Variable Coefficients. The remainder of this paper will examine perturbation methods applied to second order ordinary differential equations with variable coefficients. Many interesting phenomenon can develop when the coefficients are not constant. We may have situations with multiple boundary layers, boundary layers within the interval of interest, and cases where more than two scales are required to approximate the solution. Four more examples will be presented in this chapter to examine these types of problems. First we will look at a problem with an interior boundary layer and then at one with multiple distinguished limits within a boundary layer. Then we will solve a variable coefficient problem in general form by the methods of matched asymptotic expansions and finally we will compare this last result to the solution of the same problem by the method of multiple scales.

EXAMPLE 7.5: In this example we will use matched asymptotic expansions to construct an approximation to a problem with an interior boundary layer. Consider

$$\epsilon\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0 \tag{7.65}$$

$$y(0) = 0$$
$$y(1) = 1$$

for special functions $p_1$and p_0. If $p_1(x) = (x - \frac{1}{2})$ and $p_0(x) = -(x - \frac{1}{2})$, Nayfeh[8], pp. 292-296, we get

$$\epsilon\frac{d^2y}{dx^2} + (x - \frac{1}{2})\frac{dy}{dx} - (x - \frac{1}{2})y = 0 \tag{7.66}$$

which is an equation with a simple zero at $x = \frac{1}{2}$. In this case $p_1(x) < 0$ for $x < \frac{1}{2}$and $p_1(x) > 0$ for $x > \frac{1}{2}$. It turns out that our usual assumption about the location of the boundary layer based on the sign of $p_1(x)$ leads us to assume the boundary layer is located at $x = \frac{1}{2}$ and no boundary layers at the end points. Success will determine the validity of our assumption. A more detailed discussion on finding the location of a boundary layer may be found in Hinch[3], pp. 63-65. We seek an outer expansion of the form

$$y^o = y_0(x) + \epsilon y_1(x) + \cdots. \tag{7.24}$$

The outer expansion is derived by substituting (7.24) into (7.66) and taking the reduced equation giving

$$(x-\frac{1}{2})\frac{dy_0}{dx}-(x-\frac{1}{2})y_0=0$$

whose general solution is

$$y_0=c_0e^x$$

which we expect to be valid everywhere except in a small neighborhood of $x=\frac{1}{2}$. In the interval $x>\frac{1}{2}$, y^o must satisfy

$$y^o(1)=1 \quad \Rightarrow \quad c_0e=1$$

and thus

$$y_r^o=e^{x-1}+\cdots. \tag{7.67}$$

where the subscript r indicates the right side of the interval. In the interval $x<\frac{1}{2}$, must satisfy $y^o(0)=0$. Hence $c_0=0$ and

$$y_l^o=0. \tag{7.68}$$

The subscript l indicates the left part of the interval. Next, we investigate the neighborhood of $x=\frac{1}{2}$ by introducing the stretching transformation

$$\xi=\frac{(x-\frac{1}{2})}{\epsilon^\nu}, \qquad \nu>0. \tag{7.69}$$

Then (7.66) becomes

$$\epsilon^{1-2\nu}\frac{d^2y^i}{d\xi^2}+\xi\frac{dy^i}{d\xi}-\epsilon^\nu\xi\,y^i=0. \tag{7.70}$$

The distinguished limit corresponds to $\nu=\frac{1}{2}$, and the least degenerate form (by undetermined gauges) is

$$\frac{d^2y^i}{d\xi^2}+\xi\frac{dy^i}{d\xi}=0. \tag{7.71}$$

Equation (7.71) is a first order differential equation in $dy^i/d\xi$ and hence, solvable. Putting

$$u=\frac{dy^i}{d\xi}$$

in (7.71), we have

$$\frac{du}{d\xi} + \xi\, u = 0$$

whose general solution is

$$u = a_0 e^{-(1/2)\xi^2}.$$

Therefore,

(7.72) $$y^i = a_0 \int_0^{\xi} e^{-(1/2)\tau^2} d\tau + b_0$$

where the lower limit is taken to be $\xi\ = 0$ corresponding to the assumed location, $x = \frac{1}{2}$, of the boundary layer. We now match the one term outer expansion (7.67) with the one term inner expansion (7.72).

