CHAPTER VII

TRANSITION LAYER PROBLEMS

The remaining chapters of this paper concern differential equations of the form

$$\epsilon \frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

and other equations for which the order drops when $\epsilon = 0$. This class of problems is very different from the oscillatory problems studied up to this point. The primary challenge for those problems was to find solutions valid for long intervals of time. Regular expansions gave solutions valid for finite intervals, and more subtle methods were able to extend validity to expanding intervals or for all time in some cases. In the problems to be studied now, it is difficult to find solutions that are even valid for finite intervals, because the solutions undergo rapid transitions as they pass from one part of their domain to another. However, it is often fairly easy to find valid approximate solutions restricted to one part of the domain. The problem then, is to fit these solutions together to create an approximation valid over the entire domain. These problems have been grouped by Murdock[6], pp. 343, into three types:

1. Boundary Layer problems. These are boundary value problems in which y' is relatively large in a region or "layer" near one or both boundaries of its domain. To emphasize the nature of the boundaries in this type problem we will use "x" as the independent variable instead of "t".

2. Initial layer problems. These are initial value problems in which y' is relatively large immediately after being started at the initial point. A problem of this type will be considered in example 7.4.

3. Internal layer problems. Initial or boundary value problems in which y' is relatively large in one or more regions in the interior of its domain. Together these three types of problems are called *transition layer problems*.

Until now, the reduced problem has always been a simpler, exactly solvable special case of the perturbed problem. In transition layer problems the exact solution does not exist when $\epsilon = 0$. We will see that the problem obtained by setting $\epsilon = 0$ is self contradictory and is not a suitable starting point for developing an asymptotic series. This difficulty is resolved by dropping some of the initial or boundary conditions when $\epsilon = 0$ in order to define a suitable reduced problem from which to begin. It is a characteristic feature of layer type problems that the reduced equation has a different qualitative character than the original problem.

The concept of a boundary layer will be introduced by a simple "model problem" that is also exactly solvable. We will examine the exact solution to motivate the main lines of a singular perturbation procedure that works for a large class of problems in which a small parameter multiplies the highest derivative in a differential equation. The variety of singular perturbation problems that one is likely to meet is so large as to preclude the hope of a comprehensive recipe. In fact, none is available. However, we will present several methods that are effective in many applications.

Boundary Layer Problem.

EXAMPLE 7.1: Consider the following equation for $y(x)$ from Lin and Segel[5], pp. 285-288.

(7.1a) $$\epsilon\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0,\quad 0<x<1,\quad 0<\epsilon\ll 1$$

(7.1b) $$y(0)=0$$

(7.1c) $$y(1)=1$$

To obtain a first approximation to this problem we will apply the zeroth order regular expansion, $y(x;\epsilon)\sim y_0$. Setting $\epsilon=0$ we have the reduced problem

(7.2) $$2\frac{dy}{dx}+y=0$$

whose general solution is

$$y_0=Ce^{-x/2}.$$

If we attempt to find C at $x=0$ we get

(7.3) $$y_0=0.$$

But if we compute C at $x=1$ we find

(7.4) $$y_0=e^{(1/2-x/2)}.$$

So the regular expansion gives us a contradiction in that (7.3) and (7.4) cannot both be the solution of (7.2). This should not be a surprise since it would be a fortunate coincidence that a *first* order equation could satisfy *two* boundary conditions. Hence, one of the boundary conditions cannot be satisfied and must be dropped in our approximation. Consequently, the resulting approximation is not expected to be valid at or near the end point where the boundary condition has been dropped. To gain further insight into this type of problem we will now examine its exact solution and then use it to obtain an approximate solution for small, positive ϵ.

Since (7.1) is linear with constant coefficients its general solution is given by

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} \tag{7.5}$$

where k_1 and k_2 are the two roots of

$$\epsilon k^2 + 2k + 1 = 0. \tag{7.6}$$

When the boundary conditions are imposed on (7.5) we obtain the final exact solution

$$y^e = \left(e^{k_1 x} - e^{k_2 x}\right) / \left(e^{k_1} - e^{k_2}\right). \tag{7.7}$$

Equation (7.6) is similar to the problem considered in example 2.2, so when ϵ is small we see that we may approximate k_1 and k_2 by

$$k_1 = -\frac{1}{2}, \qquad k_2 = -\frac{2}{\epsilon}.$$

Since ϵ is positive, we may neglect e^{k_2} compared to e^{k_1} in the denominator of (7.7). Our approximate solution takes the form

$$y(x;\epsilon) \sim e^{1/2}\left(e^{-x/2} - e^{-2x/\epsilon}\right), \quad 0 < \epsilon \ll 1. \tag{7.8}$$

As required by the boundary condition (7.1b), this approximation has the value zero at $x = 0$, the terms in the brackets canceling exactly. However, as x increases the second term decreases rapidly. When $x = \epsilon$ this term is about one-seventh of its value at $x = 0$, and at $x = 2\epsilon$ it is less than 2 percent of this value. Thus, if $x > 2\epsilon$, $e^{-2x/\epsilon}$ is negligible and (7.8) may be further approximated outside of $O(\epsilon)$ by

$$y(x;\epsilon) \sim e^{(1/2 - x/2)} \tag{7.9}$$

which is exactly the same solution given by the reduced equation evaluated at the right boundary ($x = 1$). We see from the exact equation that the solution changes rapidly in the layer near $x = 0$ whose thickness is $O(\epsilon)$. This region of rapid change near a boundary is called a *boundary layer*.

For small positive ϵ, the solution rises rapidly from its assigned value of zero at $x = 0$ until it merges with equation (7.4), which satisfies boundary condition (7.1c) at $x = 1$. The solution of the reduced equation is depicted by a dashed line in Figure 8 and the approximate of the exact solution, (7.8), is depicted by a solid line. Since the approximation given by equation (7.4) is valid <u>outside</u> the boundary layer it is referred to as the *outer solution* and is labeled y^o. For Figure 8, $\epsilon = 0.1$.

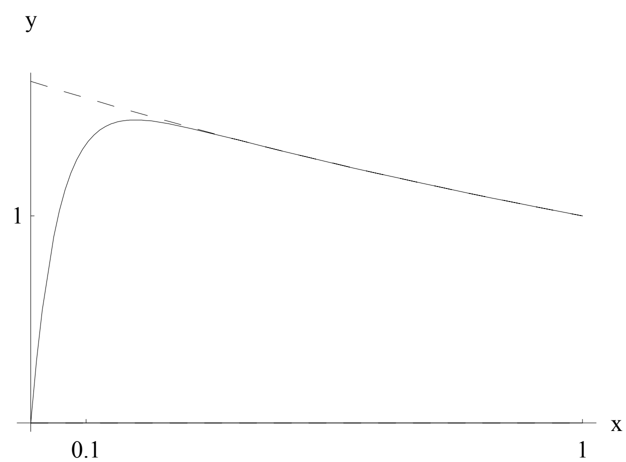

Figure 8.
Exact Layer Solution vs. Outer Solution

We now know the behavior of the exact solution and have a way of approximating an outer solution which is only valid outside the boundary layer. We need to find an *inner solution*, y^i, that is valid inside the boundary layer. Since the inner solution will probably not be valid outside the boundary layer, we will also need to find a way to combine (match) the inner and outer solutions to form a *composite solution*, y^c, that is uniform over the entire domain of interest of the independent variable. This method is known as the method of *matched asymptotic expansions*. With this plan in mind let us look again at the same example and proceed systematically through the approximation process.

Step 1. <u>Determine the outer approximation</u>.

The outer approximation will always be given by the regular expansion of the original equation evaluated at the outer boundary. The only dilemma is that in general we do not know the location of the boundary layer. If knowledge of the physics behind the problem formulation gives no clue (but it usually will), then the boundary layer must be found by trial and error. The procedure is to assume that the boundary layer is located at one of the boundaries and then perform the remaining steps in this analysis using the first approximation. If a contradiction results somewhere along the way, then we picked the wrong boundary. For a second order linear O.D.E. with variable coefficients and

$c(x) > 0$ if $b(x) > 0$ then the boundary is on the left and if $b(x) < 0$ then the boundary is on the right.

Step 2. Introduce a change of variable.

