CHAPTER V

MULTIPLE SCALES

This chapter and the next concern initial value problems of oscillatory type on long intervals of time. Until Chapter VII we will study autonomous oscillatory second order initial value problems of the form

$$\frac{d^2u}{dt^2} + k^2u = \epsilon f(u, u', \epsilon)$$

$$u(0) = \alpha$$

$$\frac{du}{dt}(0) = \beta.$$

It is clear from the results of the damped linear oscillator in Chapter IV that to obtain a regular expansion which is uniformly valid, we must pull a part of the effect of ϵ into the first approximation of $u(t, \epsilon)$. The solution to the reduced problem of

$$\frac{d^2u}{dt^2} + 2\epsilon\frac{du}{dt} + u = 0 \tag{3.19}$$

$$u(0) = \alpha$$
$$\frac{du}{dt}(0) = \beta$$

can be written in polar form

$$u = \rho\cos(t - \psi) \tag{5.1}$$

or Cartesian form

$$u = \alpha\cos(t) + \beta\sin(t). \tag{5.2}$$

The results of Chapter III show us that the zeroth approximation has an error $O(\epsilon)$ uniformly for t in a finite interval $0 \le t \le T$. The first task of the method of multiple scales is to improve this zeroth approximation to produce a first approximation having error $O(\epsilon)$ uniformly for t in an expanding interval of the form $0 < t < L/\epsilon$. This is a different problem from that of improving the accuracy to $O(\epsilon^2)$ on a finite interval, a problem handled satisfactorily by regular perturbation methods. Up to now, the solution of the reduced problem was also the leading term in the asymptotic series being constructed. However, in the methods of multiple scales, the leading term of the asymptotic solution is not the solution of the reduced problem but is already an improvement on it. So there is no zeroth term in this method, the first term is the first approximation. The first approximation will now take the form of (5.1) or (5.2) except that the quantities α, β, ρ, and ψ which were constants of integration now become slowly varying functions of time. They are of the form

$$u = \alpha(\epsilon t)\cos t + \beta(\epsilon t)\sin t$$

or

$$u = \rho(\epsilon t)\cos(t - \psi(\epsilon t))$$

where the functions of ϵt are found as the solutions of differential equations developed in the method of multiple scales. It is customary to write $\tau = \epsilon t$ or use the notation $T_0 = t$ and $T_1 = \epsilon t$. We will refer to τ and T_1 as *slow time* and t and T_0 as *fast time*. It may be seen upon examination of the exact solution of the damped linear oscillator that the damping half life is of the order T_1, or slow time, and the period of the oscillations are of the order T_0, or fast time. T_0 and T_1 are also referred to as two *time scales*. One of the distinctive properties of most perturbation problems is that no single scale can be used to characterize the complete solution of the problem.

In most applications of the method of multiple scales, the first approximation is sufficient. There are several different methods of multiple scales and they all give the same first approximation. Beyond the first approximation the different methods do not necessarily give the same result. This is possible because generalized asymptotic expansions are not a unique approximation of the exact result. Murdock[6], pp. 230, groups higher order multiple scale methods for oscillatory problems into three types:

1. Two scale methods using t and τ. The solutions are written in the form $u_0(t,\tau) + \epsilon u_1(t,\tau) + \cdots$
2. Two scale methods using slow time, $\tau = \epsilon t$, and strained time $T := t(w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots)$. The solutions appear as $u_0(T,\tau) + \epsilon u_1(T,\tau) + \cdots$
3. Multiple scale methods using M scales where $T_m = \epsilon^m t$, $m = 1, 2, \ldots, M$. The solutions are written $u_0(T_0, T_1, \ldots, T_m) + \epsilon u_1(T_0, T_1, \ldots, T_m) + \cdots$. A variation of this method omits one time scale for each successive term so that a three scale three term solution would be written $u_0(T_0, T_1, T_2) + \epsilon u_1(T_0, T_1) + \epsilon^2 u_2(T_0)$.

