CHAPTER IV

STRAINED PARAMETERS

In Chapter III we introduced the regular perturbation expansion using the Duffing equation as an example of a conservative, nonlinear oscillator. This chapter continues the study of nearly linear second order differential equations focusing on periodic solutions of oscillatory equations of the form

$$u'' + k^2 u = \epsilon f(t, u, u', \epsilon)$$

where f is either periodic in t or independent of t. We will develop a method of rendering the approximate solutions to some of the differential equations mentioned in Chapter III uniformly valid by introducing near-identity transformations of the independent variable. This technique goes back to the nineteenth century when astronomers, such as Lindstedt(1882), devised techniques to avoid the appearance of secular terms in perturbation solutions of equations such as

$$u'' + w_0^2 u = \epsilon f(u, u'),\ \epsilon \ll 1.$$

The fundamental idea in Lindstedt's technique is based on the observation that the nonlinearities alter the frequency of the system from the linear one, w_0, to $w(\epsilon)$. To account for this change in frequency, he introduced a new variable $\tau = wt$ and expanded w and u in powers of ϵ as

$$u = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \cdots$$

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots$$

Then he chose the parameters $w_i, i \geq 1,$ to prevent the appearance of secular terms. Poincaré(1892) proved that the expansions obtained by Lindstedt's technique are asymptotic.

Various forms of this idea have been utilized to obtain approximate solutions to problems in physics and engineering. The idea is to find a parameter in the problem that is altered by the perturbations and then expand both the dependent variables as well as this parameter using an appropriate sequence of gauge functions derived from the perturbation(say, powers of the strength of the perturbation). The perturbations in the parameter are then chosen to render the expansion uniformly valid. Thus, this technique is called the method of *strained parameters*.

The *Lindstedt method*, applies only to the periodic solutions; more general methods given in Chapters V and VI can handle the transient solutions as well. But for the periodic solutions, the Lindstedt method has a distinct advantage over the other methods, both in simplicity and accuracy. The regular expansion will continue to be referenced for purposes of comparison but the remainder of this paper will focus on techniques developed specifically to improve on the shortcomings of the regular method.

We will begin with the Duffing equation. In Chapter III we found that the exact solution of the Duffing equation is different from the regular perturbation approximation because the exact period is a function of ϵ and the period of the regular approximation is not. This discrepancy slowly drives $u(t,\epsilon)$ and its approximation of the form

$$\widehat{u}(t,\epsilon) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots + \epsilon^n u_n$$

apart. Suppose then that the period is a function $T(\epsilon)$, of ϵ and that

$$u(t+T(\epsilon),\epsilon) = u(t,\epsilon) \quad \text{for all } t \text{ and for } |\epsilon| < \epsilon_0.$$

Associated with the period $T(\epsilon)$ there is a frequency $w(\epsilon) = 2\pi/T(\epsilon)$. The idea of the Lindstedt method is to introduce a new time variable such that the given function works out to have a period independent of ϵ when expressed in the new variable. Usually the new constant period is constructed to be 2π. To make this happen, the new time variable must depend on ϵ. We introduce the transformation

$$\tau(\epsilon) = w(\epsilon)t. \tag{4.1}$$

Where τ is called the *strained time*. The function u expressed in strained time now becomes

$$\varphi(\tau,\epsilon) = u(\tau/w(\epsilon),\epsilon) \tag{4.2a}$$

or

$$u(t,\epsilon) = \varphi(w(\epsilon)t,\epsilon). \tag{4.2b}$$

From the periodicity of $u(t,\epsilon)$ it follows that $\varphi(\tau+2\pi,\epsilon) = \varphi(\tau,\epsilon)$. So, by theorem 3.1 $\varphi(\tau,\epsilon) = \varphi_0(\tau) + \cdots + \epsilon^k\varphi_k(\tau) + O(\epsilon^{k+1})$ uniformly for all τ. Therefore

$$u(t,\epsilon) = \varphi_0(w(\epsilon)t) + \cdots + \epsilon^k\varphi_k(w(\epsilon)t) + O(\epsilon^{k+1}) \tag{4.3}$$

uniformly for all t.

The above expansion is not a Taylor expansion since ϵ appears not only in the powers but also in the coefficients of each power. Such an expansion is called a *generalized asymptotic power series*. A generalized power series such as (4.3), where the coefficient of each ϵ^n is a periodic function with ϵ entering only through the frequency, is called a *Lindstedt expansion*.

