2.5 Modeling: Free Oscillations (Mass–Spring System)

Homogeneous linear differential equations with constant coefficients have basic engineering applications. In this section we consider an important application from mechanics (a vibrating mass on an elastic spring). We model the system (i.e., set up its mathematical equation), solve it, and discuss the types of motion, which—interestingly enough—will correspond to Cases I–III in Secs. 2.2 and 2.3. Incidentally, our mechanical system has a complete analog in electric circuits, as we shall discover later (in Sec. 2.12).

Setting up the Model

We take an ordinary spring that resists compression as well as extension and suspend it vertically from a fixed support (Fig. 32). At the lower end of the spring we attach a body of mass m. We assume m to be so large that we may disregard the mass of the spring. If we pull the body down a certain distance and then release it, it undergoes a motion. We assume that the body moves strictly vertically.

We want to determine the motion of our mechanical system. For this purpose we consider the forces[6] acting on the body during the motion. This will lead to a differential equation, whose solution $y(t)$ will give the displacement of the mass as a function of time t.

We choose the *downward direction* as the positive direction and thus regard downward forces as positive and upward forces as negative.

The most obvious force acting on the body is the *attraction of gravity*

$$F_1 = mg \tag{1}$$

where m is the mass of the body and g (= 980 cm/sec^2) is the acceleration of gravity.

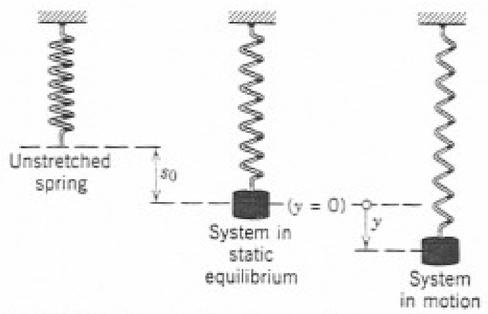

Fig. 32. Mechanical system under consideration

[6] For **systems of units** and conversion factors, see inside of front cover.

We next consider the *spring force* F_2 acting on the body. Experiments show that within reasonable limits, its magnitude is proportional to the change in the length of the spring. Its direction is upward if the spring is extended and downward if the spring is compressed. Thus,

$$F_2 = -ks \qquad \textbf{(Hooke's[7] law)}, \tag{2}$$

where s is the vertical displacement of the body (recall that the upper end of the spring is fixed), the constant of proportionality k is called the *spring modulus*, and the minus sign makes F_2 negative (upward) for positive s (extension of the spring) and positive (downward) for negative s (compression of the spring).

If $s = 1$, then $F_2 = -k$. The stiffer the spring, the larger k.

When the body is at rest (motionless), gravitational force and spring force are in equilibrium, their resultant is the zero force,

$$F_1 + F_2 = mg - ks_0 = 0 \tag{3}$$

where s_0 is the extension of the spring corresponding to this position, which is called the *static equilibrium position.*

We denote by $y = y(t)$ [t time] the displacement of the body from the static equilibrium position ($y = 0$), with the positive direction downward (Fig. 32). This displacement causes an additional force $-ky$ on the body, by Hooke's law. Hence the resultant of the forces on the body at position $y(t)$ is [see (3)]

$$F_1 + F_2 - ky = -ky. \tag{4}$$

Undamped System: Equation and Solution

If the damping of the system is so small that it can be disregarded, then (4) is the resultant of *all* the forces acting on the body. The differential equation will now be obtained by the use of **Newton's second law**

$$\text{Mass} \times \text{Acceleration} = \text{Force}$$

where *force* means the resultant of the forces acting on the body at any instant. In our case, the acceleration is $y'' = d^2y/dt^2$ and that resultant is given by (4). Thus

$$my'' = -ky.$$

Hence the motion of our system is governed by the linear differential equation with constant coefficients

$$\boxed{my'' + ky = 0.} \tag{5}$$

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[7] ROBERT HOOKE (1635—1703), English physicist, a forerunner of Newton with respect to the law of gravitation.

By the method in Sec. 2.3 (see Example 3) we get the general solution

$$\boxed{y(t) = A\cos\omega_0 t + B\sin\omega_0 t} \qquad \omega_0 = \sqrt{k/m}. \tag{6}$$

The corresponding motion is called a **harmonic oscillation.** Figure 33 shows typical forms of (6) corresponding to some positive initial displacement $y(0)$ [which determines $A = y(0)$ in (6)] and different initial velocities $y'(0)$ [each of which determines a value of B in (6), since $y'(0) = \omega_0 B$].