The theory of higher order approximations by the first form is well understood. This form is applicable to a wide variety of problems and gives approximations to any order of accuracy which are valid on expanding intervals of length $O(\epsilon^{-1})$. The second and third forms are intended to give approximations on expanding intervals longer than $O(\epsilon^{-1})$ but they are not always successful and there is no adequate theory to explain the success they do have.

We return now to the damped linear oscillator. The method of strained parameters failed to improve the solution over the regular method due to its inability to handle transient characteristics. Both methods use only one time scale in their solution. The exponential decay of the oscillator due to the damping introduces a characteristic time (the half life of the decay) that occurs on a longer scale than the oscillations. We will then introduce a longer time scale right from the beginning and see if a two scale method can resolve this difficulty.

Two Scale Method.

EXAMPLE 5.1: Damped Linear Oscillator.

We seek an approximate solution of the IVP

(5.3)
$$u'' + 2\epsilon u' + u = 0$$

$$u(0) = \alpha$$
$$u'(0) = \beta$$

in the form

(5.4)
$$u(t,\tau,\epsilon) = u_0(t,\tau) + \epsilon u_1(t,\tau) + \epsilon^2 u_2(t,\tau) + \cdots + \epsilon^n u_n(t,\tau) + O\left(\epsilon^{n+1}\right)$$

where

(5.5)
$$\tau = \epsilon t.$$

$(' = \frac{d}{dt})$. In this example, only the first term of (5.4) will be determined. Using (5.5) and the chain rule we have

(5.6)
$$u' = u_t + \epsilon u_\tau$$

(5.7)
$$u'' = u_{tt} + 2\epsilon u_{t\tau} + \epsilon^2 u_{\tau\tau}$$

where partial derivatives are indicated by subscripted variables. Substituting (5.6) and (5.7) into the P.D.E. of (5.3) and substituting (5.4) into the initial conditions we obtain

$$(5.8) \qquad u_{tt} + u + 2\epsilon(u_{t\tau} + u_t) + \epsilon^2(u_{\tau\tau} + 2u_\tau) = 0$$

$$u_0(0,0) + \epsilon u_1(0,0) + \epsilon^2 u_2(0,0) + \cdots = \alpha$$

$$\begin{aligned} u_{0t}(0,0) &+ \epsilon[u_{0\tau}(0,0) + u_{1t}(0,0)] \\ &+ \epsilon^2[u_{1\tau}(0,0) + u_{2t}(0,0)] + \cdots = \beta. \end{aligned}$$

Now substitute (5.4) into the P.D.E. of (5.8) and use the fundamental theorem to get the following sequence of partial differential equations and initial conditions:

$$(5.9) \qquad u_{0tt} + u_0 = 0$$

$$\begin{aligned} u_0(0,0) &= \alpha \\ u_{0t}(0,0) &= \beta \end{aligned}$$

$$(5.10) \qquad u_{1tt} + u_1 = -2(u_{0t\tau} + u_{0t})$$

$$\begin{aligned} u_1(0,0) &= 0 \\ u_{1t}(0,0) + u_{0\tau}(0,0) &= 0 \end{aligned}$$

$$(5.11) \qquad u_{2tt} + u_2 = -2(u_{1t\tau} + u_{1t}) - u_{0\tau\tau} - 2u_{0\tau}$$

$$\begin{aligned} u_2(0,0) &= 0 \\ u_{2t}(0,0) + u_{1\tau}(0,0) &= 0 \end{aligned}$$

We note that (5.4) is a generalized asymptotic expansion since ϵ enters both through the gauges and also through the coefficients u_n by way of τ. Although there is no general theorem allowing the differentiation of a generalized asymptotic expansion term by term it is nevertheless reasonable to construct the coefficients of (5.4) on the assumption that such differentiation is possible, and then to justify the resulting series by *direct error estimation* afterwards. It is also important to note that even the step of equating coefficients of equal powers of ϵ (the fundamental theorem) is not justified by any theorem about generalized asymptotic expansions since they have no uniqueness properties. (5.9) is a partial differential equation. The general solution is obtained as usual except we let the arbitrary constants become arbitrary functions of τ.

