Notes on Perturbation Techniques for ODEs

James A. Tzitzouris

The idea behind the perturbation method is a simple one. Faced with a problem that we cannot solve exactly, but that is close (in some sense) to an auxiliary problem that we can solve exactly, a good approximate solution to the original problem should be close (in a related sense) to the exact solution of the auxiliary problem.

Suppose we know the solution to the (possibly nonlinear) first-order differential equation given by

$$f(\dot{y}_0, y_0, t) \;\; = \;\; 0, \quad y_0(t_0) = a. \tag{1}$$

Now, consider the *perturbed* first-order differential equation given by

$$f(\dot{y}, y, t) + \varepsilon g(\dot{y}, y, t) \;\; = \;\; 0, \quad y(t_0) = a, \tag{2}$$

where $g(\dot{y}, y, t)$ is some known function and $|\varepsilon|$ is taken to be very small. We assume without loss of generality that y_0 is **not** a solution to the equation $g(\dot{y}, y, t) = 0$. Suppose we cannot solve (2) exactly. For small $|\varepsilon|$, since (2) is close to (1), we should have that y_0 is close to y (i.e., $y(t) \approx y_0(t)$). In fact, as we consider smaller and smaller $|\varepsilon|$, the closer the perturbed and auxiliary problems become. However, we can derive a better approximation of $y(t)$ as follows.

Assume a solution exists for (2), although we cannot solve for it exactly. Let us denote this solution by $y(t; \varepsilon)$ to denote the fact that y is a function of t, but *depends* on the parameter ε. Since we know ε is close to zero, we can take a first-order Taylor expansion of $y(t; \varepsilon)$ around $\varepsilon = 0$ to obtain

$$y(t; \varepsilon) \;\; \approx \;\; \underbrace{y(t; 0)}_{y_0(t)} + \varepsilon \underbrace{\left. \frac{dy(t; \varepsilon)}{d\varepsilon} \right|_{\varepsilon = 0}}_{y_1(t)}.$$

The above approximation carries with it an error term proportional to ε^2, so this implies we should ignore all terms that are proportional to quadratic or higher powers of ε. Defining y_0 and y_1 as indicated in the Taylor expansion above, we define our revised approximation $\hat{y}$ as follows

$$\hat{y} \;\; = \;\; y^0 + \varepsilon y^1. \tag{3}$$

The function $\hat{y}$ should be interpreted as follows. The function y_0 is the roughest approximation to y that we can make. We arrive at y_0 by setting ε to zero. The function y_1 represents a correction term, a refinement to our rough estimate y_0. It is multiplied by ε because the true solution y should approach the rough estimate y_0 as ε approaches zero. Thus, we require a smaller correction as ε approaches zero.

Since $\hat{y} \approx y$, we require that $\hat{y}(t_0) = y_0(t_0) + \varepsilon y_1(t_0) = a$. Since this relation must hold for any choice of ε, we have the modified initial conditions $y_0(t_0) = a$ and $y_1(t_0) = 0$. Substituting (3) into (2), we have

$$f(\dot{y}_0 + \varepsilon\dot{y}_1, y_0 + \varepsilon y_1, t) + \varepsilon g(\dot{y}_0 + \varepsilon\dot{y}_1, y_0 + \varepsilon y_1, t) \quad = \quad 0, \quad y_0(t_0) = a, \quad y_1(t_0) = 0.$$

Defining f_y and $f_{\dot{y}}$ as the partial derivatives of f with respect to y and $\dot{y}$ respectively, we have the following linearization of the above equation around $\varepsilon = 0$

$$f(\dot{y}_0, y_0, t) + \varepsilon\left(f_{\dot{y}}(\dot{y}_0, y_0, t)\dot{y}_1 + f_y(\dot{y}_0, y_0, t)y_1 + g(\dot{y}_0, y_0, t)\right) \quad = \quad 0, \quad y_0(t_0) = a, \quad y_1(t_0) = 0.$$

Since the above equation must hold for all ε close to zero, we must have the following initial value system

$$\left.\begin{aligned} f(\dot{y}_0, y_0, t) = 0, \quad y_0(t_0) = a, \\ \underbrace{f_{\dot{y}}(\dot{y}_0, y_0, t)}_{p(t)}\,\dot{y}_1 + \underbrace{f_y(\dot{y}_0, y_0, t)}_{q(t)}\,y_1 + \underbrace{g(\dot{y}_0, y_0, t)}_{r(t)} = 0, \quad y_1(t_0) = 0. \end{aligned}\right\} \tag{4}$$