Sometimes the solution frequency is known in advance and at other times it must be constructed as part of the solution process. When the frequency is known, recursive calculations yield $\varphi_0, \varphi_1, \ldots, \varphi_k,$ up to the desired order, and (4.3) provides an approximation to $u(t,\epsilon)$ which is uniformly valid for all time. When the frequency is not known, the method of strained parameters is a systematic procedure for determining successively more accurate approximations to $w(\epsilon) = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots$. The best approximation that can be constructed with this information is

(4.4) $$u(t,\epsilon) \sim \widehat{\varphi}(\widehat{w}(\epsilon)t, \epsilon)$$

where

(4.5) $$\widehat{\varphi}(\tau,\epsilon) = \varphi_0(\tau) + \epsilon\varphi_1(\tau) + \cdots + \epsilon^k\varphi_k(\tau)$$

(4.6) $$\widehat{w}(\epsilon) = w_0 + \epsilon w_1 + \cdots + \epsilon^k w_k.$$

The hat in these expressions denotes that the series is truncated after $k+1$ terms and is an approximation to the corresponding full series.

Duffing equation.

EXAMPLE 4.1: With (4.1) and the chain rule applied to

(4.7a) $$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0$$

$$u(0) = \alpha$$

$$\frac{du}{dt}(0) = \beta.$$

(4.7a) becomes

(4.7b) $$w^2\frac{d^2u}{d\tau^2} + u + \epsilon u^3 = 0$$

$$u(0) = \alpha$$

$$\frac{du}{d\tau}(0) = \beta/w.$$

We note that the actual frequency of the system, $w(\epsilon)$, now appears explicitly in the equation and we choose $w(0) = w_0 = 1$. We seek approximations for u and w in the form of truncated power series in ϵ. From (4.5) and (4.6) we take

$$\widehat{\varphi} = \varphi_0(\tau) + \epsilon\varphi_1(\tau) \tag{4.8}$$

$$\widehat{w}(\epsilon) = 1 + \epsilon w_1. \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7) and collecting terms to the first order we obtain

$$\varphi_0'' + \varphi_0 + \epsilon\left(\varphi_1'' + \varphi_1 + \varphi_0^3 + 2w_1\varphi_0''\right) + O\left(\epsilon^2\right) = 0.$$

($' := \frac{d}{d\tau}$). Applying the fundamental theorem gives the following sequence of IVP's

$$\varphi_0'' + \varphi_0 = 0 \tag{4.10}$$

$$\varphi_0(0) = \alpha$$
$$\varphi_0'(0) = \beta/w$$

$$\varphi_1'' + \varphi_1 = -\varphi_0^3 - 2w_1\varphi_0'' \tag{4.11}$$

$$\varphi_1(0) = 0$$
$$\varphi_1'(0) = 0.$$

The solution of (4.10) in polar form is

$$\varphi_0(\tau) = \rho\cos(\tau - \psi) \tag{4.12}$$

where ρ and ψ are constants determined by

$$\rho = \left(\alpha^2 + \beta^2/w^2\right)^{\frac{1}{2}}$$

$$\tan(\psi) = \beta/\alpha w.$$

Then (4.11) becomes

$$\varphi_1'' + \varphi_1 = -\frac{\rho^3}{4}\cos 3(\tau - \psi) + \left(2w_1\rho - \frac{3}{4}\rho^3\right)\cos(\tau - \psi). \tag{4.13}$$

Recalling from Chapter III that the resonant term, $\cos(\tau - \psi)$, produced the secular term in the final solution we choose w_1 to suppress it. Therefore let

$$w_1 = \frac{3}{8}\rho^2. \tag{4.14}$$

A particular solution to (4.13) is

(4.15) $$\varphi_1 = \frac{1}{32}\rho^3 \cos 3(\tau - \psi).$$

Substituting (4.14) into (4.9), (4.9) into (4.1) and then (4.1), (4.12), and (4.15) into (4.8) we get our uniform first order approximation free from secular terms

$$\widehat{\varphi}(\widehat{w}(\epsilon)t, \epsilon) = \rho \cos\left[\left(1 + \frac{3}{8}\epsilon\rho^2\right)t - \psi\right] + \frac{\epsilon\rho^3}{32}\cos 3\left[\left(1 + \frac{3}{8}\epsilon\rho^2\right)t - \psi\right].$$

As we compute higher order approximations to $\widehat{\varphi}(\widehat{w}(\epsilon)t, \epsilon)$ the right hand sides of each of the DE's producing $\varphi_2, \varphi_3, \ldots$ will contain a resonance producing term which must be suppressed by a proper choice of $w_2, w_3, \ldots$.
Nayfeh[8], pp. 118-120 and Murdock[6], pp. 160-162.