$$(5.12) \qquad u_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t = \rho(\tau)\cos(t - \psi(\tau))$$

The initial conditions impose the following restrictions on the arbitrary functions in (5.12) :

$$A(0) = \alpha \tag{5.13}$$

$$B(0) = \beta$$

$$\rho(0) = \sqrt{\alpha^2 + \beta^2}$$

$$\tan[\psi(0)] = \beta/\alpha.$$

At this point we see that we get no information on $A(\tau),\ B(\tau),\ \rho(\tau),$ or $\psi(\tau)$ except their initial values. Thus, the initial value problem for u_0 does not completely determine u_0. The polar form of (5.12) will expedite calculations but the computational advantage for higher order terms lies with the Cartesian form. Therefore, we will compute the first order approximation in Cartesian form because we will need the functions $A(\tau)$ and $B(\tau)$ to compute the second order term. We write the general solution to (5.9) as

$$u_0(t, \tau) = A(\tau)\cos t + B(\tau)\sin t$$

where

$$A(0) = \alpha \tag{5.15}$$

$$B(0) = \beta.$$

The equation for u_1 is

$$u_{1tt} + u_1 = 2[(A'(\tau) + A(\tau))\sin t - (B'(\tau) + B(\tau))\cos t]. \tag{5.16}$$

Since τ is bounded on expanding intervals of $O(\epsilon^{-1})$, it is permissible for u_1 to contain so called τ-*secular terms* which involve factors of τ, but not t-secular terms which involve factors of t. The terms resonant in t which will produce t-secular terms are the sin t and cos t terms. The coefficients of these terms must be set to zero. Thus $A(\tau)$ and $B(\tau)$ must satisfy the O.D.E.s

$$A'(\tau) = -A(\tau)$$

$$B'(\tau) = -B(\tau).$$

The solutions of these equations satisfying (5.15) are

$$A(\tau) = \alpha e^{-\tau}$$

$$B(\tau) = \beta e^{-\tau}.$$

So, in Cartesian form, the first approximation is computed to be

$$u_0(t,\tau) = e^{-\tau}(\alpha \cos t + \beta \sin t). \tag{5.17}$$

We can convert (5.17) directly to polar form by taking

$$\begin{aligned} \alpha &= \rho \cos \psi \\ \beta &= \rho \sin \psi \\ \rho &= \sqrt{\alpha^2 + \beta^2} \\ \tan(\psi) &= \beta/\alpha. \end{aligned} \tag{5.18}$$

then

$$u_0(t,\tau) = e^{-\tau}\rho \cos(t - \psi(\tau)).$$

We found that the entire right side of (5.10) was resonant so we eliminated it by our choice of coefficients. (5.10) then becomes

$$\begin{aligned} &u_{1tt} + u_1 = 0 \\ &u_1(0,0) = 0 \\ &u_{1t}(0,0) = -u_{0\tau}(0,0) = \alpha \end{aligned}$$

whose solution is

$$\begin{aligned} &u_1(t,\tau) = C(\tau)\cos t + D(\tau)\sin t \\ &C(0) = 0 \\ &D(0) = \alpha. \end{aligned}$$

As with the first order approximation, we do not have enough information to determine $C(\tau)$ and $D(\tau)$ unless we use the expression for (5.11) and suppress all terms in u_2 which would produce t-secular terms in u_1. We can now write (5.11) as

$$\begin{aligned} u_{2tt} + u_2 &= -2(u_{1t\tau} + u_{1t}) - u_{0\tau\tau} - 2u_{0\tau} \\ &= [2C' + 2C + \beta e^{-\tau}]\sin t + [-2D' - 2D + \alpha e^{-\tau}]\cos t. \end{aligned}$$