By applying the addition formula for the cosine, the student may verify that (6) can be written [see also (13) in Appendix 3]

$$y(t) = C\cos(\omega_0 t - \delta) \qquad \left(C = \sqrt{A^2 + B^2}, \quad \tan\delta = \frac{B}{A}\right). \tag{6*}$$

Since the period of the trigonometric functions in (6) is $2\pi/\omega_0$, the body executes $\omega_0/2\pi$ cycles per second. The quantity $\omega_0/2\pi$ is called the **frequency** of the oscillation and is measured in cycles per second. Another name for cycles/sec is hertz (Hz).[8]

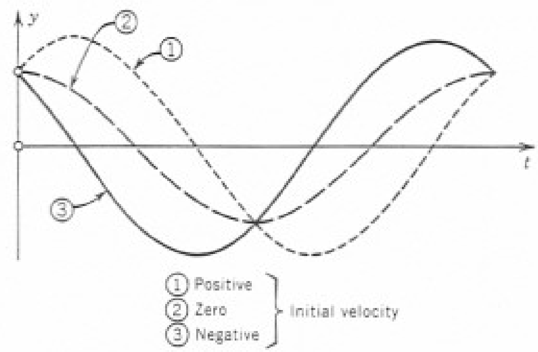

Fig. 33. Harmonic oscillations

EXAMPLE 1 Undamped system. Harmonic oscillations

If an iron ball of weight $W = 89.00$ nt (about 20 lb) stretches a spring 10.00 cm (about 4 inches), how many cycles per minute will this mass-spring system execute? What will its motion be if we pull down the weight an additional 15.00 cm (about 6 inches)?

Solution. From (2) we obtain the value $k = 89.00/0.1000 = 890.0$ [nt/meter]. The mass is $m = W/g = 89.00/9.8000 = 9.082$ [kg]. This gives the frequency

$$\omega_0/2\pi = \sqrt{890.0/9.082}/2\pi = 9.899/2\pi = 1.576 \text{ Hz}$$

or 94.5 cycles per minute. From (6) and the initial conditions, $y(0) = A = 0.1500$ [meter] and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.1500\cos 9.899t \text{ [meters]} \qquad \text{or} \qquad 0.492\cos 9.899t \text{ [ft]}.$$

If you have a chance of experimenting with a mass-spring system, don't miss it. You will be surprised about the good agreement between theory and experiment, usually within a fraction of one percent if you measure carefully. ∎

[8]HEINRICH HERTZ (1857—1894), German physicist, who discovered electromagnetic waves and made important contributions to electrodynamics.

Damped System: Equation and Solutions

If we connect the mass to a dashpot (Fig. 34), then we have to take the corresponding viscous damping into account. The corresponding damping force has the direction opposite to the instantaneous motion, and we assume that it is proportional to the velocity $y' = dy/dt$ of the body. This is generally a good approximation, at least for small velocities. Thus the damping force is of the form

$$F_3 = -cy'.$$

Let us show that the *damping constant* c is positive. If y' is positive, the body moves downward (in the positive y-direction) and $-cy'$ must be an upward force, that is, by agreement, $-cy' < 0$, which implies $c > 0$. For negative y' the body moves upward and $-cy'$ must represent a downward force, that is, $-cy' > 0$, which implies $c > 0$.

The resultant of the forces acting on the body is now [see (4)]

$$F_1 + F_2 + F_3 = -ky - cy'.$$

Hence, by Newton's second law,

$$my'' = -ky - cy',$$

and we see that the motion of the damped mechanical system is governed by the linear differential equation with constant coefficients

$$\boxed{my'' + cy' + ky = 0.} \tag{7}$$

The corresponding characteristic equation is

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

The roots are

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

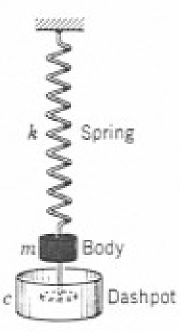

Fig. 34. Damped system

Using the short notations

$$\alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}, \tag{8}$$

we can write

$$\lambda_1 = -\alpha + \beta \quad \text{and} \quad \lambda_2 = -\alpha - \beta.$$

The form of the solution of (7) will depend on the damping, and, as in Secs. 2.2 and 2.3, we now have the following three cases:

Case I.	$c^2 > 4mk$.	*Distinct real roots* λ_1, λ_2.	***(Overdamping)***
Case II.	$c^2 = 4mk$.	*A real double root.*	***(Critical damping)***
Case III.	$c^2 < 4mk$.	*Complex conjugate roots.*	***(Underdamping)***

Let us discuss these three cases separately.

