CHAPTER II

POLYNOMIAL EQUATIONS

Finding roots of polynomials is the simplest problem which may be addressed by perturbation theory. Nevertheless, this type of problem may give us insight into proper formulation of the perturbation problem, singular and regular cases, uniform and nonuniform solutions, rescaling coordinates and rescaling parameters. We will consider the following examples adapted from Simmonds and Mann[10], pp. 3-17.

Regular Expansion.

EXAMPLE 2.1: Consider

$$z^2 - 2z + .01 = 0. \tag{2.1}$$

with roots $z_1(\epsilon)$ and $z_2(\epsilon)$. These roots may be found exactly by the quadratic formula and a perturbation method is not required. But our methods will also work for polynomials of higher degree which are not solvable by the quadratic formula. In this case we will "cheat" and use the quadratic formula to verify our results. The following discussion outlines the general method of attack in solving a problem using perturbation techniques.

First, we notice that the constant .01 is relatively small when compared with the other constants in the equation. If we replace it by zero, the equation is easily factored with roots

$$z_1(0) = 0$$

$$z_2(0) = 2.$$

Second, we create a family of problems intermediate between the easy, factorable problem and the original problem by replacing .01 with ϵ. Equation (2.1) now has the form

$$z^2 - 2z + \epsilon = 0 \tag{2.2}$$

and it represents a family of equations, one for each value of ϵ. When $\epsilon = 0$ we have the factorable or so called *reduced* problem and when $\epsilon = .01$ we have the *target* problem.

Third, we find an approximate solution to (2.2) assuming each root has a *regular perturbation expansion* of the form

$$z(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + O(\epsilon^3) \tag{2.3}$$

then

$$z^2(\epsilon) = a_0^2 + 2a_0a_1\epsilon + \big(2a_0a_2 + a_1^2\big)\epsilon^2 + O\big(\epsilon^3\big)$$

substituting these expressions into (2.2) we get

$$a_0^2 + 2a_0a_1\epsilon + \big(2a_0a_2 + a_1^2\big)\epsilon^2 - 2a_0 - 2a_1\epsilon - 2a_2\epsilon^2 + \epsilon + O\big(\epsilon^3\big) = 0$$

combining terms and using the fundamental theorem of perturbation theory

$$\begin{aligned}
&a_0^2 - 2a_0 = 0 && \Rightarrow \quad a_0 = && 0, && 2 \\
&(2a_0a_1 - 2a_1 + 1)\epsilon = 0 && \Rightarrow \quad a_1 = && \tfrac{1}{2}, && -\tfrac{1}{2} \\
&(2a_0a_2 + a_1^2 - 2a_2)\epsilon^2 = 0 && \Rightarrow \quad a_2 = && \tfrac{1}{8}, && -\tfrac{1}{8}
\end{aligned}$$

therefore,

$$z_1(\epsilon) = \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + O\big(\epsilon^3\big)$$

$$z_2(\epsilon) = 2 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O\big(\epsilon^3\big).$$

We see that these are the roots of the reduced equation when $\epsilon = 0$.

Fourth, whenever possible, say something about the error of these approximations. In this case we may compare our approximation when $\epsilon = .01$ to the actual roots of the target equation.

Exact Roots:

$$z_1 = .00501256$$

$$z_2 = 1.9949874$$

Approximate Roots:

$$z_1(.01) = .0050125$$

$$z_2(.01) = 1.9949875$$

with error of $.000001$. If the approximation is not good enough with six decimal digits of accuracy then one may compute more terms of the root expansions.

This short example reveals an important aspect of regular perturbation problems. That is, these methods can only be applied when the target problem is close to a solvable problem (the reduced problem). In fact, we see that the target problem is replaced by a sequence of problems of which the reduced problem is the first. Since the solution of the reduced problem gives us the first term of the series solution it must be solvable for the target problem to be solvable by the regular perturbation method.

Singular Expansion.

