Review of First and Second Order Linear ODEs

1) Consider the linear first order ODE with arbitrary coefficients:

$$y' + a(x)y = b(x)$$

Its complete solution (homogeneous and particular) is simply

$$y = e^{-\int a\,\mathrm{d}x}\left[C_1 + \int b e^{+\int a\,\mathrm{d}x}\mathrm{d}x\right]$$

2) Consider the second order linear ODE with constant coefficients:

$$y'' + ay' + by = 0$$

Assuming a general solution of the form $y = \exp(\lambda x)$, the resulting characteristic or auxiliary equation for the argument λ is the quadratic polynomial

$$\lambda^2 + a\lambda + b = 0$$

with roots

$$\lambda_1 = \tfrac{1}{2}(-a + \sqrt{a^2 - 4b}) \qquad \lambda_2 = \tfrac{1}{2}(-a - \sqrt{a^2 - 4b})$$

Three general cases can occur and these are summarized below.

Case	Roots	General solution
I: two distinct real	λ_1 , λ_2 given above	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
II: double real	$\lambda_{1,2} = -\tfrac{1}{2}a$, $a = \pm 2\sqrt{b}$	$y = (C_1 + C_2 x)e^{-ax/2}$
III: complex conjugate	$\lambda_{1,2} = -\tfrac{1}{2}a \pm i\omega$, $\omega = \tfrac{1}{2}\sqrt{4b - a^2}$	$y = (C_1 \cos\omega x + C_2 \sin\omega x)e^{-ax/2}$