MATCHING

One term outer expansion: $y_r^o \sim e^{x-1}$

Rewritten in inner variable: $= e^{\epsilon^{(1/2)}\xi - \frac{1}{2}}$

Expanded for small ϵ: $= e^{-(1/2)}(1 + \epsilon^{(1/2)}\xi + \cdots)$

Therefore

(7.73) $$y_r^{oi} = e^{-(1/2)}.$$

One term inner expansion: $y^i \sim b_0 + a_0 \int_0^{\xi} e^{-(1/2)\tau^2} d\tau$

Rewritten in outer variable: $= b_0 + a_0 \int_0^{(x-\frac{1}{2})/\epsilon^{(1/2)}} \epsilon^{-(1/2)\tau^2} d\tau$

Expanded for small ϵ: $= b_0 + a_0 \int_0^{\infty} \epsilon^{-(1/2)\tau^2} d\tau$

Therefore

(7.74) $$y_r^{io} = b_0 + \frac{a_0\sqrt{\pi}}{\sqrt{2}}.$$

Equating (7.73) and (7.74) we have

(7.75) $$b_0 + \frac{a_0\sqrt{\pi}}{\sqrt{2}} = e^{-(1/2)}$$

which is an equation with two unknowns, a_0 and b_0. To determine a second equation with unknowns a_0 and b_0, we match the inner expansion (7.72) to the outer expansion, (7.68).

One term outer expansion: $y_l^o \sim 0$

Therefore

$$y_l^{oi} = 0. \tag{7.76}$$

One term inner expansion: $y^i \sim b_0 + a_0 \int_0^{\xi} e^{-(1/2)\tau^2} d\tau$

Rewritten in outer variable: $= b_0 + a_0 \int_0^{(\frac{1}{2}-x)/\epsilon^{(1/2)}} \epsilon^{-(1/2)\tau^2} d\tau$

Expanded for small ϵ: $= b_0 + a_0 \int_0^{-\infty} \epsilon^{-(1/2)\tau^2} d\tau$

Therefore

$$y_l^{oi} = b_0 - \frac{a_0\sqrt{\pi}}{\sqrt{2}}. \tag{7.77}$$

Equating (7.76) and (7.77) we have

$$b_0 - \frac{a_0\sqrt{\pi}}{\sqrt{2}} = 0. \tag{7.78}$$

Solving (7.75) and (7.78) simultaneously yields

$$a_0 = \frac{1}{\sqrt{2\pi}} e^{-(1/2)}$$

$$b_0 = \frac{1}{2} e^{-(1/2)}$$

and thus

$$y^i = \frac{1}{2} e^{-(1/2)} + (2\pi)^{-(1/2)} e^{-(1/2)} \int_0^{\xi} e^{-(1/2)\tau^2} d\tau. \tag{7.79}$$

In this problem we cannot form a single composite expansion that is uniformly valid over the whole interval. Instead, we form two composite expansions, one valid in $[0, \frac{1}{2}]$ and the other valid in $[\frac{1}{2}, 1]$. Thus we put

$$y_l^c = y_l^o + y^i - y_l^{oi} \quad \Rightarrow$$

$$y_l^c = y^i.$$

Similarly

$$y_r^c = y_r^o + y^i - y_r^{oi} \quad \Rightarrow$$

(7.81) $$y_r^c = e^{x-1} + y^i - e^{-(1/2)}.$$

EXAMPLE 7.6: Triple Decks

Consider the following example from Nayfeh[8], pp. 307-307.

(7.82) $$\epsilon^3 \frac{d^2y}{dx^2} + x^3 \frac{dy}{dx} + (x^3 - \epsilon) = 0$$

$$y(0) = \alpha$$
$$y(1) = \beta.$$

When more than two distinguished limits exist in a given boundary layer, the resulting expansion consists of two inner expansions in addition to the outer expansion. The domains of validity of each of them are often called decks, and thus, a problem with three decks is usually referred to as a *triple deck problem*. Since the coefficient of dy/dx is positive the boundary layer is expected to be at the origin. Therefore, the outer expansion will satisfy the boundary condition $y(1) = \beta$ but not $y(0) = \alpha$.