EXAMPLE 2.2: Find the roots of the singular problem

$$\epsilon z^2 - 2z + 1 = 0. \tag{2.4}$$

Substituting (2.3) we get

$$\epsilon\left(a_0^2 + 2a_0a_1\epsilon + \left(2a_0a_2 + a_1^2\right)\epsilon^2\right) - 2\left(a_0 + a_1\epsilon + a_2\epsilon^2\right) + 1 = 0$$

collecting terms and using the fundamental theorem

$$\begin{aligned}
-2a_0 + 1 = 0 &\quad\Rightarrow\quad a_0 = \tfrac{1}{2}\\
(a_0^2 - 2a_1)\epsilon = 0 &\quad\Rightarrow\quad a_1 = \tfrac{1}{8}\\
(2a_0a_1 - 2a_2)\epsilon^2 = 0 &\quad\Rightarrow\quad a_2 = \tfrac{1}{16}
\end{aligned}$$

so, we discover only one root!

$$z_1(\epsilon) = \frac{1}{2} + \frac{1}{8}\epsilon + \frac{1}{16}\epsilon^2 + O\left(\epsilon^3\right)$$

What are the characteristics of the missing root? We try to find the second root by using the quadratic formula. Applied to (2.4), the quadratic formula gives

$$z_1(\epsilon) = \frac{1-\sqrt{1-\epsilon}}{\epsilon} \qquad z_2(\epsilon) = \frac{1+\sqrt{1-\epsilon}}{\epsilon}. \tag{2.5}$$

Taking $|\epsilon| < 1$ we can expand $\sqrt{1-\epsilon}$ in a power series in ϵ. Using the binomial expansion

$$\sqrt{1-\epsilon} = 1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 - \cdots$$

and substituting into (2.5) we find

$$z_1(\epsilon) = \frac{1}{2} + \frac{1}{8}\epsilon + O(\epsilon^2)$$

$$z_2(\epsilon) = \frac{2}{\epsilon} - \frac{1}{2} - \frac{1}{8}\epsilon + O(\epsilon^2).$$

So, the missing root $z_2(\epsilon)$ has the characteristic that it approaches infinity as $\epsilon \to 0$, indicating a singularity in the model. The singularity arises when $\epsilon = 0$. We say that the expansion for $z_2(\epsilon)$ does not hold uniformly for all ϵ, but the expansion for $z_1(\epsilon)$ is uniform. The form of $z_2(\epsilon)$ suggests the change of variable

(2.6) $$\omega(\epsilon) = \epsilon z(\epsilon).$$

This change (*rescaling coordinates*) converts (2.4) to

$$\omega^2 - 2\omega + \epsilon = 0$$

where the problem has been transformed from singular to regular behavior. The general procedure of singular perturbation theory is to extract the singular behavior of a solution and by a change of variable and/or parameter to reduce the singular problem to a regular one. We have already solved this reduced problem. The solutions of example 2.2 were

$$\omega_1(\epsilon) = \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + O(\epsilon^3)$$

$$\omega_2(\epsilon) = 2 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3).$$

Restoring the original variable using (2.6) we obtain

$$z_1(\epsilon) = \frac{1}{2} + \frac{1}{8}\epsilon + O(\epsilon^2)$$

$$z_2(\epsilon) = \frac{2}{\epsilon} - \frac{1}{2} - \frac{1}{8}\epsilon + O(\epsilon^2).$$

where we have reproduced the roots of the singular problem by a change of variable without recourse to the quadratic formula. This is satisfactory since our new method will work for polynomials of degree > 2. We also notice that it was necessary, by a change of variable, to include negative powers of the parameter in the assumed form of our perturbation expansion, (2.3). $z_2(\epsilon)$ is not a regular expansion because it is not expressed in positive integer powers of the expansion parameter as it must be to be of the form (2.3). As a result it does not approach a finite value as $\epsilon \to 0$. The change of variable

$\omega(\epsilon) = \epsilon z(\epsilon)$ worked for the specific case of (2.4) but will not work in general. In the general case we will proceed by making the change of variable $\omega(\epsilon) = \epsilon^{\mathrm{P}} z(\epsilon)$ and allow the specifics of the problem to determine a value for P which will transform singular behavior to regular. This is a specific case of the method of *undetermined gauges*.