Notice that the first equation above is simply the auxiliary equation (1). Upon first look, the second equation appears to be nonlinear. However, note that after solving the first equation, $y_0(t)$ becomes a known function of time. Thus, the second equation is a first-order linear inhomogeneous equation, which we can usually solve using some kind of integrating factor. In fact, if we have some function $h(t)$ that satisfies

$$\frac{d}{dt}\left(h(t)p(t)\right) \quad = \quad h(t)q(t),$$

then

$$y_1(t) \quad = \quad -\frac{1}{h(t)p(t)}\int_{t_0}^{t} h(\tau)r(\tau)d\tau.$$

We leave it as an exercise to the reader to show that if, instead of (1) and (2), we have

$$f(\ddot{y}_0, \dot{y}_0, y_0, t) \quad = \quad 0, \quad y_0(t_0) = a, \tag{5}$$

and

$$f(\ddot{y}_0, \dot{y}_0, y_0, t) + \varepsilon g(\ddot{y}_0, \dot{y}_0, y_0, t) \quad = \quad 0, \quad y(t_0) = a,, \tag{6}$$

then (4) becomes

$$\left.\begin{aligned} f(\ddot{y}_0, \dot{y}_0, y_0, t) = 0, \quad y_0(t_0) = a, \\ \underbrace{f_{\ddot{y}}(\ddot{y}_0, \dot{y}_0, y_0, t)}_{p(t)}\,\ddot{y}_1 + \underbrace{f_{\dot{y}}(\ddot{y}_0, \dot{y}_0, y_0, t)}_{q(t)}\,\dot{y}_1 + \underbrace{f_y(\ddot{y}_0, \dot{y}_0, y_0, t)}_{r(t)}\,y_1 + \underbrace{g(\ddot{y}_0, \dot{y}_0, y_0, t)}_{s(t)} = 0, \quad y_1(t_0) = 0. \end{aligned}\right\} \tag{7}$$

Note that the second equation in (7) is in general a second-order, linear, variable-coefficient, inhomogeneous equation. That may seem like a mouthful, but what it means is that we may not be able to solve this problem. In particular, the "variable-coefficient" part casts a little bit of doubt. However, there are many problems of this type that can be solved.

This leads us to an important observation. In (4) and (7) above, we may have difficulty depending on the functions p, q, r and s. However, when these functions are constant, we can **always** solve these systems. This corresponds to the case when the auxiliary problems, (1) and (5), are both linear, constant-coefficient differential equations. This bears repeating. **If the auxiliary problem is a linear, constant-coefficient ordinary differential equation, then we can always solve the approximating system of equations for y_0 and y_1.**

While there certainly exist some nonlinear differential equations that we can solve exactly, we cannot do so for most nonlinear differential equations exactly. In contrast, it is always the case that we can solve problems involving linear, constant-coefficient differential equations. For this reason, we consider solving **vector** initial value problems whose general form is given by

$$\dot{\vec{y}} = A\vec{y} + \varepsilon g(\vec{y}, t), \quad \vec{y}(t_0) = \vec{a}, \tag{8}$$

where $\vec{y}$ is an n-dimensional vector-valued function of t, A is an $n \times n$ matrix and g is an n-dimensional vector-valued function of two arguments, $\vec{y}$ and t. Substituting $\hat{\vec{y}} = \vec{y}_0 + \varepsilon\vec{y}_1$ into (8) and expanding g in a *zeroth-order* Taylor expansion around $\varepsilon = 0$ we have

$$\dot{\vec{y}}_0 + \varepsilon\dot{\vec{y}}_1 = A\vec{y}_0 + \varepsilon A\vec{y}_1 + \varepsilon g(\vec{y}_0, t), \quad \vec{y}_0(t_0) + \varepsilon\vec{y}_1(t_0) = \vec{a}.$$

Since the above equation must hold for all ε close to zero, we must have that

$$\dot{\vec{y}}_0 = A\vec{y}_0, \quad \vec{y}_0(t_0) = \vec{a},$$

$$\dot{\vec{y}}_1 = A\vec{y}_1 + g(\vec{y}_0, t), \quad \vec{y}_1(t_0) = \vec{0}.$$

Therefore, in order to obtain an approximate solution to the nonlinear initial value problem given by (8) we need only solve the following two **linear** initial value problems

$$\dot{\vec{y}}_0 = A\vec{y}_0, \quad \vec{y}_0(t_0) = \vec{a},$$

$$\dot{\vec{y}}_1 = A\vec{y}_1 + g(\vec{y}_0, t), \quad \vec{y}_1(t_0) = \vec{0}.$$

Example 1.