The outer approximation does not vary rapidly with its slow independent variable x which is of order $O(1)$. This slow scale is not appropriate in the boundary layer as the solution there changes rapidly in a distance of magnitude $\delta(\epsilon)$. This layer thickness approaches zero as $\epsilon \to 0$ so we require that $\delta \to 0$ as $\epsilon \to 0$. To mathematically exploit this analysis we will introduce a faster boundary layer variable, ξ, which will permit us to measure proportional distances into the layer. Let

$$\xi = \pm \frac{x-a}{\delta(\epsilon)}. \tag{7.10}$$

Here we are rescaling x with an appropriately chosen function for $\delta(\epsilon)$. "a" represents the value for x at the location of the boundary layer. In our present example, $a = 0$. $\xi \to 0$ as $x \to$ (the boundary) and the sign is chosen to make ξ positive. Before ξ is completely determined we must determine $\delta(\epsilon)$. This task is accomplished by means of the method of undetermined gauges last seen in Chapter II. Let us introduce (7.10) into (7.1) and replace the function y by the function y^i (for "y-inner" per Lagerstrom and Casten[4]). We obtain

$$\frac{\epsilon}{\delta^2}\frac{d^2y^i}{d\xi^2} + \frac{2}{\delta}\frac{dy^i}{d\xi} + y^i = 0 \tag{7.11}$$

$$y^i(0) = 0.$$

If we assume that $\delta(\epsilon) = \epsilon^{\mathrm{P}}$, $\mathrm{P} > 0$, then we can quickly find by undetermined gauges that

$$\delta(\epsilon) = \epsilon. \tag{7.12}$$

The value of the function $\delta(\epsilon)$ that satisfies the requirements of undetermined gauges is sometimes referred to as the *distinguished limit*(Nayfeh[8], pp. 277 and Cole[2], pp. 10).

Step 3. Determine y^i.

With (7.12) in hand, (7.11) becomes

$$\frac{d^2y^i}{d\xi^2} + 2\frac{dy^i}{d\xi} + \epsilon y^i = 0. \tag{7.13}$$

Here we see that the change of variable has allowed us to change a singular equation into a regular equation. Just as the regular expansion of (7.1) expressed in terms of the slow variable and evaluated at the outer boundary gave us the outer solution, we expect that the regular expansion of (7.13) expressed in terms of the fast variable and evaluated at the inner boundary will give us the inner solution valid in the boundary layer. Setting $\epsilon = 0$ in (7.13) gives

$$\frac{d^2y^i}{d\xi^2} + 2\frac{dy^i}{d\xi} = 0. \tag{7.14}$$

This equation, which is derived when δ equals the distinguished limit, is referred to as the *least degenerate form* of the limiting equation (7.11) for all choices of δ. It turns out that the proper scale provided by the distinguished limit always yields the least degenerate limiting form of the equation in the boundary layer. Therefore, we will always determine the proper scale by choosing distinguished limits. The solution to (7.14) evaluated at the inner boundary is

$$y^i(\xi) = C\big(1 - e^{-2\xi}\big). \tag{7.15}$$

The inner equation is second order and we might expect it to be able to satisfy both boundary conditions. The reason that it cannot is because its regular expansion is valid only in a small interval which includes the boundary layer and this interval does not include the outer boundary in general. We need the "extra" arbitrary constant to enable us to complete the matching procedure.

Step 4. Match the inner and outer approximations.

We must now try to combine (7.4) and (7.15) to make an approximate solution of (7.1). The constant C will be determined in the process of making (7.4) and (7.15) match at the edge of the boundary layer, which is located somewhere between $O(\epsilon)$ and $O(1)$. In this *intermediate region* we hope to be able to satisfy the requirement that

$$\lim_{x\to 0} y^o = \lim_{\xi\to\infty} y^i. \tag{7.16}$$

Direct application of (7.16) to (7.4) and (7.15) gives

$$C = e^{1/2}.$$

We might call (7.16) the "quick matching procedure" following Simmonds and Mann[10], pp. 99. In general, matching is more complicated and must be carried out in a more formal way by introducing an intermediate variable appropriate for the intermediate region. But for now (7.15) becomes

$$y^i(\xi) = e^{1/2}\big(1 - e^{-2\xi}\big).$$

Written in terms of the original variable, x, our approximation is now given by two pieces.

$$y(x;\epsilon) \sim \begin{cases} e^{1/2}\big(1 - e^{-2x/\epsilon}\big), & x < O(\epsilon) \\ e^{(1/2 - x/2)}, & x > O(\epsilon) \end{cases} \tag{7.17}$$

Equation (7.17) is awkward because we must decide when to stop using one expression and start using the other. A uniformly valid composite solution, y^c, may be found by adding the inner and outer approximations and subtracting the part which is common to both. This common part is the common limit given by (7.16). Thus

$$y^c = y^o + y^i - \lim_{x\to 0} y^o$$

which evaluates to

$$y^c = e^{1/2}\big(e^{-x/2} - e^{-2x/\epsilon}\big). \tag{7.18}$$

We recognize this as the same approximation, (7.8), which we derived from the exact solution.

The basic idea of layer type problems is that an approximate solution to a given problem is sought not as a single expansion in terms of a single scale but as two or more separate expansions in terms of two or more scales each of which is valid in part of the domain. The scales are chosen so that, (a) the expansion as a whole covers the entire domain of interest and, (b) the domains of validity of neighboring expansions overlap. Because the domains overlap, the neighboring expansions can be matched.

Neighboring expansions obtained by using different scales do not necessarily have overlapping domains. Moreover, the union of overlapping domains does not necessarily cover the entire domain of interest. In our present example we could compute expansions with the scales x/ϵ^2 and x to illustrate the first point and expansions with x and $x/\epsilon^{1/2}$ to illustrate the second. We do not need to worry about these concerns in practical application as the method outlined above will provide us with the distinguished limit in most cases. In problems where our methods fail to produce a valid uniform approximation, the fault lies in a violation of one of these two requirements, (a) or (b).

Van Dyke Method. The Van Dyke matching method is based on three hypothesis which can be stated in terms of *expansion operators* as in Murdock[6], pp. 358-362. The k-term inner expansion operator, I_k, when placed in front of any function y of (t, ϵ) or (ξ, ϵ) means, "express the function in terms of the inner variable ξ and then expand in powers of ϵ, keeping the first k terms." O_k, the k-term outer expansion operator is defined to mean, "express the function in terms of the outer variable t and expand in powers of ϵ, keeping the first k terms."

The first Van Dyke hypothesis

(7.19) $$O_k I_k y^e = I_k O_k y^e$$

says that if you express a function y in terms of ξ, expand in ϵ, truncate at k terms, express the result in terms of t, expand again in ϵ, and truncate at k terms, you will(often) get the same result as if you perform these operations in the opposite order. The first Van Dyke hypothesis concerns only the exact solution y^e. The remaining two hypotheses relate the exact solution y^e to the k-term inner solution, y^i, and the k-term outer solution, y^o.

The next two hypotheses are

(7.20) $$I_k y^e = y^i$$

(7.21) $$O_k y^e = y^o$$

which say that the inner and outer expansions of the exact solution are equal respectively to the inner and outer perturbation solutions, which are computed by formal perturbation methods without knowledge of the exact solutions. Since all three of the Van Dyke hypotheses refer to the exact solution y^e, which for any real problem in perturbation theory is unknown, we wonder how they could be of any practical use. The answer is found by substituting (7.20) and (7.21) into (7.19) to obtain

(7.22a) $$O_k y^i = I_k y^o$$

which is most often seen in the simpler looking but less precise form

(7.22b) $$y^{io} = y^{oi}.$$

The symbol y^{io} denotes the "outer expansion of the inner solution," called the *inner-outer solution*; similarly, y^{oi} is the "inner expansion of the outer solution" called the *outer-inner solution*. Either form of (7.22) states that the k-term outer expansion of the k-term inner solution equals the k-term inner expansion of the k-term outer solution when the undetermined constants in these solutions are chosen correctly. The Van Dyke method uses equation (7.22) as a means of fixing these constants and thereby achieving the "first objective" of matching. The "second objective", the creation of a composite solution, is then achieved by the formula

(7.23) $$y^c = y^i + y^o - y^{io}.$$

The symbols I_p and O_q are also used in more general contexts for p-term inner and q-term outer expansions with respect to any specified set of gauge functions. In some problems it is necessary to use non-integer gauges in the inner and/or outer regions. Then, achieving a given degree of accuracy may require that a different number of terms be used in each expansion. One may need to work with a value of p that is unequal to q. In most of our examples $p = q$ and we call them both k. When using gauges that are non-integer powers, p and q still denote the number of terms retained, but this no longer equals the order of the first omitted term. In these instances a more precise rendering of (7.22) is

(7.22c) The q-term outer expansion of the p-term inner expansion equals the p-term inner expansion of the q-term outer expansion.

Equations (7.22) and (7.23) express the essence of the Van Dyke method. Let us now present two examples using the expansion operators I_k and O_k to gain some familiarity with the Van Dyke matching method.

EXAMPLE 7.2: Find a second order approximation to problem (7.1) using Van Dyke matching.

