AUBURN UNIVERSITY

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COLLEGE OF ENGINEERING

AEROSPACE

AERO 4970/7970 **Advanced Perturbation Methods** **SET II**

Advanced WKB Analysis

1. Using the simplified test function $F = \frac{1}{2}\pi r^2$, apply the WKB Method to solve the following second order linear ODE over the interval $0 \le r \le 1$:

$$\varepsilon \frac{\mathrm{d}^2 R_n}{\mathrm{d}r^2} + \frac{(\varepsilon + F)}{r}\frac{\mathrm{d}R_n}{\mathrm{d}r} + \left[\mathrm{i}S - \frac{2(n+1)}{r}F'\right]R_n = 0; \quad \begin{cases} R_n(1) = 1 \\ R_n'(0) = 0 \end{cases}$$

   Here $\mathrm{i} = \sqrt{-1}$, ε is the primary perturbation parameter, S is the Strouhal number (usualy a large quantity of orders varying between $\varepsilon^{-1/2}$ and ε^{-1}), and n is a natural integer. With δ as the gauge parameter in your expansion, $R_n(r) = \exp\left[\delta^{-1}Z_0(r) + Z_1(r) + \delta Z_2(r) + ...\right]$, find the first three terms of the WKB solution corresponding to the distinguished limit $\delta \sim \sqrt{\varepsilon} \sim S^{-1}$.

2. Redo Problem 1 with $\delta \sim \varepsilon \sim S^{-1}$. The quadratic Eikonal equation will yield two roots. Retain only the root leading to $Z_{01} = -\int_1^r \frac{1}{2x}\left[F(x) - \sqrt{F(x)^2 - 4iS\varepsilon x^2}\right]\mathrm{d}x$; discard the other (unphysical) root $Z_{02} = -\int_1^r \frac{1}{2x}\left[F(x) + \sqrt{F(x)^2 - 4iS\varepsilon x^2}\right]\mathrm{d}x$.

3. For Problem 1, examine the behavior of the solution at the origin by evaluating the limit $\lim_{r\to 0} R_n$ at both $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$.

4. For Problem 2, examine the behavior of the solution at the origin by evaluating the limit $\lim_{r\to 0} R_n$ at both $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$.