OUTER EXPANSION. Assuming an outer expansion of the form

(7.24) $$y^o = y_0(x) + \epsilon y_1(x) + \cdots$$

we substitute (7.24) into (7.82) and $y(1) = \beta$ to obtain

(7.83) $$x^3 \frac{dy_0}{dx} + x^3 y_0 = 0$$

$$y_0(1) = \beta$$

whose general solution is

$$y_0 = c_0 e^{-x}.$$

Applying $y_0(1) = \beta$ we obtain

(7.84) $$y^o = \beta e^{1-x} + \cdots$$

which does not satisfy the boundary condition at the origin.

INNER EXPANSION. To investigate the behavior at the origin we introduce stretching transformation

(7.85) $$\xi = \frac{x}{\epsilon^\nu}$$

into (7.112) and derive

(7.86)
$$\epsilon^{3-2\nu}\frac{d^2y}{d\xi^2}+\epsilon^{2\nu}\xi^3\frac{dy}{d\xi}+(\epsilon^{3\nu}\xi^3-\epsilon)y=0.$$

The distinguished limits may be determined by the method of undetermined gauges or can be found graphically.

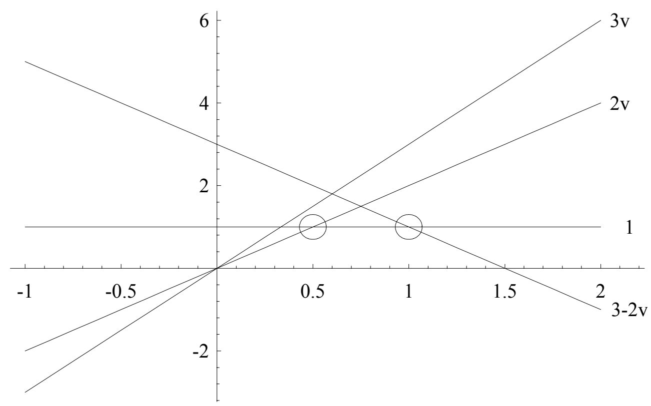

Figure 10.
Distinguished Limits

We find that $\nu=\{\frac{1}{2},1\}$ will satisfy the criteria. The higher the value of ν in (7.85), the closer the domain is to the origin. Therefore we will refer to the deck corresponding to $\nu=1$ as the left deck, $\nu=\frac{1}{2}$ as the middle deck, and the outer expansion as the right deck. Taking $\nu=1$,

(7.87)
$$\xi=\frac{x}{\epsilon}$$

we obtain from (7.116)

(7.88)
$$\epsilon\frac{d^2y}{d\xi^2}+\epsilon^2\xi^2\frac{dy}{d\xi}+(\epsilon^3\xi^3-\epsilon)y=0.$$

Dividing by ϵ and taking the limit as $\epsilon\to 0$ we see that the left deck expansion is governed by the *dominant part*

$$\frac{d^2Y_0}{d\xi^2}-Y_0=0$$

whose general solution is

$$Y_0 = a_0 e^{-\xi} + b_0 e^{\xi}. \tag{7.89}$$

We note that b_0 must be zero or the expansion would grow without bound with increasing ξ and be unmatchable to the middle deck. Since the left deck is valid at the origin it must match $y_0 = \alpha$. From (7.87) $x = 0$ corresponds to $\xi = 0$ and (7.89) implies

$$y^l = \alpha e^{-\xi} + \cdots. \tag{7.90}$$

MIDDLE EXPANSION. With $\nu = \frac{1}{2}$ we obtain the middle variable

$$\zeta = \frac{x}{\epsilon^{\frac{1}{2}}} \tag{7.91}$$

and from (7.86) we get

$$\epsilon^2 \frac{d^2 y}{d\zeta^2} + \epsilon \zeta^3 \frac{dy}{d\zeta} + (\epsilon^{(3/2)} \zeta^3 - \epsilon) y = 0. \tag{7.92}$$