OUTER EXPANSION

We seek an outer expansion of the form

(7.24) $$y^o = y_0(x) + \epsilon y_1(x) + \cdots$$

This expansion is expected to satisfy the boundary condition $y(1) = 1$ because the boundary layer is at the origin. Substituting (7.24) into (7.1) and equating coefficients of like powers of ϵ we have

(7.25) $$2\frac{dy_0}{dx} + y_0 = 0 \qquad y_0(1) = 1$$

(7.26) $$2\frac{dy_1}{dx} + y_1 = -\frac{d^2y_0}{dx^2} \qquad y_1(1) = 0.$$

Previously, we found the solution of (7.25) to be

$$y_0 = e^{(1/2 - x/2)}.$$

Then (7.26) becomes

$$2\frac{dy_1}{dx} + y_1 = -\frac{1}{4}e^{(1/2-x/2)}$$

whose general solution is

$$y_1 = C_1 e^{-x/2} - \frac{1}{8}xe^{(1/2-x/2)}.$$

Evaluated at the right boundary we have

$$y_1 = \frac{1}{8}e^{(1/2-x/2)} - \frac{1}{8}xe^{(1/2-x/2)}.$$

Therefore,

$$y^o = e^{(1/2-x/2)} + \epsilon\left(\frac{1}{8}e^{(1/2-x/2)} - \frac{1}{8}xe^{(1/2-x/2)}\right) + O(\epsilon^2). \tag{7.27}$$

INNER EXPANSION

In the previous section we determined the distinguished limit and the least degenerate form for this problem. Using this information we will now change the independent variable from x to $\xi = x/\epsilon$ and determine the appropriate inner expansion. Then (7.1) becomes

$$\frac{d^2y^i}{d\xi^2} + 2\frac{dy^i}{d\xi} + \epsilon y^i = 0. \tag{7.28}$$

Since the boundary layer is at the origin and $x = 0 \Rightarrow \xi = 0$ we have from (7.1)

$$y^i(0) = 0. \tag{7.29}$$

We seek an inner expansion of the form

$$y^i = Y_0(\xi) + \epsilon Y_1(\xi) + \cdots \tag{7.30}$$

Substituting (7.30) into (7.28) and (7.29) and equating coefficients of like powers of ϵ, we have

$$\frac{d^2Y_0}{d\xi^2} + 2\frac{dY_0}{d\xi} = 0 \qquad Y_0(0) = 0$$

$$\frac{d^2Y_1}{d\xi^2} + 2\frac{dY_1}{d\xi} = -Y_0 \qquad Y_1(0) = 0. \tag{7.31}$$

The general solution of Y_0 is

$$Y_0 = A + Be^{-2\xi}$$

which evaluated at the inner boundary gives

$$Y_0 = -B + Be^{-2\xi}.$$

Then (7.31) becomes

$$\frac{d^2Y_1}{d\xi^2} + 2\frac{dY_1}{d\xi} = B - Be^{-2\xi}$$

whose general solution evaluated at the inner boundary is

$$Y_1 = -D + De^{-2\xi} + \left(\frac{B}{2}\right)\xi + \left(\frac{B}{2}\right)\xi e^{-2\xi}.$$

Therefore,

$$y^i = -B + Be^{-2\xi} + \epsilon\left(-D + De^{-2\xi} + \left(\frac{B}{2}\right)\xi + \left(\frac{B}{2}\right)\xi e^{-2\xi}\right) + O(\epsilon^2) \tag{7.32}$$

where the constants B and D need to be determined by the matching conditions.

Here is a good place to point out that in a formal generalized asymptotic expansion such as (7.32) the apparent order of the terms can be misleading. For instance, $\epsilon\left(\frac{B}{2}\right)\xi$ is actually an $O(1)$ term but it appears with the $O(\epsilon)$ terms. There are certainly $O(\epsilon)$ terms that would appear with the $O(\epsilon^2)$ expansion terms but the Van Dyke rules require us to stop expanding after determining Y_1 in this case. This apparent anomaly turns out not to be a problem.

MATCHING

To match the outer expansion and the inner expansion using the Van Dyke rules we apply the matching condition $y^{oi} = y^{io}$. In this particular case we will use (7.22) where $p = q = k = 2$. First, to compute $y^{oi} = I_2 y^o$ we write the outer solution (7.27) in the inner variable, giving

$$y^o(\xi) = e^{(1/2 - \epsilon\,\xi/2)} + \epsilon\left(\frac{1}{8}e^{(1/2-\epsilon\,\xi/2)} - \frac{1}{8}\epsilon\,\xi\,e^{(1/2-\epsilon\,\xi/2)}\right).$$

Next, this expression must be expanded for small ϵ with ξ held fixed or treated as a constant. Therefore,

$$y^{oi} = e^{1/2}\left(1 - \frac{1}{2}\epsilon\xi\right) + \epsilon\, e^{1/2}\left[\frac{1}{8}\left(1 - \frac{1}{2}\epsilon\,\xi\right) - \frac{1}{8}\epsilon\,\xi\left(1 - \frac{1}{2}\epsilon\,\xi\right)\right]$$

$$= e^{1/2} - \frac{1}{2}\epsilon\,\xi\, e^{1/2} + \frac{1}{8}\epsilon\, e^{1/2} + O(\epsilon^2).$$

Truncating all but the first two terms we have

(7.33) $$y^{oi} = e^{1/2} + \epsilon\, e^{1/2}\left(\frac{1}{8} - \frac{1}{2}\xi\right).$$

Now, to compute $y^{io} = O_2 y^i$, the inner expansion with its undetermined constants must be expressed in the outer variable. (7.32) becomes

(7.34) $$y^i(x) = -B + Be^{-2x/\epsilon} + \epsilon\left(-D + De^{-2x/\epsilon} + \frac{B}{2}\left(\frac{x}{\epsilon}\right) + \frac{B}{2}\left(\frac{x}{\epsilon}\right)e^{-2x/\epsilon}\right)$$

Next, this must be expanded in powers of ϵ to two terms. Since $e^{-2x/\epsilon}$ is not defined when $\epsilon = 0$, a *limit process expansion* must be used. That is, let $\epsilon \to 0$ and find the constant term $B\left(\frac{x}{2} - 1\right)$. Subtract this from y^i, divide by ϵ, and again let $\epsilon \to 0$ to find that the next coefficient is $-D$. In practice, one may simply discard the terms containing $e^{-2x/\epsilon}$ as being *transcendentally small* and handle the rest as a Taylor series. Therefore,

(7.35) $$y^{io} = B\left(\frac{x}{2} - 1\right) - \epsilon D.$$

Equating (7.33) and (7.35) according to the matching principle we have

$$e^{1/2} + \frac{1}{8}\epsilon\, e^{1/2} - \frac{1}{2}\epsilon\,\xi\, e^{1/2} = B\left(\frac{x}{2} - 1\right) - \epsilon D$$

which, since $\xi = x/\epsilon$ can be rewritten

(7.36) $$e^{1/2} + \frac{1}{8}\epsilon\, e^{1/2} - \frac{1}{2}xe^{1/2} = B\left(\frac{x}{2} - 1\right) - \epsilon D.$$

Equating coefficients of like powers of ϵ in (7.36) yields

$$D = -\frac{1}{8}e^{1/2}$$

$$-B + \frac{1}{2}xB = e^{1/2} - \frac{1}{2}xe^{1/2}.$$

Notice that the coefficient B is over determined; that is, D is determined by equating powers of ϵ but B is determined twice when equating powers of x. Although this seems to be a lucky coincidence it is actually because the exact solution satisfies the Van Dyke hypotheses. That is, the Van Dyke hypotheses imply the existence of a solution to the over determined system of equations for the constants in y^i. Equating powers of x we find that

$$B = -e^{1/2}.$$

Hence, y^i becomes

$$y^i(x) = e^{1/2} - e^{(1/2-2x/\epsilon)} + \epsilon\left(\frac{1}{8}e^{1/2} - \frac{1}{8}e^{(1/2-2x/\epsilon)} - \frac{x}{\epsilon}e^{1/2} - \frac{x}{\epsilon}e^{(1/2-2x/\epsilon)}\right).$$

With B and D determined we can quickly check and see that

$$y^{io} = y^{oi} = -e^{1/2}\left(\frac{x}{2} - 1\right) + \epsilon\left(\frac{1}{8}\right)e^{1/2}.$$