Elimination of resonant terms requires that

$$C(\tau) = -\frac{\beta}{2}\tau e^{-\tau}$$

$$D(\tau) = \frac{\alpha}{2}\tau e^{-\tau} + \alpha e^{-\tau}$$

so that

$$u_1(t,\tau) = -\frac{\beta}{2}\tau e^{-\tau}\cos t + \left(\frac{\alpha}{2}\tau e^{-\tau} + \alpha e^{-\tau}\right)\sin t.$$

Substituted into (5.4), the complete second approximation becomes

$$\begin{aligned} u(t,\epsilon) &= \alpha e^{-\tau}\cos t + \beta\, e^{-\tau}\sin t \\ &\quad + \epsilon\left[-\frac{\beta}{2}\tau e^{-\tau}\cos t + \left(\frac{\alpha}{2}\tau e^{-\tau} + \alpha e^{-\tau}\right)\sin t\right] + O\left(\epsilon^2\right) \end{aligned}$$

or in polar form

$$\begin{aligned} u(t,\epsilon) &= \rho e^{-\tau}\cos(t-\psi) \\ &\quad + \epsilon\left[\frac{1}{2}\rho\tau e^{-\tau}\sin(t-\psi) + \alpha e^{-\tau}\sin t\right] + O\left(\epsilon^2\right). \end{aligned}$$

Adapted from Murdock[6], pp. 233-237 and Simmonds and Mann[10], pp. 65-69.

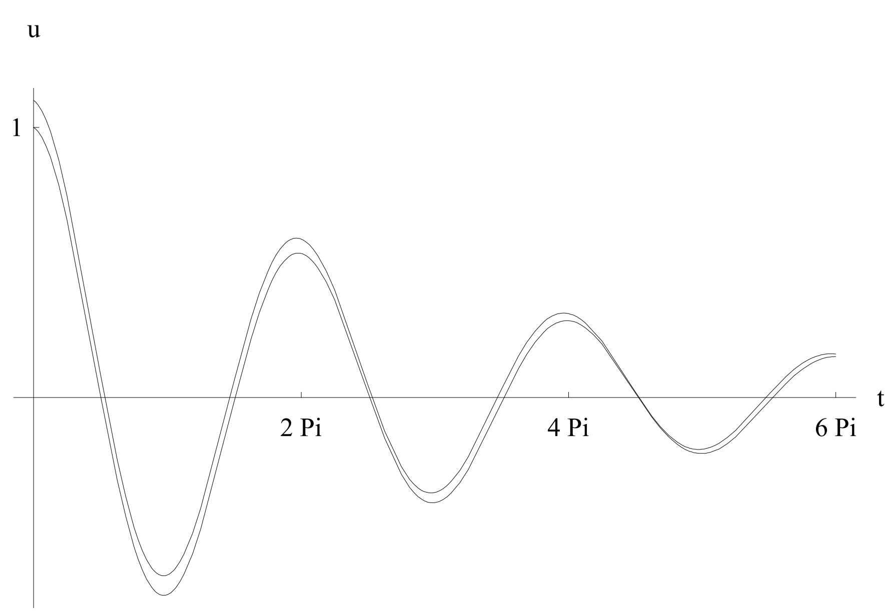

Figure 6.
Exact Linear Solution vs. Two Scale Expansion

Two Scales with Strained Time.

The term proportional to $\tau e^{-\tau}$ which at first gives the appearance of a τ-secular term turns out not to be a problem. This is because the exponential damping acts to reduce not only the size of the solution but also the error as $t \to \infty$. Because of this, the error, rather than being $O(\epsilon^2)$ on expanding intervals of $O(\epsilon^{-1})$, is actually $O(\epsilon^2)$ for all time. Although the term $\tau e^{-\tau}$ does no harm in this case, its appearance is due to the fact that fast time in the exact solution (3.33) is not t but the strained time $t\sqrt{1-\epsilon}$. Recall that damping alters the period of oscillation. Failure to introduce a strained time results in the appearance of such terms as $\tau^n e^{-\tau}$ in u_n. Just as secular terms were suppressed by strained time in the Lindstedt expansion, strained time can be used to suppress τ-secular terms in multiple scale expansions. We will now examine this approach even though the only apparent reason for suppressing these terms is that doing so will simplify the equations.