Undetermined Gauges.

EXAMPLE 2.3: Find an expansion for the roots of

(2.7) $$z^2 - 2\epsilon z - \epsilon = 0.$$

Using expansion (2.3) in (2.7) we have

$$a_0^2 + 2a_0 a_1 \epsilon - 2\epsilon(a_0 + a_1\epsilon) - \epsilon + O(\epsilon^2) = 0$$

which gives us

$$a_0^2 = 0 \qquad \Rightarrow \qquad a_0 = \quad 0$$

$$(2a_0 a_1 - 2a_0 - 1)\epsilon = 0 \quad \Rightarrow \quad -1 = \quad 0$$

a contradiction. Thus, we find no roots, indicating that this equation has no roots of form (2.3). Consider the change of variable

$$z(\epsilon) = \epsilon^{\mathrm{P}} \omega(\epsilon)$$

where

$$\omega(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots$$

with $a_0 \neq 0$. Then (2.7) becomes

(2.8) $$\epsilon^{2\mathrm{P}} \omega^2(\epsilon) - 2\epsilon^{\mathrm{P}+1}\omega(\epsilon) - \epsilon^1 = 0.$$

We solve for P by equating all possible pairs of gauge function exponents. Consider the three possibilities that result.

$$\text{i)} \quad 2\mathrm{P} = \mathrm{P} + 1 \quad \Rightarrow \quad \mathrm{P} = 1$$

$$\text{ii)} \quad \mathrm{P} + 1 = 1 \quad \Rightarrow \quad \mathrm{P} = 0$$

$$\text{iii)} \quad 2\mathrm{P} = 1 \quad \Rightarrow \quad \mathrm{P} = \tfrac{1}{2}$$

Case i) would leave ϵ in the first and second terms and the singular behavior remains.
Case ii) $\Rightarrow z(\epsilon) = \omega(\epsilon)$ and the singular behavior remains.

Case iii) let P $= \frac{1}{2}$, then (2.8) becomes

(2.9) $$\epsilon\omega^2(\epsilon) - 2\epsilon^{3/2}\omega(\epsilon) - \epsilon = 0 \qquad \Rightarrow$$

$$\omega^2(\epsilon) - 2\epsilon^{1/2}\omega(\epsilon) - 1 = 0.$$

Make a change of parameter (*rescaling parameter*)

$$\beta = \epsilon^{1/2}$$

to get rid of the fractional exponent. Then (2.9) becomes

(2.10) $$\omega^2(\beta) - 2\beta\,\omega(\beta) - 1 = 0.$$

The regular expansion for ω in parameter β

$$\omega(\beta) = b_0 + b_1\beta + b_2\beta^2 + O\big(\beta^3\big)$$

is substituted into (2.10) to get

$$b_0^2 + 2b_0b_1\beta + \big(2b_0b_2 + b_1^2\big)\beta^2 - 2b_0\beta - 2b_1\beta^2 - 1 + O\big(\beta^3\big) = 0.$$

Collecting terms and using the fundamental theorem we find

$$\begin{array}{lclll} b_0^2 - 1 = 0 & \Rightarrow & b_0 = & 1, & -1 \\ (2b_0b_1 - 2b_0)\beta = 0 & \Rightarrow & b_1 = & 1, & 1 \\ (2b_0b_2 + b_1^2 - 2b_1)\beta^2 = 0 & \Rightarrow & b_2 = & \frac{1}{2}, & -\frac{1}{2}. \end{array}$$

Therefore,

$$\omega_1(\beta) = 1 + \beta + \frac{1}{2}\beta^2 + O\big(\beta^3\big)$$
$$\omega_2(\beta) = -1 + \beta - \frac{1}{2}\beta^2 + O\big(\beta^3\big).$$

Restore original parameter ϵ

$$\omega_1(\epsilon) = 1 + \epsilon^{1/2} + \frac{1}{2}\epsilon + O\big(\epsilon^{3/2}\big)$$
$$\omega_2(\epsilon) = -1 + \epsilon^{1/2} - \frac{1}{2}\epsilon + O\big(\epsilon^{3/2}\big)$$

and then restore original variable z to obtain the final result.