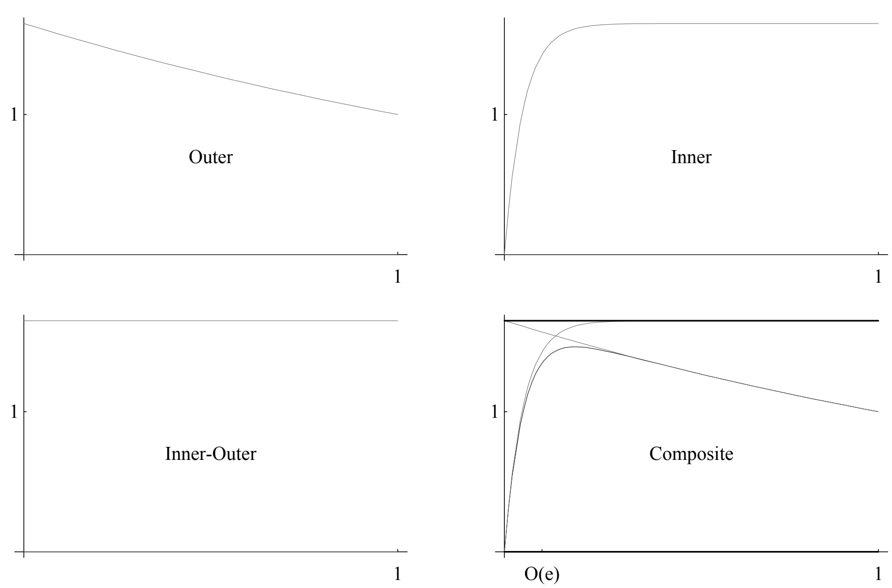

Figure 9.
Van Dyke Matching

COMPOSITE EXPANSION

Once the inner and outer expansions have been determined and matched we can form a composite expansion, y^c, that is uniform everywhere on the domain of interest. We use formula (7.23) and find

$$y^c = y^i + y^o - y^{io}$$

(7.37)
$$\begin{aligned} y^c = & -e^{(1/2-2x/\epsilon)} - \frac{x}{2}e^{1/2} + e^{(1/2-x/2)} - xe^{(1/2-2x/\epsilon)} \\ & - \frac{1}{8}\epsilon e^{(1/2-2x/\epsilon)} + \frac{1}{8}\epsilon e^{(1/2-x/2)} - \frac{1}{8}\epsilon\, xe^{(1/2-x/2)} \end{aligned}$$

as a single uniform expansion.

Since $y^{oi} = y^{io}$ we can see that (7.23) could also take the form

(7.38)
$$y^c = y^i + y^o - y^{oi}.$$

Equations (7.23) and (7.38) are usually presented as definitions. But we can at least check the plausibility of (7.23) to see if y^c agrees with the outer and inner expansions in their respective domains of validity. We have from (7.23)

$$(y^c)^o = (y^i)^o + (y^o)^o - (y^{io})^o$$

but $(f^o)^o = f^o$ so that

$$(y^o)^o = y^o \text{ and } (y^{io})^o = y^{io}$$

therefore,

$$(y^c)^o = y^o.$$

Likewise, from (7.38)

$$(y^c)^i = (y^i)^i + (y^o)^i - (y^{oi})^i$$

but $(f^i)^i = f^i$ so that

$$(y^i)^i = y^i \text{ and } (y^{oi})^i = y^{oi}$$

therefore,

$$(y^c)^i = y^i.$$

Since y^c reproduces the outer expansion in the outer domain and the inner expansion in the inner domain and the domains have a functional overlap, we postulate that y^c is valid everywhere on the domain of interest.

EXAMPLE 7.3: Intermediate Variables.

An alternative to matching by the Van Dyke method is a method using an intermediate variable(Murdock[6], pp. 371-374). In the case of problem (7.1) we have found the inner and outer scales to be $\xi = x/\epsilon$ and x, with their associated domains of validity. An intermediate variable, η, would be a function of x and ϵ such that it would yield a scale intermediate to x and ξ on the overlap domain between the inner and outer regions. Such a variable could be defined as

$$\eta := \frac{x}{\epsilon^{\nu}} \qquad 0 < \nu < 1$$

We will also need an intermediate expansion operator, N_k, which acts upon any function to express it in terms on the intermediate variable η, expand it in powers of ϵ holding η constant, and retain k terms of the expansion. The intermediate expansion will usually be a limit process expansion. Also, if gauges are used for y^i and y^o where it happens that $p = q$ (p and q are the number of terms retained in each expansion) then

$$I_p y^e = y^i \text{ and } O_q y^e = y^o$$

and it will not necessarily follow that k is the same number. The number of terms of the intermediate expansion must be chosen so that the expected error is of the same order as in y^i and y^o so that

(7.39) $$N_k y^i = N_k y^o$$

or

$$y^{in} = y^{on}.$$

If this can be accomplished then

$$y^c = y^i + y^o - y^{in}.$$

We will now apply this method of matching to the problem of Example 7.2 and begin matching by intermediate variables after equation (7.32). Substituting $x = \eta\epsilon^{\nu}$ into (7.34) we find

$$y^i(\eta;\epsilon) = -B + Be^{-2\eta\epsilon^{\nu-1}} + \epsilon\left(-D + De^{-2\eta\epsilon^{\nu-1}} + \frac{B}{2}\eta\epsilon^{\nu-1} + \frac{B}{2}\eta\epsilon^{\nu-1}e^{-2\eta\epsilon^{\nu-1}}\right).$$

Expanding in powers of ϵ we obtain

$$N_3 y^i = -B + \frac{B}{2}\eta\epsilon^{\nu} - \epsilon D.$$

where $e^{-2\eta\epsilon^{\nu-1}}$ is neglected as it is transcendentally small. Here it becomes obvious that we need three terms to retain accuracy up to $O(\epsilon)$. Additionally we do not even see the unknown, D, appear until the third term. Now, from (7.27) we substitute the intermediate variable and obtain

$$y^o(\eta;\epsilon) = e^{(1/2-\eta\,\epsilon^\nu/2)} + \frac{1}{8}\epsilon\, e^{(1/2-\eta\,\epsilon^\nu/2)} - \frac{1}{8}\eta\epsilon^{\,\nu+1}e^{(1/2-\eta\,\epsilon^\nu/2)} + O\big(\epsilon^{2\nu}\big).$$

Expanding in powers of ϵ we obtain

$$y^o(\eta;\epsilon) = e^{1/2}\left(1 - \frac{1}{2}\eta\,\epsilon^\nu\right) + \frac{1}{8}\epsilon\, e^{1/2}\left(1 - \frac{1}{2}\eta\,\epsilon^\nu\right) - \frac{1}{8}\eta\epsilon^{\,\nu+1}e^{1/2}\left(1 - \frac{1}{2}\eta\,\epsilon^\nu\right)$$

$$N_3 y^o = e^{1/2} - \frac{1}{2}\eta\,\epsilon^\nu e^{1/2} + \frac{1}{8}\epsilon\, e^{1/2}$$

where we have additionally required that

$$\frac{1}{2} < \nu < 1$$

so that we may neglect other terms of order higher than $O(\epsilon)$. Condition (7.39) is now met if

$$D = \,-\frac{1}{8}e^{1/2}$$

$$B = -e^{1/2}$$

as we found previously by the Van Dyke method.

This matching method is more difficult to use in applications than Van Dyke matching, but is more natural since it does not rest on a hypothesis about the commutativity of two expansion operators. Most theoretical studies of the validity of matching use the method of intermediate variables. In practice, however, matching by the method of intermediate variables is an unnecessary complication and we will not use it further.

Two Scale Method.

EXAMPLE 7.4: It appears that a uniform expansion of a singular perturbation problem cannot be expressed in terms of a single scale, making it an ideal problem for application of the method of multiple scales. Since the domain is finite, ϵx stays small and hence, nonuniformities will not arise from the presence of the secular terms ϵx, $\epsilon^2 x^2$, $\epsilon^3 x^3, \ldots,$ in contrast with the infinite domains considered in oscillatory problems.