It follows from (7.92) that the leading term $\widehat{Y}_0$ in the middle deck expansion y^m is governed by the dominant part

$$\zeta^3 \frac{d\widehat{Y}_0}{d\zeta} - \widehat{Y}_0 = 0.$$

Separating variables we have

$$\widehat{Y}_0 = d_0 e^{-(1/2)\zeta^2}$$

and

$$y^m = d_0 e^{-(1/2)\zeta^2} + \cdots. \tag{7.93}$$

where d_0 will be determined by matching.

MATCHING. Note that y^l cannot be matched directly with y^o since

$$(y^l)^o = 0 \quad \text{and} \quad (y^o)^l = \beta e.$$

To match y^m with y^o we note that

$$(y^m)^o = d_0 \quad \text{and} \quad (y^o)^m = \beta e$$

implying that

$$d_0 = \beta e \tag{7.94}$$

and

(7.95) $$y^m = \beta e^{1-(1/2)\zeta^2} + \cdots$$

To match y^m with y^l we note that $(y^m)^l = 0$ and $(y^l)^m = 0$ indicating that y^m and $y^l$are matchable.

COMPOSITE EXPANSION. The composite expansion is formed from the relation

$$y^c = y^o + y^m + y^l - y^{om} - y^{lm}.$$

Therefore,

(7.96) $$y^c = \beta e^{1-x} + \beta e^{1-(1/2)x^2/\epsilon} + \alpha e^{-x/\epsilon} - \beta e + \cdots.$$

Checking our previous results we can quickly compute

$$(y^c)^l = \alpha e^{-\xi} = y^l$$

$$(y^c)^m = \beta e^{1-(1/2)\zeta^2} = y^m$$

$$(y^c)^o = \beta e^{1-x} = y^o.$$

Showing that our composite expansion y^c is valid in the inner, middle, and outer regions and is thus valid everywhere as their regions must overlap to make matching possible.

EXAMPLE 7.7: From Nayfeh[8], pp. 286-289, we will now derive a solution of

(7.65) $$\epsilon \frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0$$

$$y(0) = 0$$
$$y(1) = 1$$

with only the assumption that $p_1(x)$ is positive everywhere in $[0,1]$. We suppose that the boundary layer is on the left at $y(0) = \alpha$.

OUTER EXPANSION. Assuming an outer expansion of the form

(7.24) $$y^o = y_0(x) + \epsilon y_1(x) + \cdots$$

we substitute (7.24) into (7.65), equate coefficients of e^0 and obtain

(7.97) $$p_1 \frac{dy_0}{dx} + p_0 y_0 = 0.$$

Separating variables and integrating we derive

$$y_0 = c_0 \exp\left[-\int_1^x p_0(\tau)/p_1(\tau)d\tau\right]$$

where the lower limit of integration is taken to be 1 to enable satisfaction of the boundary condition $y_0(1) = \beta$. Hence,

$$y_0 = \beta e^{f(x)}$$

where we define

(7.98) $$f(x) := \int_x^1 (p_0/p_1)d\tau.$$

Thus

(7.99) $$y^{\text{o}} = \beta e^{f(x)} + \cdots.$$

INNER EXPANSION. To determine the inner expansion valid at the origin we introduce the stretching transformation

(7.85) $$\xi = \frac{x}{\epsilon^{\nu}}.$$

Then (7.95) becomes

(7.100) $$\epsilon^{1-2\nu}\frac{d^2y^i}{d\xi^2} + \epsilon^{-\nu}p_1(\epsilon^{\nu}\xi)\frac{dy^i}{dx} + p_0(\epsilon^{\nu}\xi)y^i = 0.$$

As $\epsilon \to 0$, $p_1(\epsilon^{\nu}\xi) \to p_1(0)$ and $p_0(\epsilon^{\nu}\xi) \to p_0(0)$ where p_0 is assumed to be regular at the origin. By undetermined gauges we find that $\nu = 1$, in which case the dominant part of (7.100) is

(7.101) $$\frac{d^2y^i}{d\xi^2} + p_1(0)\frac{dy^i}{d\xi} = 0$$

whose general solution is

(7.102) $$y^i = a_0 + b_0 e^{-p_1(0)\xi}.$$

Since $x = 0 \Rightarrow \xi = 0$ and $y(0) = \alpha \Rightarrow y^i = \alpha$ then from (7.102) we see that

$$\alpha = a_0 + b_0$$

and that

(7.103) $$y^i = \alpha - b_0 + b_0 e^{-p_1(0)\xi}.$$

MATCHING.