Case I. Overdamping

If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are distinct real roots, and the general solution of (7) is

$$\boxed{y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.} \tag{9}$$

We see that in this case the body does not oscillate. For $t > 0$ both exponents in (9) are negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (9) approach zero as t approaches infinity. Practically speaking, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). This is understandable since the damping takes energy from the system and there is no external force that keeps the motion going. Figure 35 shows (9) for some typical initial conditions.

Case II. Critical damping

If $c^2 = 4mk$, then $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$, and the general solution is

$$\boxed{y(t) = (c_1 + c_2 t)e^{-\alpha t}.} \tag{10}$$

Since the exponential function is never zero and $c_1 + c_2 t$ can have at most one positive zero, it follows that the motion can have at most one passage through the equilibrium position ($y = 0$). If the initial conditions are such that c_1 and c_2 have the same sign, there is no such passage at all. This is similar to Case I. Figure 36 shows typical forms of (10).

Case II marks the border between nonoscillatory motions and oscillations; this explains its name.

Case III. Underdamping

This is the most interesting case. If the damping constant c is so small that $c^2 < 4mk$, then β in (8) is pure imaginary, say,

$4mk$.

$- \beta$.

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(Underdamping)

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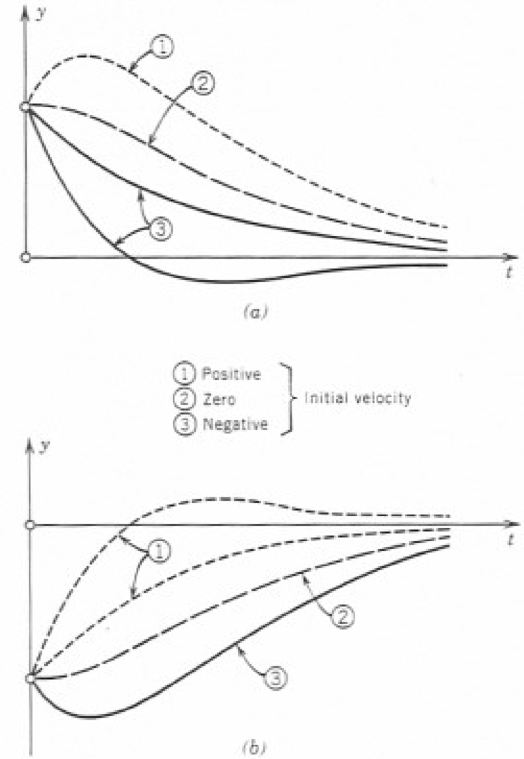

Fig. 35. Typical motions in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

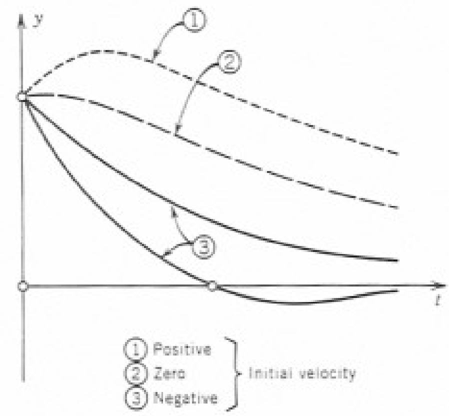

Fig. 36. Critical damping

$$\text{(11)} \quad \beta = i\omega^* \quad \text{where} \quad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \quad (> 0).$$

(We write ω^* to reserve ω for Sec. 2.11.) The roots of the characteristic equation are complex conjugate,

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^*$$

9. **(Pendulum)** Determine the frequency of oscillation of the pendulum of length L in Fig. 39. Neglect air resistance and the weight of the rod. Assume that θ is small enough that $\sin\theta \approx \theta$.

10. A clock has a 1-meter pendulum. The clock ticks once for each time the pendulum completes a swing, returning to its original position. How many times a minute does the clock tick?

11. Suppose that the system in Fig. 40 consists of a pendulum as in Prob. 9 and two springs with constants k_1 and k_2 attached to the vibrating body and two vertical walls such that $\theta = 0$ remains the position of static equilibrium and $\theta(t)$ remains small during the motion. Find the period T.