Thus, it is sufficient to introduce the stretched, or fast, scale $\xi = x/\epsilon$, which is the same as the inner variable and the normal, slow scale, x, which is the outer variable. Substituting these into

(7.1a) $$\epsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0, \quad 0 < x < 1, \quad 0 < \epsilon \ll 1$$

(7.1b) $$y(0) = 0$$

(7.1c) $$y(1) = 1$$

we obtain

(7.40) $$\epsilon\left(\frac{\partial^2 y}{\partial x^2} + \frac{2}{\epsilon}\frac{\partial^2 y}{\partial x \partial \xi} + \frac{1}{\epsilon^2}\frac{\partial^2 y}{\partial \xi^2}\right) + 2\left(\frac{\partial y}{\partial x} + \frac{1}{\epsilon}\frac{\partial y}{\partial \xi}\right) + y = 0$$

$$y(0,0,\epsilon) = 0$$

$$y\left(1, \frac{1}{\epsilon}, \epsilon\right) = 1.$$

We seek a first order uniform expansion in y of the form

(7.41) $$y = y_0(x,\xi) + \epsilon y_1(x,\xi) + \dots$$

To determine a first approximation we recall that we need to investigate the y_1 term to determine the arbitrary functions that appear in y_0. Substituting (7.41) into (7.40) and equating coefficients of like powers of ϵ, we have

(7.42) $$y_{0\xi\xi} + 2y_{0\xi} = 0$$

(7.43) $$y_{1\xi\xi} + 2y_{1\xi} = -2y_{0x\xi} - 2y_{0x} - y_{0.}$$

The general solution of (7.42) is

(7.44) $$y_0 = A(x) + B(x)e^{-2\xi}$$

where A and B are undetermined at this level of approximation. (7.43) may now be written

(7.45) $$y_{1\xi\xi} + 2y_{1\xi} = -(2A'(x) + A(x)) + (2B'(x) - B(x))e^{-2\xi}.$$

A particular solution of (7.45) is

$$y_1 = -\frac{1}{2}(2A'(x) + A(x))\xi - \frac{1}{2}(2B'(x) - B(x))\xi\, e^{-2\xi}$$

To eliminate resonant terms the coefficients of ξ and $\xi\, e^{-2\xi}$ must vanish independently. Therefore,

(7.46) $$2A'(x) + A(x) = 0$$

$$2B'(x) - B(x) = 0.$$

The solutions of (7.46) are

$$A = C_1 e^{-x/2}$$

$$B = C_2 e^{x/2}.$$

Then (7.44) becomes

$$y_0 = C_1 e^{-x/2} + C_2 e^{x/2} e^{-2x/\epsilon}.$$

Solving this equation for the boundary conditions in (7.40) yields

(7.47) $$C_1 = -C_2$$

$$1 = C_1 e^{-1/2} + C_2 e^{1/2} e^{-2/\epsilon}.$$

Neglecting the transcendentally small term in (7.47), $\left(e^{-2/\epsilon}\right)$, we have

$$C_1 = e^{1/2}$$

$$C_2 = -e^{1/2}$$

and substituting for y_0 in (7.41) gives

(7.48) $$y = e^{1/2}\left(e^{-x/2} - e^{x/2} e^{-2x/\epsilon}\right).$$

Note that we assumed the boundary layer was on the left when we set $\xi = x/\epsilon$, and we guessed the correct gauge for the fast scale. Had we been in error on either account the method would have failed. Nayfeh[8], pp. 268-270.

Before concluding this section we want to compare (7.48) with (7.18) found by the method of matched asymptotic expansions.

(7.18) $$y^c = e^{1/2}\big(e^{-x/2} - e^{-2x/\epsilon}\big)$$

(7.48) $$y = e^{1/2}\big(e^{-x/2} - e^{x/2}e^{-2x/\epsilon}\big).$$

Whereas the method of matched asymptotic expansions yields a composite expansion that is separable in the outer and inner scales, the method of multiple scales yields a non separable expansion. Letting $\epsilon \to 0$ with x being kept fixed we find that (7.48) and (7.18) yield the same outer expansion. Putting $x = \epsilon\xi$ and letting $\epsilon \to 0$ with x fixed yields the same inner expansion. Thus, (7.48) and (7.18) agree in the inner and outer domains. However, the presence of the extra factor, $e^{x/2}$, in (7.48) makes it agree more closely with the exact solution in the intermediate region. It appears that the method of multiple scales is superior to the method of matched asymptotic expansions. In this application it is but we can find many examples, Nayfeh[7], where the application of multiple scales to nonlinear differential equations and partial differential equations is not straightforward. We need both methods.

Initial Layer Problem.

EXAMPLE 7.5: Consider a linear spring-mass system with mass m, damping coefficient μ, and spring constant k. This is similar to the problem in Example 3.2 with the exception that now we wish to impart an initial momentum (or impulse) of C kg m/sec to the mass at time $T = 0$ and thus set the system in motion. Since momentum equals mass times velocity, the initial velocity is C/m. The reason for specifying the initial *momentum* rather than initial velocity is that we wish to think of a specific device which delivers a given impulse irregardless of the amount of mass present. We are interested in the case in which m is very small; the smaller m is the larger will be the initial velocity C/m. For this class of problems, C is regarded as a constant and m is treated as a variable. m is almost the perturbation parameter in the problem but dimensional analysis will reveal a better nondimensional small parameter. The final complete description of the system is given by the initial value problem

(7.49) $$m\frac{d^2X}{dT^2} + \mu\frac{dX}{dT} + kX = 0$$

$$X(0) = 0$$

$$\frac{dX}{dT}(0) = C/m$$

NONDIMENSIONALIZATION

As a result of our study of the boundary value problem that is similar to (7.49) we have an idea of what to expect for the solution characteristics of this layer type problem. It is interesting, nevertheless, to see how the properties of this perturbation problem are brought out by the purely mathematical process of nondimensionalization. In this process we are led to the recognition that there are two distinctly different perturbation problems associated with (7.49), each of which is suitable over a different portion of the domain. Furthermore, we are led to the specific rescaling which relates these two problems. We begin with a table of the quantities in (7.49) and their dimensions.

Variable	units
X	m
T	sec
m	kg
μ	kg/sec
k	kg/sec^2
C	kg m/sec

The first two items in this table are coordinates and the last four are natural parameters. The next step is to form combinations of the natural parameters having the same dimensions as the coordinates. We find one possibility for characteristic length and two for characteristic time.

Scale	units
C/μ	m
μ/k	sec
m/μ	sec

Nondimensional lengths and times are defined as the ratios of the coordinates X and T to the characteristic quantities with the same dimensions. Therefore we define

(7.50)
$$x := \mu X/C$$

$$t := kT/\mu$$

$$\tau := \mu T/m$$

To nondimensionalize the mass we seek a combination of the remaining three natural parameters having dimensions of mass. The only such quantity is μ^2/k. Therefore, we introduce the nondimensional perturbation parameter

(7.51) $$\epsilon := km/\mu^2.$$

Although it is possible to form other nondimensional quantities from those appearing in (7.49), i.e. we can form an infinite number of time scales from $(\mu/k)^p(m/\mu)^q$ where $p+q=1,$ the four functions defined in (7.50) and (7.51) are sufficient to express (7.49) in nondimensional form. It turns out that there are two ways to do this depending on whether t or τ is used as the time variable. Substituting (7.50) into (7.49) and (7.51) and using the chain rule we derive

(7.52) $$\epsilon\frac{d^2x}{dt^2}+\frac{dx}{dt}+x=0$$

$$x(0)=0$$
$$\frac{dx}{dt}(0)=\frac{1}{\epsilon}$$

and

(7.53) $$\frac{d^2x}{d\tau^2}+\frac{dx}{d\tau}+\epsilon x=0$$

$$x(0)=0$$
$$\frac{dx}{d\tau}(0)=1.$$

One may verify that (7.50) and (7.51) imply

(7.54) $$\tau=\frac{t}{\epsilon}$$

and that the change of independent variable indicated by (7.54) transforms (7.52) into (7.53). This rescaling may also be found as before by substituting $\tau=\epsilon^\nu t$ into (7.52) and choosing ν so that ϵ no longer multiplies the highest derivative.

Let's review what has been accomplished by the process of nondimensionalization. First, the number of parameters has been reduced from four $(m,\ \mu,\ k,\ C)$ to one (ϵ); the perturbation parameter has been found not to be m but $\epsilon=km/\mu^2$. From this we see that it does not matter whether m is small, k is small, or μ is large; the same perturbation applies to all of them. Finally, we can see that there are two natural time scales in the problem and we are led to the two formulations in t-time and τ-time where τ is "fast" and t is "normal".