$$z_1(\epsilon) = \epsilon^{1/2} + \epsilon + \frac{1}{2}\epsilon^{3/2} + O(\epsilon^2) \tag{2.11}$$
$$z_2(\epsilon) = -\epsilon^{1/2} + \epsilon - \frac{1}{2}\epsilon^{3/2} + O(\epsilon^2).$$

We now see that in the case of singular problems it is necessary to include fractional powers of ϵ as well as negative powers in our perturbation expansions. These unknown gauge exponents are determined after substitution into the target problem. Since the regular series expansion will not work, the method of undetermined gauges is classified as a singular perturbation method. As a check on these results we see that the roots of the reduced equation are accurately represented by (2.11) when $\epsilon \to 0$.

We will now state and prove a theorem giving us a procedure to obtain an approximation for any polynomial perturbation. We will then illustrate the method with an example.

THEOREM 2.1: Each root of

$$P_\epsilon(z) = \epsilon^{\alpha_0}(a_0 + b_0\epsilon + c_0\epsilon^2 + \cdots) + \epsilon^{\alpha_1}(a_1 + b_1\epsilon + c_1\epsilon^2 + \cdots)z + \cdots + \epsilon^{\alpha_n}(a_n + b_n\epsilon + c_n\epsilon^2 + \cdots)z^n = 0 \tag{2.12}$$

is of the form

$$z(\epsilon) = \epsilon^{\mathrm{P}} w(\epsilon) \tag{2.13}$$

where $w(\epsilon)$ is a continuous, analytic function of ϵ for ϵ in some neighborhood of 0, $w(0) \neq 0$, and $\alpha_j \in \mathbb{Q}, \quad j = 0, \ldots, n$. Without loss of generality, α_j is nonnegative and at least one of them is zero. Clearly we may also assume $a_k \neq 0$ unless $a_k = b_k = \cdots = 0$.

Proof: First we find the *proper values* of P which cause theorem 2.1 to be satisfied. Substituting (2.13) into (2.12) we have

$$P_\epsilon\left[\epsilon^{\mathrm{P}} w(\epsilon)\right] = Q_\epsilon(w) + R_\epsilon(w) = 0$$

where

$$Q_\epsilon(w) = \epsilon^{\alpha_0} a_0 + \epsilon^{\alpha_1 + \mathrm{P}} a_1 w + \cdots + \epsilon^{\alpha_n + n\mathrm{P}} a_n w^n. \tag{2.14}$$

Note that (2.14) collects all the terms with the lowest power of ϵ for each power of w. Then if $\epsilon^{\mathrm{P}} w(\epsilon)$ is a root of (2.12)

$$P_\epsilon\left[\epsilon^{\mathrm{P}} w(\epsilon)\right] = 0 \quad \text{for all } \epsilon$$

(2.15) $$\Rightarrow \qquad \lim_{\epsilon \to 0} Q_\epsilon(w(\epsilon)) = 0.$$

If we are to satisfy $w(0) \neq 0$, (2.15) implies that at least two of the exponents of the set

$$E = \{\alpha_0, \alpha_1 + \mathrm{P}, \alpha_2 + 2\mathrm{P}, \ldots, \alpha_n + n\mathrm{P}\}$$

must have identical, minimal values. We agree that if the coefficient of $z^r = 0$ then $\alpha_r + rP$ does not appear in set E. If we take a value of P for which there is only a single minimal value in E, say $\alpha_k + k\mathrm{P}$, then we may divide (2.12) by the gauge function $\epsilon^{\alpha_k + k\mathrm{P}}$ and from (2.14) we have

(2.16) $$\lim_{\epsilon \to 0} \epsilon^{\alpha_k + k\mathrm{P}} P_\epsilon\left[\epsilon^{\mathrm{P}} w(\epsilon)\right] = a_k w^k(0) = 0.$$

Since $w(0) \neq 0,\ a_k = 0$ is implied. This is a contradiction because if $a_k = 0$ then its gauge exponent would not be in the set E. So there must be more than one exponent in E with identical minimal values for each proper value of P. We also see that if the exponent value is not minimal then the resulting limit (2.16) would be undefined.