The regular method has only one time scale and could not account for the damping nor for the frequency change in the period of oscillation caused by the damping. We have just seen how the two scale method successfully handled the exponential decay with the introduction of a second longer time scale. Now, following Lindstedt, we will see if we can more closely match the actual frequency of the perturbed solution by introducing a strained time.

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EXAMPLE 5.2: Damped Linear Oscillator.

We will use the damped linear oscillator problem once again so that we may compare our results with previous methods and to simplify calculations. Instead of fast time, t, and slow time, τ, we will now use strained time, T, and slow time, τ, defined by

(5.19)
$$T := t\big(w_0 + w_1\epsilon + w_2\epsilon^2\big)$$

$$\tau := \epsilon t.$$

We seek a solution of the form

(5.20) $$u(T,\tau,\epsilon) = u_0(T,\tau) + \epsilon u_1(T,\tau) + \cdots$$

To save some time and space, we report that the end result of using (5.19) in its current form is $w_0 = 1$ and $w_1 = 0$. So instead of (5.19) we will use strained time to be

$$T := t\big(1 + w_2\epsilon^2\big).$$

Substituting (5.20) into (5.3), using the chain rule and the fundamental theorem results in the following sequence of partial differential equations:

(5.21) $$u_{0TT} + u_0 = 0$$

$$\begin{aligned} u_0(0,0) &= \alpha \\ u_{0T}(0,0) &= \beta \end{aligned}$$

(5.22) $$u_{1TT} + u_1 = -2u_{0T\tau} - 2u_{0T}$$

$$\begin{aligned} u_1(0,0) &= 0 \\ u_{1T}(0,0) &= -u_{0\tau}(0,0) \end{aligned}$$

(5.23) $$u_{2TT} + u_2 = -2w_2u_{0TT} - u_{0\tau\tau} - 2u_{1T\tau} - 2u_{0\tau} - 2u_{1T}$$

$$\begin{aligned} u_2(0,0) &= 0 \\ u_{2T}(0,0) &= -w_2u_{0T}(0,0) - u_{1\tau}(0,0) \end{aligned}$$

The solution to (5.21) is

(5.24) $$u_0 = A(\tau)\cos T + B(\tau)\sin T$$

$$\begin{aligned} A(0) &= \alpha \\ B(0) &= \beta. \end{aligned}$$

(5.22) becomes

$$\begin{aligned} u_{1TT} + u_1 = &-2[-A'(\tau)\sin T + B'(\tau)\cos T] \\ &-2[A(\tau)\sin T + B(\tau)\cos T] \end{aligned}$$

requiring

$$A'(\tau) + A(\tau) = 0$$

$$B'(\tau) + B(\tau) = 0$$

to suppress resonant terms. With initial conditions (5.24) the solutions are

$$A(\tau) = \alpha e^{-\tau}$$

$$B(\tau) = \beta e^{-\tau}.$$

(5.22) now becomes

$$u_{1TT} + u_1 = 0$$

$$\begin{aligned} u_1(0,0) &= 0 \\ u_{1T}(0,0) &= \alpha \end{aligned}$$

whose solution is

$$u_1 = C(\tau)\cos T + D(\tau)\sin T$$

$$\begin{aligned} C(0) &= 0 \\ D(0) &= \alpha. \end{aligned}$$

(5.23) becomes

$$\begin{aligned} u_{2TT} + u_2 = {} & \left[2w_2A - A'' - 2A'' - 2D' - 2D\right]\cos T \\ & + [2w_2B - B'' - 2B + 2C' + 2C]\sin T. \end{aligned}$$