INNER EXPANSION

Setting $\epsilon = 0$ causes the second order differential equation (7.52) to drop to first order, so that its solutions cannot satisfy two initial conditions. Therefore, (7.52) will give the outer solution (*out* away from the initial conditions) and (7.53) will give the inner solutions. Let

$$x^i(\tau, \epsilon) = x_0^i(\tau) + \epsilon x_1^i(\tau) + \cdots \tag{7.55}$$

Substitute (7.55) into (7.53), ignore ϵ^2 and obtain the sequence of problems

$$\frac{d^2 x_0^i}{d\tau^2} + \frac{dx_0^i}{d\tau} = 0 \tag{7.56}$$

$$x_0^i(0) = 0$$
$$\frac{dx_0^i}{d\tau}(0) = 1$$

$$\frac{d^2 x_1^i}{d\tau^2} + \frac{dx_1^i}{d\tau} = -x_0^i \tag{7.57}$$

$$x_1^i(0) = 0$$
$$\frac{dx_1^i}{d\tau}(0) = 0.$$

The solution of (7.56) is

$$x_0^i = 1 - e^{-\tau}. \tag{7.58}$$

Substituting (7.58) into (7.57) gives an inhomogeneous problem with solution

$$x_1^i = 2 - \tau - (2 + \tau)e^{-\tau}$$

thus the final two term inner expansion is

$$x^i(\tau,\epsilon) = (1 - e^{-\tau}) + \epsilon[(2 - \tau) - (2 + \tau)e^{-\tau}].$$

OUTER EXPANSION

Let

$$x^o(t, \epsilon) = x_0^o(t) + \epsilon x_1^o(t). \tag{7.59}$$

Substitute (7.59) into (7.52), separate terms of order ϵ^0 and ϵ^1 and get

(7.60) $$\frac{dx_0^o}{dt} + x_0^o = 0$$

(7.61) $$\frac{dx_1^o}{dt} + x_1^o = -\frac{d^2x_0^o}{dt^2}.$$

The general solution of (7.60) is

$$x_0^o = Ae^{-t}$$

after which we may find the solution of (7.61) to be

$$x_1^o = (-At + B)e^{-t}$$

and the two term outer expansion is

$$x^o(t, \epsilon) = Ae^{-t} + \epsilon(-At + B)e^{-t}.$$

MATCHING

A limit process expansion must be used to find

(7.62) $$x^{io} = (1 - t) + 2\epsilon.$$

Writing the outer expansion in terms of the inner variable and expanding for small ϵ we obtain

(7.63) $$x^{oi} = A + \epsilon(-A\tau + B)$$

$$= (A - At) + \epsilon B.$$

Equating (7.62) and (7.63) we find

$$A = 1, \quad B = 2$$

where A is overdetermined.

COMPOSITE EXPANSION

Using our formula

$$x^c = x^i + x^o - x^{oi}$$

we find the composite expansion to be

(7.64) $$x^c = \left(e^{-t} - e^{-t/\epsilon} - te^{-t/\epsilon}\right) + \epsilon\left(-te^{-t} + 2e^{-t} - 2e^{-t/\epsilon}\right).$$

Murdock[6], pp. 343-364.

Variable Coefficients. The remainder of this paper will examine perturbation methods applied to second order ordinary differential equations with variable coefficients. Many interesting phenomenon can develop when the coefficients are not constant. We may have situations with multiple boundary layers, boundary layers within the interval of interest, and cases where more than two scales are required to approximate the solution. Four more examples will be presented in this chapter to examine these types of problems. First we will look at a problem with an interior boundary layer and then at one with multiple distinguished limits within a boundary layer. Then we will solve a variable coefficient problem in general form by the methods of matched asymptotic expansions and finally we will compare this last result to the solution of the same problem by the method of multiple scales.

EXAMPLE 7.6: In this example we will use matched asymptotic expansions to construct an approximation to a problem with an interior boundary layer. Consider

(7.65)
$$\epsilon\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_0(x)y=0$$

$$y(0)=0$$
$$y(1)=1$$

for special functions $p_1$and p_0. If $p_1(x)=(x-\frac{1}{2})$ and $p_0(x)=\,-\,(x-\frac{1}{2})$, Nayfeh[8], pp. 292-296, we get

(7.66)
$$\epsilon\frac{d^2y}{dx^2}+(x-\frac{1}{2})\frac{dy}{dx}-(x-\frac{1}{2})y=0$$

which is an equation with a simple zero at $x=\frac{1}{2}$. In this case $p_1(x)<0$ for $x<\frac{1}{2}$and $p_1(x)>0$ for $x>\frac{1}{2}$. It turns out that our usual assumption about the location of the boundary layer based on the sign of $p_1(x)$ leads us to assume the boundary layer is located at $x=\frac{1}{2}$ and no boundary layers at the end points. Success will determine the validity of our assumption. A more detailed discussion on finding the location of a boundary layer may be found in Hinch[3], pp. 63-65. We seek an outer expansion of the form

(7.24)
$$y^o=y_0(x)+\epsilon y_1(x)+\cdots.$$

The outer expansion is derived by substituting (7.24) into (7.66) and taking the reduced equation giving

$$(x - \frac{1}{2})\frac{dy_0}{dx} - (x - \frac{1}{2})y_0 = 0$$

whose general solution is

$$y_0 = c_0 e^x$$

which we expect to be valid everywhere except in a small neighborhood of $x = \frac{1}{2}$. In the interval $x > \frac{1}{2}$, y^o must satisfy

$$y^o(1) = 1 \quad \Rightarrow \quad c_0 e = 1$$

and thus

(7.67) $$y_r^o = e^{x-1} + \cdots.$$

where the subscript r indicates the right side of the interval. In the interval $x < \frac{1}{2}$, must satisfy $y^o(0) = 0$. Hence $c_0 = 0$ and

(7.68) $$y_l^o = 0.$$

The subscript l indicates the left part of the interval. Next, we investigate the neighborhood of $x = \frac{1}{2}$ by introducing the stretching transformation

(7.69) $$\xi = \frac{(x - \frac{1}{2})}{\epsilon^\nu}, \qquad \nu > 0.$$

Then (7.66) becomes

(7.70) $$\epsilon^{1-2\nu}\frac{d^2y^i}{d\xi^2} + \xi\frac{dy^i}{d\xi} - \epsilon^\nu \xi\, y^i = 0.$$

The distinguished limit corresponds to $\nu = \frac{1}{2}$, and the least degenerate form (by undetermined gauges) is

(7.71) $$\frac{d^2y^i}{d\xi^2} + \xi\frac{dy^i}{d\xi} = 0.$$

Equation (7.71) is a first order differential equation in $dy^i/d\xi$ and hence, solvable. Putting

$$u = \frac{dy^i}{d\xi}$$

in (7.71), we have

$$\frac{du}{d\xi} + \xi\, u = 0$$

whose general solution is

$$u = a_0 e^{-(1/2)\xi^2}.$$

Therefore,

$$y^i = b_0 + a_0 \int_0^{\xi} e^{-(1/2)\tau^2} d\tau \tag{7.72}$$

where the lower limit is taken to be $\xi\ = 0$ corresponding to the assumed location, $x = \frac{1}{2}$, of the boundary layer. We now match the one term outer expansion (7.67) with the one term inner expansion (7.72).

MATCHING

One term outer expansion: $y_r^o \sim e^{x-1}$

Rewritten in inner variable: $= e^{\epsilon^{(1/2)}\xi - (1/2)}$

Expanded for small ϵ: $= e^{-(1/2)}(1 + \epsilon^{(1/2)}\xi + \cdots)$

Therefore

$$y_r^{oi} = e^{-(1/2)}. \tag{7.73}$$

One term inner expansion: $y^i \sim b_0 + a_0 \int_0^{\xi} e^{-(1/2)\tau^2} d\tau$

Rewritten in outer variable: $= b_0 + a_0 \int_0^{(x-\frac{1}{2})/\epsilon^{(1/2)}} \epsilon^{-(1/2)\tau^2} d\tau$

Expanded for small ϵ: $= b_0 + a_0 \int_0^{\infty} \epsilon^{-(1/2)\tau^2} d\tau$

Therefore

$$y_r^{io} = b_0 + \frac{a_0\sqrt{\pi}}{\sqrt{2}}. \tag{7.74}$$

Equating (7.73) and (7.74) we have

$$b_0 + \frac{a_0\sqrt{\pi}}{\sqrt{2}} = \beta e^{-(1/2)} \tag{7.75}$$

which is an equation with two unknowns, a_0 and b_0. To determine a second equation with unknowns a_0 and b_0, we match the inner expansion (7.72) to the outer expansion, (7.68).

One term outer expansion: $y_l^o \sim 0$

Therefore

$$y_l^{oi} = 0. \tag{7.76}$$

One term inner expansion: $y^i \sim b_0 + a_0 \int_0^{\xi} e^{-(1/2)\tau^2} d\tau$

Rewritten in outer variable: $= b_0 + a_0 \int_0^{(\frac{1}{2}-x)/\epsilon^{(1/2)}} \epsilon^{-(1/2)\tau^2} d\tau$

Expanded for small ϵ: $= b_0 + a_0 \int_0^{-\infty} \epsilon^{-(1/2)\tau^2} d\tau$

Therefore

$$y_l^{oi} = b_0 - \frac{a_0\sqrt{\pi}}{\sqrt{2}}. \tag{7.77}$$

Equating (7.76) and (7.77) we have

$$b_0 - \frac{a_0\sqrt{\pi}}{\sqrt{2}} = 0. \tag{7.78}$$

Solving (7.75) and (7.78) simultaneously yields

$$a_0 = \frac{1}{\sqrt{2\pi}} e^{-(1/2)}$$

$$b_0 = \frac{1}{2} e^{-(1/2)}$$

and thus

$$y^i = \frac{1}{2} e^{-(1/2)} + (2\pi)^{-(1/2)} e^{-(1/2)} \int_0^{\xi} e^{-(1/2)\tau^2} d\tau. \tag{7.79}$$

In this problem we cannot form a single composite expansion that is uniformly valid over the whole interval. Instead, we form two composite expansions, one valid in $[0, \frac{1}{2}]$ and the other valid in $[\frac{1}{2}, 1]$. Thus we put

$$y_l^c = y_l^o + y^i - y_l^{oi} \quad \Rightarrow$$

$$y_l^c = y^i.$$

Similarly

$$y_r^c = y_r^o + y^i - y_r^{oi} \quad \Rightarrow$$

(7.81) $$y_r^c = e^{x-1} + y^i - e^{-(1/2)}.$$

EXAMPLE 7.7: Triple Decks

Consider the following example from Nayfeh[8], pp. 304-307.