One term outer expansion: $y \sim \beta e^{f(x)}$

Rewritten in inner variable : $= \beta e^{f(\epsilon\xi)}$

Expanded for small ϵ : $= \beta e^{f(0)} + \cdots$

Therefore

$$y^{oi} = \beta e^{f(0)}. \tag{7.104}$$

One term inner expansion: $y \sim \alpha - b_0 + b_0 e^{-p_1(0)\xi}$

Rewritten in outer variable: $= \alpha - b_0 + b_0 e^{-p_1(0)x/\epsilon}$

Expanded for small ϵ : $= \alpha - b_0$

Therefore

$$y^{io} = \alpha - b_0. \tag{7.105}$$

Note that exp$(-p_1(0)x/\epsilon)$ is exponentially small as $\epsilon \to 0$ because $p_1(0)$ is positive. If it were negative then exp$(-p_1(0)x/\epsilon)$ would have been exponentially large as $\epsilon \to 0$ and matching would not have been possible. In that case the boundary layer would be at the right end. The absence of exponential growth is essential for matching.

Equating (7.104) and (7.105) we have

$$b_0 = \alpha - \beta e^{f(0)}.$$

Therefore

$$y^i = \beta e^{f(0)} + (\alpha - \beta e^{f(0)})e^{-p_1(0)\xi} + \cdots. \tag{7.106}$$

COMPOSITE EXPANSION. Recalling that

$$y^c = y^o + y^i - y^{oi}$$

then

$$y^c = \beta e^{f(x)} + (\alpha - \beta e^{f(0)})e^{-p_1(0)\xi} + \cdots. \tag{7.107}$$

EXAMPLE 7.8: Multiple Scales.

We will now solve (7.65) by the method of multiple scales following Simmonds and Mann[10], pp. 103-106. This is done primarily to show the wide range of applicability of the two scale method and also to compare results with the method of asymptotic expansions. The method of matched asymptotic expansions is less sensitive to the form of $p_1(x)$ and $p_0(x)$ than is the two scale method and is normally the only recourse if the two scale method fails. Consider

(7.65)
$$\epsilon\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_0(x)y=0$$

$$y(0)=\alpha$$
$$y(1)=\beta.$$

We assume that the solution to (7.65) depends on a slow variable x and a fast variable

$$\xi=\frac{g(x)}{\epsilon}.$$

We seek an approximate solution in the form

(7.109)
$$y(x,\xi,\epsilon)=y_0(x,\xi)+\epsilon y_1(x,\xi)+O(\epsilon^2).$$

In this example only the first term of (7.139) will be determined. Taking the derivatives of (7.109) and using definition (7.108) we derive

(7.110)
$$\frac{dy}{dx}=\frac{\partial y}{\partial x}+\frac{\partial y}{\partial \xi}\frac{g'}{\epsilon}$$

$$\frac{d^2y}{dx^2}=\frac{\partial^2 y}{\partial x^2}+2\frac{\partial^2 y}{\partial x\partial \xi}\frac{g'}{\epsilon}+\frac{\partial y}{\partial \xi}\frac{g''}{\epsilon}+\frac{\partial^2 y}{\partial \xi^2}\frac{g'^2}{\epsilon^2}$$

where $' := \dfrac{d}{dx}$.