To suppress resonant terms we require

$$C' + C = -\left(w_2 - \frac{1}{2} + 1\right)\alpha e^{-\tau}$$

$$D' + D = \left(w_2 - \frac{1}{2} + 1\right)\beta e^{-\tau}.$$

We see here another important principle of perturbation theory. There is no unique way to distribute the original independent variable, t, between the two new variables T and τ. The rule of thumb is to use any arbitrariness to simplify the equations as much as possible. In this case we choose

$$w_2 = -\frac{1}{2}$$

then

$$C(\tau) = 0$$

$$D(\tau) = \alpha e^{-\tau}$$

With this information we may now write our second approximation

$$u(T,\tau,\epsilon) = e^{-\tau}\Big[\alpha\cos t\Big(1-\frac{1}{2}\epsilon^2\Big) + \beta\sin t\Big(1-\frac{1}{2}\epsilon^2\Big) + \epsilon\alpha\sin t\Big(1-\frac{1}{2}\epsilon^2\Big)\Big] \tag{5.25}$$

which is free of t-secular and τ-secular terms according to our design. We may convert (5.25) to polar form by using the equations (5.18) and get

$$u(T,\tau,\epsilon) = e^{-\tau}\Big[\rho\cos\Big[t\Big(1-\frac{1}{2}\epsilon^2\Big)-\psi\Big] + \epsilon\alpha\sin t\Big(1-\frac{1}{2}\epsilon^2\Big)\Big]. \tag{5.26}$$

Adapted from Simmonds and Mann[10], pp. 69-70.

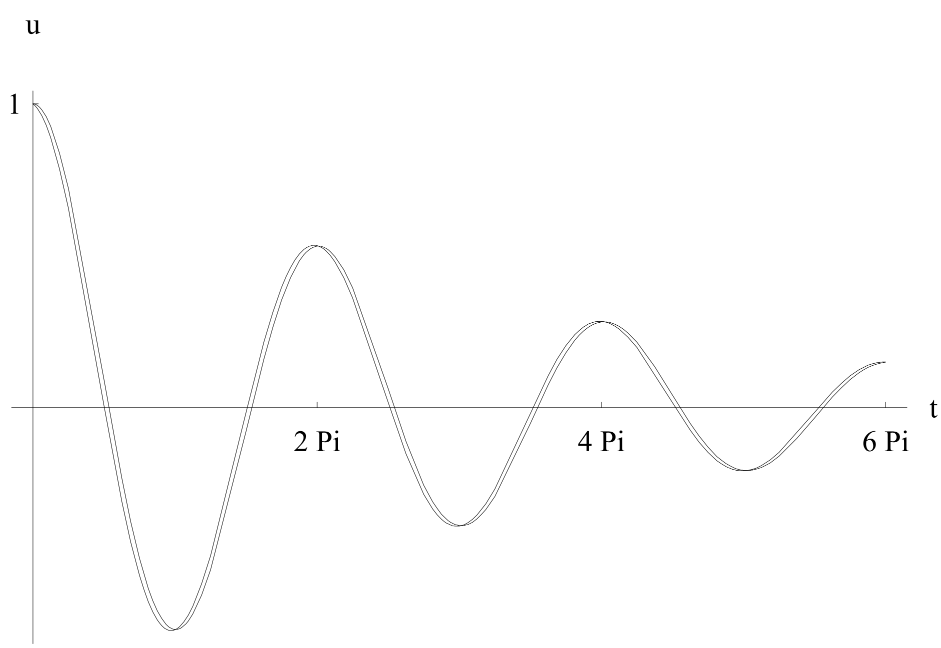

Figure 7.
Exact Linear Solution vs. Two Scales with Strained Time

Multiple Scales. The method of multiple scales has been employed to solve many applied problems and is used extensively in engineering applications. However, it has incomplete mathematical foundations. The following example from Murdock[6], pp. 245-249, is intended to illustrate the method and to allow examination of some of its irregularities. Anyone wishing to learn the method for practical use would have to study many different examples to see how the difficulties are dealt with in each specific situation.