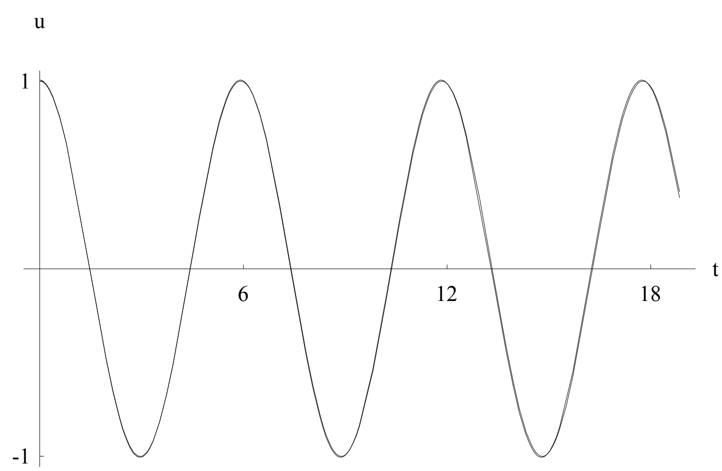

Figure 5.
Exact Duffing Solution vs. Lindstedt Expansion

Error Analysis. The approximation $\widehat{\varphi}(\widehat{w}(\epsilon)t, \epsilon)$, (4.4), differs from $u(t, \epsilon)$, (4.3), containing the same number of terms, in that the actual frequency $w(\epsilon)$ in (4.3) is replaced by $\widehat{w}(\epsilon)$, its approximation (4.6). Just as we discovered in the case of the regular expansion, a discrepancy in the frequency will cause the approximation to drift away from the actual solution over time. Thus, we cannot expect (4.4) to retain the accuracy $O(\epsilon^{k+1})$ over all time as (4.3) does. We clarify this idea in the following theorem from Murdock[6], pp. 162-164, on a general Lindstedt expansion.

THEOREM 4.1: If $u(t,\epsilon)$ is analytic, $w(\epsilon)$ is analytic and $u(t+2\pi/w(\epsilon),\epsilon) = u(t,\epsilon)$ for all t and ϵ, and if $\varphi(\tau,\epsilon)$, $\widehat{\varphi}(\tau,\epsilon)$ and $\widehat{w}(\epsilon)$ are defined by (4.2), (4.5) and (4.6) then

$$|u(t,\epsilon) - \widehat{\varphi}(\widehat{w}(\epsilon)t,\epsilon)| \leq K\epsilon^{k+1}|t| \tag{4.16}$$

for some constant K.

Proof: Given $0 < \epsilon < \epsilon_0$, there exists $c_0 > 0$ such that

$$|w(\epsilon) - \widehat{w}(\epsilon)| \leq c_0\epsilon^{k+1}. \tag{4.17}$$

Let

$$\ell = \max\left\{\left|\frac{d\widehat{\varphi}}{d\tau}(\tau,\epsilon)\right| : 0 \leq \tau \leq 2\pi, |\epsilon| < \epsilon_0\right\}.$$

The maximum exists because the intervals are compact. Since $\widehat{\varphi}$ is 2π-periodic, ℓ is maximum over all τ. By the mean value theorem

$$|\widehat{\varphi}(\tau_1,\epsilon) - \widehat{\varphi}(\tau_2,\epsilon)| \leq \ell|\tau_1 - \tau_2| \tag{4.18}$$

for all τ_1, τ_2, and $0 < \epsilon < \epsilon_0$. Thus, ℓ is a global Lipschitz constant for $\widehat{\varphi}$ with respect to τ. From (4.17) and (4.18) we have

$$|\widehat{\varphi}(w(\epsilon)t,\epsilon) - \widehat{\varphi}(\widehat{w}(\epsilon)t,\epsilon)| \leq c_0\ell\epsilon^{k+1}|t|. \tag{4.19}$$

From (4.3) there exists $c_1 > 0$ such that

$$|u(t,\epsilon) - \widehat{\varphi}(w(\epsilon)t,\epsilon)| \leq c_1\epsilon^{k+1} \tag{4.20}$$

for all t. From (4.19), (4.20), and the triangle inequality we conclude

$$|u(t,\epsilon) - \widehat{\varphi}(\widehat{w}(\epsilon)t,\epsilon)| \leq c_1\epsilon^{k+1} + c_0\ell\epsilon^{k+1}|t|. \tag{4.21}$$

□

Now we may determine the range of t for which the error remains $O\big(\epsilon^{k+1}\big)$. If L is a positive constant then

$$|u(t,\epsilon) - \widehat{\varphi}(\widehat{w}(\epsilon)t,\epsilon)| \leq (c_1 + c_0\ell L)\epsilon^{k+1}$$

for $0 < \epsilon < \epsilon_0$ and $|t| \leq L$. Thus the approximation is uniformly of order $O\big(\epsilon^{k+1}\big)$ for $-L \leq t \leq L$. From (4.21) we can see what happens to our estimate beyond $t = L$. Up to $t = L/\epsilon$ the error is $\leq c_1\epsilon^{k+1} + c_0\ell L\epsilon^k \leq (c_1\epsilon_0 + c_0\ell L)\epsilon^k = O\big(\epsilon^k\big)$. Thus the approximation is uniformly of order $O\big(\epsilon^k\big)$ for $-L/\epsilon \leq t \leq L/\epsilon$. This time interval is called an *expanding interval* of order $O(\epsilon^{-1})$ since the interval increases as ϵ decreases. So, as ϵ decreases the estimate improves in two ways: The bound for the error decreases and the range of time for which it is valid increases. Thus the approximation on the expanding interval of order $O(\epsilon^{-1})$ is $O\big(\epsilon^k\big)$. This illustrates the *trade-off property* of accuracy for the length of the uniform interval.