Substituting (7.108), (7.109), and (7.110) into (7.65) and multiplying by ϵ we obtain

(7.111)
$$g'^2\frac{\partial^2 y}{\partial \xi^2}+p_1g'\frac{\partial y}{\partial \xi}+\epsilon\left(2\frac{\partial^2 y}{\partial x\partial \xi}g'+\frac{\partial y}{\partial \xi}g''+p_1\frac{\partial y}{\partial x}+p_0y\right)+\epsilon^2\frac{\partial^2 y}{\partial x^2}=0$$

$$y(0,0,\epsilon)=\alpha$$
$$y(1,g(1)/\epsilon,\epsilon)=\beta.$$

Now substitute (7.109) into (7.111) and get the following sequence of P.D.E.s.

$$g'^2\frac{\partial^2 y_0}{\partial\xi^2} + p_1 g'\frac{\partial y_0}{\partial\xi} = 0 \tag{7.112}$$

$$y_0(0,0) = \alpha$$
$$y_0(1, g(1)/\epsilon) = \beta.$$

$$g'^2\frac{\partial^2 y_1}{\partial\xi^2} + p_1 g'\frac{\partial y_1}{\partial\xi} = -2g'\frac{\partial^2 y_0}{\partial x\partial\xi} - g''\frac{\partial y_0}{\partial\xi} - p_1\frac{\partial y_0}{\partial x} - p_0 y_0 \tag{7.113}$$

$$y_1(0,0) = 0$$
$$y_1(1, g(1)/\epsilon) = 0.$$

If we now pick

$$g(x) = \int_0^x p_1(t)dt \quad \Rightarrow \quad g' = p_1 \tag{7.114}$$

then (7.112) reduces to

$$\frac{\partial^2 y_0}{\partial\xi^2} + \frac{\partial y_0}{\partial\xi} = 0$$

whose solution is

$$y_0(x,\xi) = A(x) + B(x)e^{-\xi}. \tag{7.115}$$

The left boundary condition at (7.112) yields

$$B(0) = \alpha - A(0). \tag{7.116}$$

At $x = 1$

$$e^{-\xi} = e^{-g(1)/\epsilon}$$

is an exponentially small term as $\epsilon \to 0$ so we can say that the first (7.112) boundary condition reduces to

$$A(1) = \beta. \tag{7.117}$$

To determine $A(x)$ and $B(x)$ we must consider the form of the solution of (7.113). With (7.114) defined we see that (7.113) reduces (after dividing by p_1^2) to

$$\frac{\partial^2 y_1}{\partial\xi^2} + \frac{\partial y_1}{\partial\xi} = e^{-\xi}\left[\frac{B'}{p_1} + \frac{(p_1' - p_0)B}{p_1^2}\right] - \left[\frac{A'}{p_1} + \frac{p_0 A}{p_1^2}\right]. \tag{7.118}$$

To avoid resonant terms of the form $F(x)\xi e^{-\xi} + G(x)\xi$ in the solution for y we must set to zero the coefficients of $e^{-\xi}$ and ξ on the right of (7.118). Therefore

$$p_1 A' + p_0 A = 0 \tag{7.119}$$

which has the solution

$$A(x) = \exp\left[\int_x^1 (p_0(t)/p_1(t))dt\right] \tag{7.120}$$

while

$$p_1 B' + (p_1' - p_0)B = 0 \tag{7.121}$$

which has the solution

$$B(x) = \frac{C}{p_1(x)} \exp\left[\int_0^x (p_0(t)/p_1(t))dt\right]. \tag{7.122}$$

The constant C is determined by applying (7.116) with $A(x)$ given by (7.120). Then

$$C = p_1(0)\left[\alpha - e^{f(1)}\right] \tag{7.123}$$

where

$$f(x) := \int_0^x \left[p_0(t)/p_1(t)\right] dt. \tag{7.124}$$

We may then write

$$A(x) = e^{[f(1)-f(x)]}.$$

The resulting approximation is

$$\begin{aligned} y &\sim y_0(\xi, x) + O(\epsilon) \\ &= e^{[f(1)-f(x)]} + \frac{p_1(0)}{p_1(x)}\left[\alpha - e^{f(1)}\right] e^{f(x)} e^{-\xi} + O(\epsilon). \end{aligned} \tag{7.126}$$