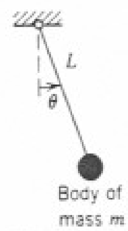

Fig. 39. Pendulum

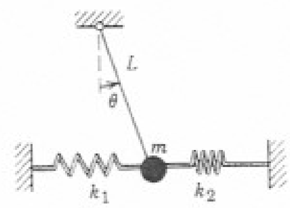

Fig. 40. Problem 11

12. **(Flat spring)** Our present equation $my'' + k_0 y = 0$ also governs the (undamped) vibrations of a body attached to a flat spring (of negligible mass) whose other end is horizontally clamped (Fig. 41); here k_0 is the spring constant in Hooke's law $F = -k_0 s$. What is the motion if the body weighs 4 nt (about 0.9 lb), the system has its equilibrium 1 cm below the horizontal line, and we let it start from this position with downward initial velocity 20 cm/sec? When will the body reach its highest position for the first time?

13. **(Torsional vibrations)** Undamped torsional vibrations (rotations back and forth) of a wheel attached to an elastic thin rod or wire (Fig. 42) are governed by the equation

$$I_0\theta'' + K\theta = 0,$$

where θ is the angle measured from the state of equilibrium, I_0 the polar moment of inertia of the wheel about its center, and K the torsional stiffness of the rod. Solve the equation for $K/I_0 = 13.69\ \text{sec}^{-2}$, initial angle $15°$ ($= 0.2618$ rad) and initial angular velocity $10°\ \text{sec}^{-1}$ ($= 0.1745\ \text{rad}\cdot\text{sec}^{-1}$).

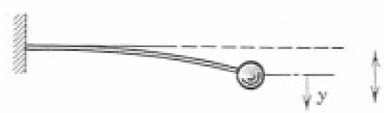

Fig. 41. Problem 12
(Flat spring)

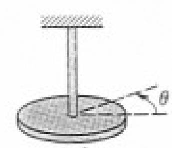

Fig. 42. Problem 13
(Torsional vibrations)

14. **(Damping)** Determine the motion $y(t)$ of the mechanical system described by (7) corresponding to initial displacement 1, initial velocity zero, mass 1, spring modulus 1, and various values of the damping constant, say, $c = 0, 1, 2, 10$.

Case I. Overdamped motion

15. Show that for (9) to satisfy initial conditions $y(0) = y_0$ and $v(0) = v_0$ we must have $c_1 = [(1 + \alpha/\beta)y_0 + v_0/\beta]/2$ and $c_2 = [(1 - \alpha/\beta)y_0 - v_0/\beta]/2$.

16. Show that in the overdamped case, the body can pass through $y = 0$ at most once (Fig. 35).

17. In Prob. 16 find conditions for c_1 and c_2 such that the body does not pass through $y = 0$ at all.

18. Show that an overdamped motion with zero initial displacement cannot pass through $y = 0$.

Case II. Critical damping

19. Find the critical motion (10) that starts from y_0 with initial velocity v_0.

20. Under what conditions does (10) have a maximum or minimum at some instant $t > 0$?

21. Represent the maximum or minimum amplitude in Prob. 20 in terms of the initial values y_0 and v_0.

Case III. Underdamped motion (Damped oscillation)

22. Find and graph the three damped oscillations of the form

$$y = e^{-t}(A \cos t + B \sin t) = Ce^{-t} \cos (t - \delta)$$

starting from $y = 1$ with initial velocity $-1, 0, 1$, respectively.

23. Show that the damped oscillation satisfying the initial conditions $y(0) = y_0$, $v(0) = v_0$ is

$$y = e^{-\alpha t}[y_0 \cos \omega^* t + \omega^{*-1}(v_0 + \alpha y_0) \sin \omega^* t].$$

24. Show that the frequency $\omega^*/2\pi$ of the underdamped motion decreases as the damping increases.

25. Show that for small damping, $\omega^* \approx \omega_0 [1 - (c^2/8mk)]$.

26. For what c (in terms of m and k) is $\omega^*/\omega_0 = 99\%$? 95%? Calculate (a) exactly, (b) by the formula in Prob. 25.

27. Determine the values of t corresponding to the maxima and minima of the oscillation $y(t) = e^{-t} \sin t$. Check your result by graphing $y(t)$.

28. Show that the maxima and minima of an underdamped motion occur at equidistant values of t, the distance between two consecutive maxima being $2\pi/\omega^*$.

29. Consider an underdamped motion of a body of mass $m = 2$ kg. If the time between two consecutive maxima is 3 sec and the maximum amplitude decreases to $\frac{1}{2}$ its initial value after 20 cycles, what is the damping constant of the system?

30. **(Logarithmic decrement)** Prove that the ratio of two consecutive maximum amplitudes of a damped oscillation (12) is constant, the natural logarithm of this ratio being $\Delta = 2\pi\alpha/\omega^*$. ($\Delta$ is called the *logarithmic decrement* of the oscillation.) Find Δ in the case of $y = e^{-t} \cos t$ and determine the values of t corresponding to the maxima and minima.