that it is a uniform asymptotic series because uniform ordering says nothing about the error. But if the last line of (1.4) holds, the rest follows. We may also remark that the error estimate of an asymptotic series as defined by (1.4) can be improved from "o of the last gauge" to "O of the next gauge" except possibly in the last line. That is, except for the last line, the error estimates in (1.4)

$$o(\delta_n(\epsilon))$$

can be replaced by

$$O(\delta_{n+1}(\epsilon))$$

except for the last line which must remain

$$o(\delta_k(\epsilon)).$$

Usually a singular perturbation problem has an asymptotic series solution of the form (1.3) which holds pointwise in the full domain of $\boldsymbol{x}$ but not uniformly, although it may hold uniformly for $\boldsymbol{x}$ in certain subsets of the domain. In order to find approximate

solutions of singular problems which are uniformly valid in the full domain of $\boldsymbol{x}$, it is necessary to use expressions of the form

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{p};\epsilon) \sim \boldsymbol{f}_0(\boldsymbol{x};\boldsymbol{p};\epsilon)\delta_0(\epsilon) + \cdots + \boldsymbol{f}_k(\boldsymbol{x};\boldsymbol{p};\epsilon)\delta_k(\epsilon) \tag{1.11}$$

in which ϵ is allowed to enter through the coefficients $\boldsymbol{f}_n$ as well as the gauges. Such an approximation is called a *generalized asymptotic expansion*. It is only necessary to require that each $\boldsymbol{f}_n(\boldsymbol{x};\boldsymbol{p};\epsilon)$ remains bounded as $\epsilon \to 0$ for fixed x and p to extend the previous results to the generalized asymptotic expansion. We begin treatment of problems using these types of expansions in Chapter IV.

EXAMPLE 1.5: A uniformly valid expansion is given by

$$\begin{aligned}\sin(x+\epsilon) &= \sin x \cos\epsilon + \cos x \sin\epsilon \\ &= \sin x\left[1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} + \cdots\right] + \cos x\left[\epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} + \cdots\right]\end{aligned}$$

Arranged in increasing powers of ϵ, we can see that the coefficients of all the powers of ϵ are bounded for all values of x. It is the rule rather than the exception that a regular expansion in powers of a parameter has regions of nonuniformity defined on the domain of the independent variable. Most perturbation techniques were developed to render nonuniform expansions uniformly valid on larger intervals of the independent variable. An estimate of the size of the region of nonuniformity can sometimes be obtained by assuming two successive terms to be of the same order. Nayfeh[7], pp. 17.

EXAMPLE 1.6: Consider the nonuniformly valid expansion

$$f(x;\epsilon) = \sqrt{x+\epsilon} = \sqrt{x}\left[1 + \frac{\epsilon}{2x} - \frac{\epsilon^2}{8x^2} + \frac{\epsilon^3}{16x^3} - \cdots\right]$$

Each term of this expansion except the first is singular at $x = 0$ and is more singular than the preceding term. An estimate of the size of the region of nonuniformity may sometimes be computed by comparing the order of any two successive terms of the expansion. If we assume that the first two terms are the same order we have

$$1 = O(1) = O\Big(\frac{\epsilon}{2x}\Big) \tag{1.12}$$

As $\epsilon/2x$ must be $O(1)$ or less to ensure uniformity we see that (1.12) implies

$$x = O(\epsilon)$$

This relation estimates the region of nonuniformity about $x = 0$. Comparing the Taylor expansion of $\sqrt{1+\epsilon/x}$ we see that the series converges for $|x| > \epsilon$. Nayfeh[7], pp. 17.

Operations with Asymptotic Series. Asymptotic approximations can be naively added, subtracted, multiplied, or divided resulting in the correct asymptotic expression for the respective sum, difference, product, or quotient of the approximated functions. If appropriate to the limiting process, one expansion may be substituted into another. Asymptotic approximations may also be integrated term by term with respect to ϵ resulting in the correct asymptotic expression for the integral. Differentiation of an asymptotic approximation with respect to an independent variable (coordinate or parameter) is essential in finding heuristic solutions to most perturbation problems, but cannot be justified by a general theorem. The reason is that an asymptotic series is actually an inequality and inequalities cannot be differentiated. Differentiation of expansions which are actually Taylor series may be justified and this is sufficient for most regular type problems. Otherwise, differentiation must be regarded as a heuristic process, justifiable only by an error estimate performed after the solution is obtained. Hinch[3], pp. 22-23 and Murdock[6], pp. 75-76.

Fundamental Theorem of Perturbation Theory.

THEOREM 1.3: If $A_0\delta_0(\epsilon) + A_1\delta_1(\epsilon) + \cdots + A_n\delta_n(\epsilon) + O(\delta_{n+1}(\epsilon)) = 0$ where $\delta_n(\epsilon)$ is an asymptotic sequence and A_i, $i = 0, 1, \ldots, n$ are independent of ϵ then

$$A_0 = A_1 = A_2 = \cdots = A_n = 0$$

Proof: We define

$$P[z(\epsilon)] = A_0\delta_0(\epsilon) + A_1\delta_1(\epsilon) + \cdots + A_n\delta_n(\epsilon) + O(\delta_{n+1}(\epsilon)) = 0 \tag{1.13}$$

Dividing both sides of (1.13) by $\delta_0(\epsilon)$ and taking the limit as $\epsilon \to 0$ we see that

$$P[z(\epsilon)] = 0 \Rightarrow A_0 = 0$$

Now we divide (1.13) by $\delta_1(\epsilon)$ and take the limit as $\epsilon \to 0$. We see that

$$P[z(\epsilon)] = 0 \Rightarrow A_1 = 0$$

Continuing in this manner we can arrive at the relation

$$A_0 = A_1 = A_2 = \cdots = A_n = 0$$ □

Simmonds and Mann[10], pp. 12.

Summary.

Asymptotically equivalent (gauge functions)

$$f(\epsilon) \approx g(\epsilon) \quad \Rightarrow \quad f(\epsilon) = O(g(\epsilon)) \text{ and } g(\epsilon) = O(f(\epsilon))$$

Asymptotic Approximation

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) \sim \boldsymbol{f}^*(\boldsymbol{x}; \boldsymbol{p}; \epsilon) \quad \Rightarrow$$

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) = \boldsymbol{f}^*(\boldsymbol{x}; \boldsymbol{p}; \epsilon) + o(\delta(\epsilon))$$

Asymptotic Series or Asymptotic Expansion

a series asymptotic approximation which satisfies

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) = \sum_{n=0}^{k} \boldsymbol{f}_n(\boldsymbol{x}; \boldsymbol{p}) \delta_n(\epsilon) + o(\delta_k(\epsilon)).$$

and we say

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) \sim \sum_{n=0}^{k} \boldsymbol{f}_n(\boldsymbol{x}; \boldsymbol{p}) \delta_n(\epsilon)$$

Generalized Asymptotic Expansion

$$\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p}; \epsilon) \sim \sum_{n=0}^{k} \boldsymbol{f}_n(\boldsymbol{x}; \boldsymbol{p}; \epsilon) \delta_n(\epsilon)$$