Consider the initial value problem

$$\dot{y} = \varepsilon e^{y^2}, \quad y(0) = 1.$$

We substitute $\hat{y} = y_0 + \varepsilon y_1$ into the above ODE to obtain

$$\dot{y}_0 + \varepsilon\dot{y}_1 = \varepsilon e^{(y_0+\varepsilon y_1)^2}, \quad y(0) = 1.$$

Expanding the nonlinear term on the right as a zeroth-order Taylor series, we obtain

$$\dot{y}_0 + \varepsilon \dot{y}_1 \quad = \quad \varepsilon e^{y_0^2}, \quad y_0(0) + \varepsilon y_1(0) = 1.$$

Since the above equation must hold for all ε close to zero, we must have that

$$\begin{aligned} \dot{y}_0 &= 0, \quad y_0(0) = 1, \\ \dot{y}_1 &= e^{y_0^2}, \quad y_1(0) = 0. \end{aligned}$$

Thus, $y_0(t) = 1$ and therefore $y_1(t) = et$; thus $\hat{y} = 1 + \varepsilon et$.

Example 2.

Consider the initial value problem

$$\ddot{y} + 4\dot{y} + 3y + \varepsilon y^p \quad = \quad 0, \quad y(0) = 1, \quad \dot{y}(0) = -3,$$

where $p \geq 2$ is an integer. We substitute $\hat{y} = y_0 + \varepsilon y_1$ into the above ODE to obtain

$$\ddot{y_0} + 4\dot{y_0} + 3y_0 + \varepsilon\left(\ddot{y_1} + 4\dot{y_1} + 3y_1 + (y_0 + \varepsilon y_1)^p\right) \quad = \quad 0,\ y_0(0) + \varepsilon y_1(0) = 1,\ \dot{y_0} + \varepsilon \dot{y_1}(0) = -3.$$

Expanding the nonlinear term as a zeroth-order Taylor series, we obtain

$$\ddot{y_0} + 4\dot{y_0} + 3y_0 + \varepsilon\left(\ddot{y_1} + 4\dot{y_1} + 3y_1 + y_0^p\right) \quad = \quad 0,\ y_0(0) + \varepsilon y_1(0) = 1,\ \dot{y}_0(0) + \varepsilon \dot{y}_1(0) = -3.$$

Since the above equation must hold for all ε close to zero, we must have that

$$\begin{aligned} \ddot{y_0} + 4\dot{y_0} + 3y_0 &= 0, \quad y_0(0) = 1, \quad \dot{y_0}(0) = -3, \\ \ddot{y_1} + 4\dot{y_1} + 3y_1 &= -y_0^p, \quad y_0(0) = 0, \quad \dot{y_0}(0) = 0. \end{aligned}$$

Solving the above equations (left as an exercise to the reader), we have that

$$\begin{aligned} y_0(t) &= e^{-3t}, \\ y_1(t) &= \frac{1}{6}\frac{e^{-3t}}{p-1} + \frac{1}{2}\frac{e^{-t}}{1-3p} - \frac{1}{3}\frac{e^{-3pt}}{(1-3p)(1-p)}. \end{aligned}$$

Thus $\hat{y} = e^{-3t} + \varepsilon \frac{1}{6}\frac{e^{-3t}}{p-1} + \frac{1}{2}\frac{e^{-t}}{1-3p} - \frac{1}{3}\frac{e^{-3pt}}{(1-3p)(1-p)}$.

Example 3.

Consider the initial value problem

$$2y\dot{y} - y^2 + \varepsilon\sqrt{y} \quad = \quad 0, \quad y(0) = 1.$$

We substitute $\hat{y} = y_0 + \varepsilon y_1$ into the above ODE to obtain

$$2(y_0 + \varepsilon y_1)(\dot{y}_0 + \varepsilon \dot{y}_1) - (y_0 + \varepsilon y_1)^2 + \varepsilon\sqrt{y_0 + \varepsilon y_1}, \quad y_0(0) + \varepsilon y_1(0) = 1.$$

Expanding the nonlinear terms as Taylor series and ignoring high-order terms we obtain

$$\left(2y_0\dot{y}_0 - y_0^2\right) + \varepsilon\left(2y_0\dot{y}_1 + 2y_1\dot{y}_0 - 2y_0y_1 + \sqrt{y_0}\right), \quad y_0(0) + \varepsilon y_1(0) = 1.$$

Since the above equation must hold for all ε close to zero, we must have that

$$\begin{aligned} 2y_0\dot{y}_0 - y_0^2 &= 0, \quad y_0(0) = 1, \\ 2y_0\dot{y}_1 + 2y_1\dot{y}_0 - 2y_0y_1 + \sqrt{y_0} &= 0, \quad y_1(0) = 0. \end{aligned}$$

Thus, $y_0(t) = e^{\frac{1}{2}t}$ and therefore $y_1(t) = \frac{2}{3}(e^{-\frac{1}{4}t} - e^{\frac{1}{2}t})$; thus $\hat{y} = e^{\frac{1}{2}t} + \frac{2\varepsilon}{3}(e^{-\frac{1}{4}t} - e^{\frac{1}{2}t})$.