We proceed by finding all the proper values of P and their associated minimal exponents to form the set

(2.17) $$\{(\mathrm{P}_1, e_1), (\mathrm{P}_2, e_2), \ldots, (\mathrm{P}_j, e_j)\}.$$

We use the elements of this set to rewrite (2.12) in the form

(2.18) $$T_{\epsilon j}(w;\epsilon) = \epsilon^{-e_j} P_\epsilon\left(\epsilon^{\mathrm{P}_j} w\right) = T_{0j}(w) + H_{\epsilon j}(w;\epsilon)$$

where $T_{0j}(w)$ is the part that has no ϵ's and $\lim_{\epsilon \to 0} H_{\epsilon j}(w;\epsilon) = 0$. By multiplying P_ϵ by ϵ^{-e_j} and changing the variables from z to w we have extracted the *dominant part* of P_ϵ as a polynomial T_{0j} which is independent of ϵ. The solutions of $T_{0j}(w) = 0$ for each j will give values of w which, when substituted into (2.13), will give is the n roots of (2.12). □

In general, the non-zero roots of the polynomials $T_{\epsilon j}(w) = 0$ need not be regular. The α's in (2.12) and the associated proper values and exponents (P_j, e_j) may be non-integer rationals. Thus, to obtain a regular expansion, new parameters must be introduced. Let

(2.19) $$\epsilon = \beta^{q_j}$$

where q_j is the least common denominator of the set of exponents $\{\alpha_0, \ldots, \alpha_n + n\mathrm{P}_j\}$. Then from (2.18) we form

$$R_{\beta j}(w) = T_j(w;\beta^{q_j})$$

where

$$T_{\epsilon j}(w) = T_j(w;\epsilon).$$

The roots of $T_{\epsilon j}(w) = 0$ are identical to those of $R_{\beta j}(w) = 0$, but the non-zero roots of the latter will have regular expansions in β of the form

$$w(\beta) = b_0 + b_1\beta + \cdots + b_N\beta^N + O\big(\beta^{N+1}\big).$$

EXAMPLE 2.4: Construct expansions for the roots of

$$P_\epsilon(z) = 1 - \epsilon + \epsilon\big(2 + 3\epsilon^2\big)z - \epsilon^{-3}(16 - \epsilon)z^4 + \epsilon^2\big(4 - \epsilon + \epsilon^3\big)z^6 \tag{2.20}$$

Step 1.
Set $z = \epsilon^{\mathrm{P}} w$ in (2.20) and determine the set of exponents E.

$$P_\epsilon(z) = 1 - \epsilon + \epsilon^{\mathrm{P}+1}\big(2 + 3\epsilon^2\big)w - \epsilon^{-3+4\mathrm{P}}(16 - \epsilon)w^4 + \epsilon^{2+6\mathrm{P}}\big(4 - \epsilon + \epsilon^3\big)w^6$$

$$E = \{0, 1 + \mathrm{P}, -3 + 4\mathrm{P}, 2 + 6\mathrm{P}\}$$

Step 2.
Determine the pairs (P_j, e_j) of proper values with associated exponents and polynomials. Proper values and associated exponents may be determined by trial and error or with the aid of a graph. A computer program illustrating the algorithm is included as Appendix I. Figure 1 shows a plot of the exponent lines. The intersections of minimal value may be found at $\mathrm{P} = \frac{3}{4}$ and $\mathrm{P} = -\frac{5}{2.}$