EXAMPLE 5.3: Using the example of the damped linear oscillator, we will now find an approximation of the form

$$u \sim u_0(t,\, \tau,\, \sigma) + \epsilon u_1(t,\, \tau) + \epsilon^2 u_2(t) \tag{5.27}$$

where

$$\tau = \epsilon t \tag{5.28}$$

$$\sigma = \epsilon^2 t.$$

Using the chain rule, (5.27), (5.28), and (5.3) we have the following sequence of PDEs:

$$u_{0tt} + u_0 = 0 \tag{5.29}$$

$$\begin{aligned} u_0(0,0,0) &= \alpha \\ u_{0t}(0,0,0) &= \beta \end{aligned}$$

$$u_{1tt} + u_1 = -2u_{0t\tau} - 2u_{0t} \tag{5.30}$$

$$\begin{aligned} u_1(0,0) &= 0 \\ u_{1t}(0,0) &= -u_{0\tau}(0,0,0) \end{aligned}$$

$$u_{2tt} + u_2 = -2u_{0t\sigma} - u_{0\tau\tau} - 2u_{1t\tau} - 2u_{0\tau} - 2u_{1t} \tag{5.31}$$

$$\begin{aligned} u_2(0) &= 0 \\ u_{2t}(0) &= -u_{0\sigma}(0,0,0) - u_{1\tau}(0,0). \end{aligned}$$

The solution to (5.29) is written

$$u_0 = A(\tau,\sigma)\cos t + B(\tau,\sigma)\sin t$$

$$\begin{aligned} A(0,0) &= \alpha \\ B(0,0) &= \beta. \end{aligned}$$

(5.30) becomes

$$u_{1tt} + u_1 = -2[-A_\tau(\tau,\sigma)\sin t + B_\tau(\tau,\sigma)\cos t]$$
$$-2[-A(\tau,\sigma)\sin t + B(\tau,\sigma)\cos t].$$

Here we see an apparent contradiction in that $u_1(t,\tau)$ appears to be a function of t, τ, and σ. This problem is resolved, however, when we realize that the right side of (5.30) is set equal to zero to eliminate secular terms. Thus, the apparent dependence upon σ is eliminated along with the forcing term. Requiring that there be no t-secular terms in the final solution means that all t-resonant terms must vanish. Therefore

$$A_\tau(\tau,\sigma) = -A(\tau,\sigma)$$
$$B_\tau(\tau,\sigma) = -B(\tau,\sigma)$$

which leads to

$$A(\tau,\sigma) = K(\sigma)e^{-\tau}$$
$$B(\tau,\sigma) = G(\sigma)e^{-\tau}$$

(5.32) $$K(0) = \alpha$$
$$G(0) = \beta.$$

Now we see a big difference between this method and the two scale method. At this point in the latter u_0 would be completely specified, while in the present method we see a dependence upon σ which is not yet determined. Returning to (5.30) with all resonant terms eliminated we have

$$u_{1tt} + u_1 = 0$$
$$u_1(0,0) = 0$$
$$u_{1t}(0,0) = \alpha$$

whose solution is

(5.33) $$u_1 = C(\tau)\cos t + D(\tau)\sin t$$
$$C(0) = 0$$
$$D(0) = \alpha.$$

Substituting these results into (5.31) we find

$$u_{2tt} + u_2 = [-2B_\sigma - A_{\tau\tau} - 2A_\tau - 2D_\tau - 2D]\cos t + [+2A_\sigma - B_{\tau\tau} - 2B_\tau + 2C_\tau + 2C]\sin t \tag{5.34}$$