(7.82) $$\epsilon^3 \frac{d^2y}{dx^2} + x^3 \frac{dy}{dx} + (x^3 - \epsilon)y = 0$$

$$y(0) = \alpha$$
$$y(1) = \beta.$$

When more than two distinguished limits exist in a given boundary layer, the resulting expansion consists of two inner expansions in addition to the outer expansion. The domains of validity of each of them are often called decks, and thus, a problem with three decks is usually referred to as a *triple deck problem*. Since the coefficient of dy/dx is positive the boundary layer is expected to be at the origin. Therefore, the outer expansion will satisfy the boundary condition $y(1) = \beta$ but not $y(0) = \alpha$.

OUTER EXPANSION

Assuming an outer expansion of the form

(7.24) $$y^o = y_0(x) + \epsilon y_1(x) + \cdots$$

we substitute (7.24) into (7.82) and $y(1) = \beta$ to obtain

(7.83) $$x^3 \frac{dy_0}{dx} + x^3 y_0 = 0$$

$$y_0(1) = \beta$$

whose general solution is

$$y_0 = c_0 e^{-x}.$$

Applying $y_0(1) = \beta$ we obtain

(7.84) $$y^o = \beta e^{1-x} + \cdots$$

which does not satisfy the boundary condition at the origin.

INNER EXPANSION

To investigate the behavior at the origin we introduce stretching transformation

(7.85) $$\xi = \frac{x}{\epsilon^{\nu}}$$

into (7.112) and derive

(7.86) $$\epsilon^{3-2\nu}\frac{d^2y}{d\xi^2} + \epsilon^{2\nu}\xi^3\frac{dy}{d\xi} + (\epsilon^{3\nu}\xi^3 - \epsilon)y = 0.$$

The distinguished limits may be determined by the method of undetermined gauges or can be found graphically.

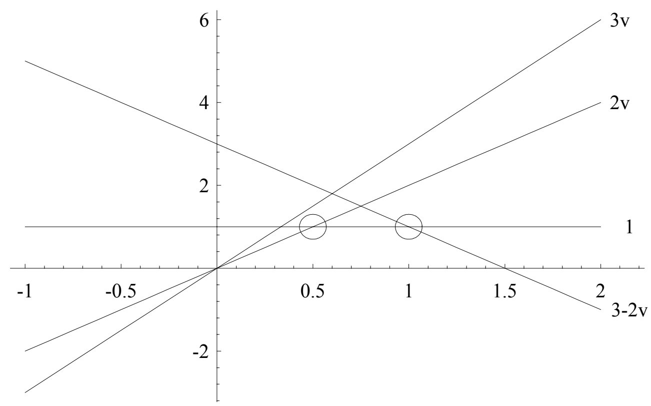

Figure 10.
Distinguished Limits

We find that $\nu = \{\frac{1}{2}, 1\}$ will satisfy the criteria. The higher the value of ν in (7.85), the closer the domain is to the origin. Therefore we will refer to the deck corresponding to $\nu = 1$ as the left deck, $\nu = \frac{1}{2}$ as the middle deck, and the outer expansion as the right deck. Taking $\nu = 1$,

(7.87) $$\xi = \frac{x}{\epsilon}$$

we obtain from (7.116)

(7.88) $$\epsilon\frac{d^2y}{d\xi^2} + \epsilon^2\xi^2\frac{dy}{d\xi} + (\epsilon^3\xi^3 - \epsilon)y = 0.$$

Dividing by ϵ and taking the limit as $\epsilon \rightarrow 0$ we see that the left deck expansion is governed by the *dominant part*

$$\frac{d^2Y_0}{d\xi^2} - Y_0 = 0$$

whose general solution is

(7.89) $$Y_0 = a_0e^{-\xi} + b_0e^{\xi}.$$

We note that b_0 must be zero or the expansion would grow without bound with decreasing ϵ (increasing ξ) and be unmatchable to the middle deck. Since the left deck is valid at the origin it must match $y_0 = \alpha$. From (7.87) $x = 0$ corresponds to $\xi = 0$ and (7.89) implies

(7.90) $$y^l = \alpha e^{-\xi} + \cdots.$$

MIDDLE EXPANSION

With $\nu = \frac{1}{2}$ we obtain the middle variable

(7.91) $$\zeta = \frac{x}{\epsilon^{\frac{1}{2}}}$$

and from (7.86) we get

(7.92) $$\epsilon^2\frac{d^2y}{d\zeta^2} + \epsilon\zeta^3\frac{dy}{d\zeta} + (\epsilon^{(3/2)}\zeta^3 - \epsilon)y = 0.$$

It follows from (7.92) that the leading term $\widehat{Y}_0$ in the middle deck expansion y^m is governed by the dominant part

$$\zeta^3\frac{d\widehat{Y}_0}{d\zeta} - \widehat{Y}_0 = 0.$$

Separating variables we have

$$\widehat{Y}_0 = d_0e^{-1/2\zeta^2}$$

and

(7.93) $$y^m = d_0e^{-1/2\zeta^2} + \cdots.$$

where d_0 will be determined by matching.

MATCHING

Note that y^l cannot be matched directly with y^o since

$$(y^l)^o = 0 \quad \text{and} \quad (y^o)^l = \beta e.$$

To match y^m with y^o we note that

$$(y^m)^o = d_0 \quad \text{and} \quad (y^o)^m = \beta e$$

implying that

(7.94) $$d_0 = \beta e$$

and

(7.95) $$y^m = \beta e^{1-1/2\zeta^2} + \cdots$$

To match y^m with y^l we note that $(y^m)^l = 0$ and $(y^l)^m = 0$ indicating that y^m and $y^l$are matchable.

COMPOSITE EXPANSION

The composite expansion is formed from the relation

$$y^c = y^o + y^m + y^l - y^{om} - y^{lm}.$$

Therefore,

(7.96) $$y^c = \beta e^{1-x} + \beta e^{1-\epsilon/2x^2} + \alpha e^{-x/\epsilon} - \beta e + \cdots.$$

Checking our previous results we can quickly compute

$$(y^c)^l = \alpha e^{-\xi} = y^l$$

$$(y^c)^m = \beta e^{1-1/2\zeta^2} = y^m$$

$$(y^c)^o = \beta e^{1-x} = y^o.$$

Showing that our composite expansion y^c is valid in the inner, middle, and outer regions and is thus valid everywhere as their regions must overlap to make matching possible.

EXAMPLE 7.8: From Nayfeh[8], pp. 286-289, we will now derive a solution of

(7.65) $$\epsilon \frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0$$

$$y(0) = \alpha$$
$$y(1) = \beta$$

with only the assumption that $p_1(x)$ is positive everywhere in $[0, 1]$. We suppose that the boundary layer is on the left at $y(0) = \alpha$.