Example 4.

Consider the initial value problem

$$\dot{y} - y - \varepsilon\frac{1}{y} = 0, \quad y(0) = 1.$$

We substitute $\hat{y} = y_0 + \varepsilon y_1$ into the above ODE to obtain

$$\dot{y}_0 + \varepsilon\dot{y}_1 - y_0 + \varepsilon y_1 + \varepsilon\frac{1}{y_0 + \varepsilon y_1}, \quad y_0(0) + \varepsilon y_1(0) = 1.$$

Multiplying the above equation by $y_0 + \varepsilon y_1$ and ignoring high-order terms we obtain

$$\left(y_0\dot{y}_0 - y_0^2\right) + \varepsilon\left(y_0\dot{y}_1 - 2y_0y_1 + \dot{y}_0y_1 - 1\right), \quad y_0(0) + \varepsilon y_1(0) = 1.$$

Since the above equation must hold for all ε close to zero, we must have that

$$\begin{aligned} y_0\dot{y}_0 - y_0^2 &= 0, \quad y_0(0) = 1, \\ y_0\dot{y}_1 - 2y_0y_1 + \dot{y}_0y_1 - 1 &= 0, \quad y_1(0) = 0. \end{aligned}$$

Thus, $y_0(t) = e^t$ and therefore $y_1(t) = \frac{1}{2}(e^t - e^{-t})$; thus $\hat{y} = e^t + \frac{\varepsilon}{2}(e^t - e^{-t})$.

Example 5.

Consider the initial value problem

$$\dot{y} + \varepsilon y^2 + \kappa\dot{y}^2 = 0, \quad y(0) = 1.$$

We substitute $\hat{y} = y_0 + \varepsilon y_1 + \kappa y_2$ into the above ODE and ignore terms multiplied by $\varepsilon^p \kappa^q$ for $p + q \geq 2$ to obtain

$$\dot{y}_0 + \varepsilon \left(\dot{y}_1 + y_0^2\right) + \kappa \left(\dot{y}_2 + \dot{y}_0^2\right) \;=\; 0, \quad y_0(0) + \varepsilon y_1(0) + \kappa y_2(0) = 1.$$

Since the above equation must hold for all ε and κ close to zero, we must have that

$$\begin{aligned} \dot{y}_0 &= 0, \quad y_0(0) = 1, \\ \dot{y}_1 + y_0^2 &= 0, \quad y_1(0) = 0, \\ \dot{y}_2 + \dot{y}_0^2 &= 0, \quad y_2(0) = 0. \end{aligned}$$

Thus, $y_0(t) = 1$ and therefore $y_1(t) = -t$ and $y_2(t) = 0$; thus $\hat{y} = 1 - \varepsilon t$.

Example 6.

Consider the initial value problem

$$\dot{y} - \varepsilon \sqrt{y} \;=\; 0, \quad y(0) = 1.$$

Note that this is a separable equation, with exact solution

$$y(t) \;=\; \frac{\varepsilon^2}{4} t^2 + \varepsilon t + 1.$$

Since the *exact* answer contains a term that is multiplied by ε^2, we should expect that our first-order approximate answer is equal to $\varepsilon t + 1$. That is, the exact answer contains a term that is lost in the approximation error. Let us see. We substitute $\hat{y} = y_0 + \varepsilon y_1$ into the above ODE and ignore high-order terms to obtain

$$\dot{y}_0 + \varepsilon \left(\dot{y}_1 - \sqrt{y_0}\right) \;=\; 0, \quad y_0(0) + \varepsilon y_1(0) = 1.$$

Since the above equation must hold for all ε close to zero, we must have that

$$\begin{aligned} \dot{y}_0 &= 0, \quad y_0(0) = 1, \\ \dot{y}_1 - \sqrt{y_0} &= 0, \quad y_1(0) = 0. \end{aligned}$$

Thus, $y_0(t) = 1$ and therefore $y_1(t) = t$; thus $\hat{y} = 1 + \varepsilon t$, as expected. Notice that

$$\begin{aligned} y_0(t) &= \left. y(t) \right|_{\varepsilon = 0}, \\ y_1(t) &= \left. \frac{\partial y(t)}{\partial \varepsilon} \right|_{\varepsilon = 0}. \end{aligned}$$