In summary, the Lindstedt method attempts to approximate a periodic function by another periodic function with a period as close to the original period as possible. If the periods are equal the approximation is uniform for all time. If the periods are different the approximation is only valid on an expanding interval. We can now more precisely characterize problems for which singular perturbation methods are appropriate. These problems have the property that an expansion in powers of a suitable small parameter, with coefficients depending only on the independent variable, are not uniformly valid for all relevant values of the independent variable.

Damped Linear Oscillator.

EXAMPLE 4.2: In Chapter III we determined that the regular method failed to produce a uniform expansion for the damped linear oscillator because it did not account for the dependence of the frequency on ϵ. To include the fact that the frequency is a function of ϵ we strain the time with the view that frequency, w, is a function of ϵ. Substitute

$$\tau(\epsilon) = w(\epsilon)t$$

in

$$\frac{d^2u}{dt^2} + 2\epsilon\frac{du}{dt} + u = 0 \tag{3.19}$$

$$u(0) = \alpha$$
$$\frac{du}{dt}(0) = \beta$$

and obtain

$$w^2(\epsilon)\frac{d^2u}{d\tau^2} + 2\epsilon w(\epsilon)\frac{du}{d\tau} + u = 0. \tag{4.22}$$

Next, we try expanding u and w in powers of ϵ.

$$u = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + O\big(\epsilon^3\big) \tag{4.23}$$

$$w(\epsilon) = 1 + \epsilon w_1 + \epsilon^2 w_2 + O\big(\epsilon^3\big). \tag{4.24}$$

Note that the first term in (4.24) is unity which will give us the unperturbed (undamped) frequency. Substituting (4.23) and (4.24) into (4.22), expanding for small ϵ and equating like powers of ϵ yields the following sequence of IVP's:

(4.25)
$$u_0'' + u_0 = 0$$
$$u_0(0) = \alpha$$
$$u_0'(0) = \beta/w$$

(4.26)
$$u_1'' + u_1 = -2w_1 u_0'' - 2u_0'$$
$$u_1(0) = 0$$
$$u_1'(0) = 0$$

(4.27)
$$u_2'' + u_2 = -2w_2 u_0'' - w_1^2 u_0'' - 2w_1 u_1'' - 2w_1 u_0' - 2u_1'$$
$$u_2(0) = 0$$
$$u_2'(0) = 0.$$

$(\,' := \frac{d}{d\tau})$. The general solution of (4.25) is then

$$u_0 = \rho\cos(\tau - \psi)$$

where ρ and ψ are constants determined by

$$\rho = \left(\alpha^2 + \beta^2/w^2\right)^{\frac{1}{2}}$$

$$\tan(\psi) = \beta/\alpha w.$$

Then (4.26) becomes

(4.28)
$$u_1'' + u_1 = -2\rho w_1 \cos(\tau - \psi) + 2\rho \sin(\tau - \psi).$$

To eliminate secular terms from the particular solution of (4.28) we need to suppress all resonance producing terms on the right. But this means that

(4.29)
$$2\rho w_1 = 0 \quad \text{and} \quad \rho = 0$$

producing

$$u_1'' + u_1 = 0.$$

Equations (4.29) cannot be satisfied simultaneously unless $\rho = 0$, in which case $u_0 = 0 \quad \Rightarrow \quad u_1 = 0 \quad \Rightarrow \quad u_2 = 0 \quad \Rightarrow$ the trivial solution. Nayfeh[8], pp. 140.

The above development shows that the Lindstedt expansion fails to yield a nontrivial uniform solution in the presence of damping. The reason is clear. Our zeroth approximation, which under the conditions of uniformity may be only slightly corrected by higher order terms, has a constant amplitude. Since the amplitude is $\rho e^{-\epsilon t}$ in the exact solution

$$u = \rho e^{-\epsilon t} \cos\left[\sqrt{1-\epsilon^2}\, t - \psi\right], \tag{3.33}$$

the only constant amplitude solution is one obtained after a long time (i.e. steady state). Therefore, although the Lindstedt expansion is effective in determining periodic solutions, it is incapable of determining transient responses.