OUTER EXPANSION

Assuming an outer expansion of the form

(7.24) $$y^o = y_0(x) + \epsilon y_1(x) + \cdots$$

we substitute (7.24) into (7.65), equate coefficients of e^0 and obtain

(7.97) $$p_1\frac{dy_0}{dx} + p_0 y_0 = 0.$$

Separating variables and integrating we derive

$$y_0 = c_0 \exp\left[-\int_1^x p_0(\tau)/p_1(\tau)d\tau\right]$$

where the lower limit of integration is taken to be 1 to enable satisfaction of the boundary condition $y_0(1) = \beta$. Hence,

$$y_0 = \beta e^{f(x)}$$

where we define

(7.98) $$f(x) := \int_x^1 (p_0/p_1)d\tau.$$

Thus

(7.99) $$y^{\text{o}} = \beta e^{f(x)} + \cdots.$$

INNER EXPANSION

To determine the inner expansion valid at the origin we introduce the stretching transformation

(7.85) $$\xi = \frac{x}{\epsilon^\nu}.$$

Then (7.95) becomes

(7.100) $$\epsilon^{1-2\nu}\frac{d^2y^i}{d\xi^2} + \epsilon^{-\nu}p_1(\epsilon^\nu\xi)\frac{dy^i}{dx} + p_0(\epsilon^\nu\xi)y^i = 0.$$

As $\epsilon \to 0$, $p_1(\epsilon^\nu\xi) \to p_1(0)$ and $p_0(\epsilon^\nu\xi) \to p_0(0)$ where p_0 is assumed to be analytic at the origin and $p_0(0) \neq 0$. By undetermined gauges we find that $\nu = 1$, in which case the dominant part of (7.100) is

(7.101) $$\frac{d^2y^i}{d\xi^2} + p_1(0)\frac{dy^i}{d\xi} = 0$$

whose general solution is

$$y^i = a_0 + b_0 e^{-p_1(0)\xi}. \tag{7.102}$$

Since $x = 0 \;\Rightarrow\; \xi = 0$ and $y(0) = \alpha \;\Rightarrow y^i = \alpha$ then from (7.102) we see that

$$\alpha = a_0 + b_0$$

and that

$$y^i = \alpha - b_0 + b_0 e^{-p_1(0)\xi}. \tag{7.103}$$

MATCHING

One term outer expansion: $y \sim \beta e^{f(x)}$
Rewritten in inner variable : $= \beta e^{f(\epsilon\xi)}$
Expanded for small ϵ : $= \beta e^{f(0)} + \cdots$
Therefore

$$y^{oi} = \beta e^{f(0)}. \tag{7.104}$$

One term inner expansion: $y \sim \alpha - b_0 + b_0 e^{-p_1(0)\xi}$
Rewritten in outer variable: $= \alpha - b_0 + b_0 e^{-p_1(0)x/\epsilon}$
Expanded for small ϵ : $= \alpha - b_0$
Therefore

$$y^{io} = \alpha - b_0. \tag{7.105}$$

Note that $\exp(-p_1(0)x/\epsilon)$ is exponentially small as $\epsilon \to 0$ because $p_1(0)$ is positive. If it were negative then $\exp(-p_1(0)x/\epsilon)$ would have been exponentially large as $\epsilon \to 0$ and matching would not have been possible. In that case the boundary layer would be at the right end. The absence of exponential growth is essential for matching.

Equating (7.104) and (7.105) we have

$$b_0 = \alpha - \beta e^{f(0)}.$$

Therefore

$$y^i = \beta e^{f(0)} + (\alpha - \beta e^{f(0)}) e^{-p_1(0)\xi} + \cdots. \tag{7.106}$$

COMPOSITE EXPANSION

Recalling that

$$y^c = y^o + y^i - y^{oi}$$

then

$$y^c = \beta e^{f(x)} + (\alpha - \beta e^{f(0)})e^{-p_1(0)\xi} + \cdots. \tag{7.107}$$

EXAMPLE 7.9: Multiple Scales.
We will now solve (7.65) by the method of multiple scales following Simmonds and Mann[10], pp. 103-106. This is done primarily to show the wide range of applicability of the two scale method and also to compare results with the method of asymptotic expansions. The method of matched asymptotic expansions is less sensitive to the form of $p_1(x)$ and $p_0(x)$ than is the two scale method and is normally the only recourse if the two scale method fails. Consider

$$\epsilon \frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0 \tag{7.65}$$

$$y(0) = \alpha$$
$$y(1) = \beta.$$

where $p_1(x)$ is analytic and positive on $(0, 1)$. We assume that the solution to (7.65) depends on a slow variable x and a fast variable

$$\xi = \frac{g(x)}{\epsilon}. \tag{7.108}$$

We seek an approximate solution in the form

$$y(x, \xi, \epsilon) = y_0(x, \xi) + \epsilon y_1(x, \xi) + O(\epsilon^2). \tag{7.109}$$

In this example only the first term of (7.139) will be determined. Taking the derivatives of (7.109) and using definition (7.108) we derive

$$\frac{dy}{dx} = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial \xi}\frac{g'}{\epsilon} \tag{7.110}$$

$$\frac{d^2y}{dx^2} = \frac{\partial^2 y}{\partial x^2} + 2\frac{\partial^2 y}{\partial x \partial \xi}\frac{g'}{\epsilon} + \frac{\partial y}{\partial \xi}\frac{g''}{\epsilon} + \frac{\partial^2 y}{\partial \xi^2}\frac{g'^2}{\epsilon^2}$$

where $' := \frac{d}{dx}$.

Substituting (7.108), (7.109), and (7.110) into (7.65) and multiplying by ϵ we obtain

$$(7.111)\quad g'^2\frac{\partial^2 y}{\partial\xi^2}+p_1 g'\frac{\partial y}{\partial\xi}+\epsilon\left(2\frac{\partial^2 y}{\partial x\partial\xi}g'+\frac{\partial y}{\partial\xi}g''+p_1\frac{\partial y}{\partial x}+p_0 y\right)+\epsilon^2\frac{\partial^2 y}{\partial x^2}=0$$

$$y(0,0,\epsilon)=\alpha$$
$$y(1,g(1)/\epsilon,\epsilon)=\beta.$$

Now substitute (7.109) into (7.111) and get the following sequence of P.D.E.s.

$$(7.112)\qquad g'^2\frac{\partial^2 y_0}{\partial\xi^2}+p_1 g'\frac{\partial y_0}{\partial\xi}=0$$

$$y_0(0,0)=\alpha$$
$$y_0(1,g(1)/\epsilon)=\beta.$$

$$(7.113)\qquad g'^2\frac{\partial^2 y_1}{\partial\xi^2}+p_1 g'\frac{\partial y_1}{\partial\xi}=-2g'\frac{\partial^2 y_0}{\partial x\partial\xi}-g''\frac{\partial y_0}{\partial\xi}-p_1\frac{\partial y_0}{\partial x}-p_0 y_0$$

$$y_1(0,0)=0$$
$$y_1(1,g(1)/\epsilon)=0.$$

If we now pick

$$(7.114)\qquad g(x)=\int_0^x p_1(t)dt\quad\Rightarrow\quad g'=p_1$$

then (7.112) reduces to

$$\frac{\partial^2 y_0}{\partial\xi^2}+\frac{\partial y_0}{\partial\xi}=0$$

whose solution is

$$(7.115)\qquad y_0(x,\xi)=A(x)+B(x)e^{-\xi}.$$

The left boundary condition at (7.112) yields

$$(7.116)\qquad B(0)=\alpha-A(0).$$

At $x=1$

$$e^{-\xi}=e^{-g(1)/\epsilon}$$

is an exponentially small term as $\epsilon\to 0$ so we can say that the first (7.112) boundary condition reduces to

(7.117) $$A(1) = \beta.$$

To determine $A(x)$ and $B(x)$ we must consider the form of the solution of (7.113). With (7.114) defined we see that (7.113) reduces (after dividing by p_1^2) to

(7.118) $$\frac{\partial^2 y_1}{\partial \xi^2} + \frac{\partial y_1}{\partial \xi} = e^{-\xi}\left[\frac{B'}{p_1} + \frac{(p_1' - p_0)B}{p_1^2}\right] - \left[\frac{A'}{p_1} + \frac{p_0 A}{p_1^2}\right].$$

To avoid resonant terms of the form $F(x)\xi e^{-\xi} + G(x)\xi$ in the solution for y we must set to zero the coefficients of $e^{-\xi}$ and ξ on the right of (7.118). Therefore

(7.119) $$p_1 A' + p_0 A = 0$$

which has the solution

(7.120) $$A(x) = \beta \text{exp}\left[\int_x^1 (p_0(t)/p_1(t))dt\right]$$

while

(7.121) $$p_1 B' + (p_1' - p_0)B = 0$$

which has the solution

(7.122) $$B(x) = \frac{C}{p_1(x)}\text{exp}\left[\int_0^x (p_0(t)/p_1(t))dt\right].$$

The constant C is determined by applying (7.116) with $A(x)$ given by (7.120). Then

(7.123) $$C = p_1(0)\left[\alpha - \beta e^{f(1)}\right]$$

where

(7.124) $$f(x) := \int_0^x \left[p_0(t)/p_1(t)\right]dt.$$

We may then write

$$A(x) = \beta e^{[f(1) - f(x)]}.$$

The resulting approximation is

(7.126) $$y \sim y_0(\xi, x) + O(\epsilon)$$

$$= \beta e^{[f(1)-f(x)]} + \frac{p_1(0)}{p_1(x)}\left[\alpha - \beta e^{f(1)}\right]e^{f(x)}e^{-\xi} + O(\epsilon).$$