where we require that the resonant terms vanish to suppress τ-secular terms in the final solution. Why do we want to get rid of τ-secular terms? The aim of a three scale method is to achieve uniformity on an expanding interval of $O(\epsilon^{-2})$. On an interval of this length, a τ-secular term will produce the same kind of problems that a t-secular term will produce on an interval of $O(\epsilon^{-1})$. We will follow the rule that $K(\sigma)$ and $G(\sigma)$ should be chosen so that $C(\tau)$ and $D(\tau)$ contain no τ-secular terms. Suppressing t-resonant terms in (5.34) we require that

$$C_\tau(\tau) + C(\tau) = \Big(-K'(\sigma) - \frac{1}{2}G(\sigma)\Big)e^{-\tau} \tag{5.35}$$

$$D_\tau(\tau) + D(\tau) = \Big(-G'(\sigma) + \frac{1}{2}K(\sigma)\Big)e^{-\tau}.$$

K and G are functions of σ alone and may be treated as constants when solving (5.35) for C and D as functions of τ. It is apparent that the forcing terms in (5.35) proportional to $e^{-\tau}$ produce a response proportional to $\tau e^{-\tau}$. In order to avoid these terms it is necessary to take

$$K'(\sigma) = -\frac{1}{2}G(\sigma) \tag{5.36}$$

$$G'(\sigma) = \frac{1}{2}K(\sigma).$$

The solutions of (5.36) with initial conditions (5.32) are

$$K(\sigma) = \alpha\cos\frac{\sigma}{2} - \beta\sin\frac{\sigma}{2}$$

$$G(\sigma) = \alpha\sin\frac{\sigma}{2} + \beta\cos\frac{\sigma}{2}.$$

Finally, u_0 is completely determined

$$u_0(t,\tau,\sigma) = \Big[\alpha\cos\frac{\sigma}{2} - \beta\sin\frac{\sigma}{2}\Big]e^{-\tau}\cos t + \Big[\alpha\sin\frac{\sigma}{2} + \beta\cos\frac{\sigma}{2}\Big]e^{-\tau}\sin t. \tag{5.37}$$

Using

(5.18)
$$\alpha = \rho \cos \psi$$

$$\beta = \rho \sin \psi$$

$$\rho = \sqrt{\alpha^2 + \beta^2}$$

$$\tan(\psi) = \beta/\alpha.$$

we may convert (5.37) to its polar form and get

$$u_0(t, \tau, \sigma) = e^{-\tau} \rho \cos\Big(t - \frac{\sigma}{2} - \psi\Big).$$

With the information from (5.35) and (5.36) we see that

$$C_\tau(\tau) = -C(\tau)$$

$$D_\tau(\tau) = -D(\tau).$$

With (5.33) these are solved as

$$C(\tau) = 0$$

$$D(\tau) = \alpha e^{-\tau}$$

and

$$u_1(t, \tau) = \alpha e^{-\tau} \sin t.$$

Since the entire right hand side of (5.31) was eliminated by our choice of terms it becomes

$$u_{2tt} + u_2 = 0$$

$$u_2(0) = 0$$
$$u_{2t}(0) = \frac{\beta}{2}$$

and its solution is

$$u_2(t) = E \cos t + F \sin t$$

where the initial conditions require that

$$E = 0$$

$$F = \frac{\beta}{2}$$

so that

$$u_2(t) = \frac{\beta}{2}\sin t.$$

The entire three term approximation may be given in the form

$$u(t,\tau,\sigma;\epsilon) = e^{-\tau}\left[\rho\cos\left(t - \frac{\sigma}{2} - \psi\right) + \epsilon\alpha\sin t + \epsilon^2\frac{\beta}{2}\sin t\right].$$

If we take $\sigma = 1$ we can see that this expansion is the same first order expansion as (5.26). The above example was adapted from Murdock[6], pp. 245-249 and Nayfeh[8], pp. 142-144.

The fundamental theorem does not hold for generalized asymptotic expansions so the theoretical foundations of the multiple scale method are very weak. Nevertheless it does provide correct and interesting results in many cases. In this example, for instance, we can see that the second scale, τ, "automatically" corrects the solution for the damping and the third scale, σ, "automatically" corrects for the frequency shift.