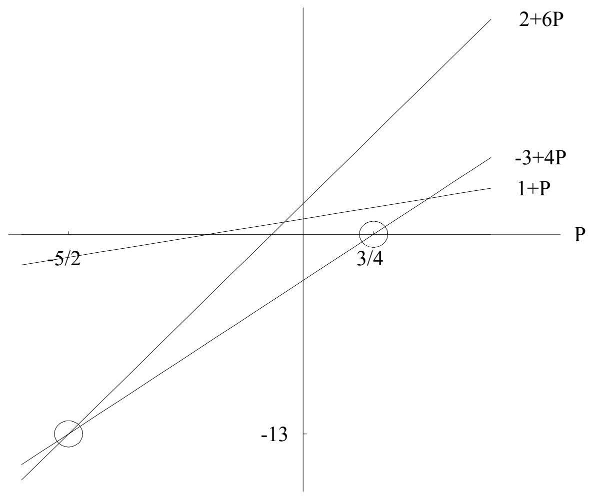

Figure 1.
Proper Values

$$\left(\frac{3}{4},0\right) : T_{\epsilon 1}(w;\epsilon) = 1-\epsilon+\epsilon^{7/4}\left(2+3\epsilon^2\right)w-(16-\epsilon)w^4 \\ +\epsilon^{13/2}\left(4-\epsilon+\epsilon^3\right)w^6$$

$$\left(-\frac{5}{2},-13\right) : T_{\epsilon 2}(w;\epsilon) = \epsilon^{13}(1-\epsilon)+\epsilon^{23/2}\left(2+3\epsilon^2\right)w-(16-\epsilon)w^4 \\ +\left(4-\epsilon+\epsilon^3\right)w^6$$

Step 3.
For each j determine q_j, where q_j is the least common denominator of the set of exponents of $T_{\epsilon j}$. Make the change of parameter $\epsilon=\beta^{q_j}$ and list the associated polynomials $R_{\beta j}(w)$.

(2.21) $$\epsilon=\beta^4 : R_{\beta 1}(w) = 1-\beta^4+\beta^7\left(2+3\beta^8\right)w-\left(16-\beta^4\right)w^4 \\ +\beta^{26}\left(4-\beta^4+\beta^{12}\right)w^6$$

(2.22) $$\epsilon=\beta^2 : R_{\beta 2}(w) = \beta^{26}\left(1-\beta^2\right)+\beta^{23}\left(2+3\beta^4\right)w-\left(16-\beta^2\right)w^4 \\ +\left(4-\beta^2+\beta^6\right)w^6$$

Step 4.
The roots of $T_{\epsilon j}(w)$ are identical to those of $R_{\beta j}(w)$ but the non-zero roots of the latter will have a regular expansion in of the form

$$w(\beta) = b_0 + b_1\beta + \cdots + b_N\beta^N + O\big(\beta^{N+1}\big). \tag{2.23}$$

Substitute (2.23) into $R_{\beta j}(w) = 0$, collect like powers of β, use the fundamental theorem and obtain a sequence of equations in $b_0, b_1, \ldots, b_N$. Solve in order for the unknowns $b_0, b_1, \ldots, b_N$. From (2.21) we get

$$1 - 16b_0^4 = 0$$

$$-1 - 64b_0^3 b_4 + b_0^4 = 0$$
$$\Rightarrow$$
$$b_0 = \sqrt[4]{\frac{1}{16}} = \frac{1}{2}e^{(k-1)i\pi/2}, \quad k = 1,2,3,4$$

$$b_4 = -\frac{15}{64}b_0$$

From (2.22) we get

$$-16b_0^4 + 4b_0^6 = 0$$

$$-64b_0^3 b_2 + b_0^4 + 24b_0^5 b_2 - b_0^6 = 0$$
$$\Rightarrow$$
$$b_0 = \sqrt{4} = 2(-1)^k, \quad k = 5,6$$

$$b_4 = \frac{3}{32}b_0$$

Step 5.
Write down the roots $z_1(\epsilon), \ldots, z_6(\epsilon)$ via the change of variable $z = \epsilon^{\mathrm{P}} w$.

$$z_k(\epsilon) = \frac{1}{2}e^{(k-1)i\pi/2}\left[1 - \frac{15}{64}\epsilon + O\big(\epsilon^{7/4}\big)\right], \quad k = 1,2,3,4$$

$$z_k(\epsilon) = 2(-1)^k\epsilon^{-5/2}\left[1 + \frac{3}{32}\epsilon + O\big(\epsilon^{2}\big)\right], \qquad k = 5,6$$