CHAPTER 4

Laplace integrals

A *Laplace integral* has the form

$$I(x) = \int_a^b f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t \tag{4.1}$$

where we assume $x > 0$. Typically x is a large parameter and we are interested in the asymptotic behaviour of $I(x)$ as $x \to +\infty$. Integrating by parts gives

$$\begin{aligned} I(x) &= \frac{1}{x}\int_a^b \frac{f(t)}{\phi'(t)} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{x\phi(t)}\right)\, \mathrm{d}t \\ &= \left[\frac{1}{x} \cdot \frac{f(t)}{\phi'(t)} \cdot \mathrm{e}^{x\phi(t)}\right]_a^b - \frac{1}{x}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{\phi'(t)}\right) \cdot \mathrm{e}^{x\phi(t)}\, \mathrm{d}t. \end{aligned} \tag{4.2}$$

If the 2nd integral term is asymptotically smaller than the boundary term, i.e.

$$\text{integral term} = \mathrm{o}(\text{boundary term}) \quad \text{as} \quad x \to +\infty,$$

then

$$I(x) \sim \frac{1}{x} \cdot \frac{f(b)}{\phi'(b)} \cdot \mathrm{e}^{x\phi(b)} - \frac{1}{x} \cdot \frac{f(a)}{\phi'(a)} \cdot \mathrm{e}^{x\phi(a)} \quad \text{as} \quad x \to +\infty \tag{4.3}$$

and we have a useful asymptotic expansion for $I(x)$ as $x \to +\infty$. In general, this will in fact be the case, i.e. (4.3) is valid, if $\phi(t)$, $\phi'(t)$ and $f(t)$ are continuous functions (possibly complex) and one of the following three conditions is satisfied:

(1) $\phi'(t) \neq 0$ for $a \leq t \leq b$ and either $f(a) \neq 0$ or $f(b) \neq 0$. These conditions ensure that the integral term in (4.2) is bounded and is asymptotically smaller than the boundary term;

(2) $\mathrm{Re}\ \phi(t) \leq \mathrm{Re}\ \phi(b)$ for $a \leq t \leq b$, $\mathrm{Re}\ \phi'(b) \neq 0$ and $f(b) \neq 0$. These conditions do *not* ensure that the integral term in (4.2) is bounded, but by *Laplace's method* (see below) they ensure

$$I(x) \sim \frac{1}{x} \cdot \frac{f(b)}{\phi'(b)} \cdot \mathrm{e}^{x\phi(b)} \quad \text{as} \quad x \to +\infty;$$

(3) $\mathrm{Re}\ \phi(t) \leq \mathrm{Re}\ \phi(a)$ for $a \leq t \leq b$, $\mathrm{Re}\ \phi'(a) \neq 0$ and $f(a) \neq 0$. Similar to the last case, the integral term in (4.2) is not necessarily bounded, but by Laplace's method we can ensure

$$I(x) \sim -\frac{1}{x} \cdot \frac{f(a)}{\phi'(a)} \cdot \mathrm{e}^{x\phi(a)} \quad \text{as} \quad x \to +\infty.$$

Further, if any one of these conditions is met then we may also continue to integrate by parts to generate further terms in the asymptotic expansion of $I(x)$; each integration by parts generates a new factor of $1/x$.

4.1. Laplace's Method

We have seen that for Laplace integrals, integration by parts fails for example, when $\phi'(t)$ has a zero somewhere in $a \le t \le b$. *Laplace's method* is a general technique that allows us to generate an asymptotic expansion for Laplace integrals for large x in such cases. Recall

$$I(x) = \int_a^b f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t$$

where we now suppose $f(t)$ and $\phi(t)$ are real, continuous functions.

Basic idea. If $\phi(t)$ has a (global) maximum at $t = c$ with $a \le c \le b$ and if $f(c) \ne 0$, then it is only the neighbourhood of $t = c$ that contributes to the full asymptotic expansion of $I(x)$ as $x \to +\infty$. This means that:

Step 1. We may approximate $I(x)$ by $I(x;\epsilon)$ where

$$I(x;\epsilon) = \begin{cases} \int_{c-\epsilon}^{c+\epsilon} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t, & \text{if } a < c < b, \\ \\ \int_{a}^{a+\epsilon} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t, & \text{if } c = a, \\ \\ \int_{b-\epsilon}^{b} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t, & \text{if } c = b, \end{cases} \tag{4.4}$$

where $\epsilon > 0$ is *arbitrary*, but sufficiently small to guarantee that each of the subranges of integration indicated are contained in the interval $[a, b]$. Such a step is valid if the asymptotic expansion of $I(x;\epsilon)$ as $x \to +\infty$ does not depend on ϵ *and* is identical to the asymptotic expansion of $I(x)$ as $x \to +\infty$. Both of these results are in fact true since (eg. when $a < c < b$)

$$\left| \int_a^{c-\epsilon} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t \right| + \left| \int_{c+\epsilon}^{b} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t \right|$$

is subdominant to $I(x)$ as $x \to +\infty$. This is because $\mathrm{e}^{x\phi(t)}$ is exponentially small compared to $\mathrm{e}^{x\phi(c)}$ for $a \le t \le c - \epsilon$ and $c + \epsilon \le t \le b$. In other words, changing the limits of integration only introduces *exponentially small errors* (all this can be rigorously proved by integrating by parts). Hence we simply replace $I(x)$ by the truncated integral, $I(x;\epsilon)$.

Step 2. Now $\epsilon > 0$ can be chosen small enough so that (now confined to the narrow region surrounding $t = c$) it is valid to replace $\phi(t)$ by the first few terms in its Taylor series expansion.

- If $\phi'(c) = 0$ with $a \le c \le b$ and $\phi''(c) \ne 0$, approximate $\phi(t)$ by
$$\phi(t) \approx \phi(c) + \frac{1}{2}\phi''(c) \cdot (t - c)^2.$$
- If $c = a$ or $c = b$ and $\phi'(c) \ne 0$, approximate $\phi(t)$ by
$$\phi(t) \approx \phi(c) + \phi'(c) \cdot (t - c).$$

In either case approximate $f(t)$ by the leading order term in its expansion about $t = c$,

$$f(t) \approx f(c) \ne 0. \tag{4.5}$$

Step 3. Having substituted the approximations for ϕ and f indicated above, we now extend the endpoints of integration to infinity, in order to evaluate the resulting integrals (again this only introduces exponentially small errors).

- If $\phi'(c) = 0$ with $a < c < b$, we must have $\phi''(c) < 0$ ($t = c$ is a maximum) and so as $x \to +\infty$

$$\begin{aligned} I(x;\epsilon) &\sim \int_{c-\epsilon}^{c+\epsilon} f(c)\mathrm{e}^{x\left(\phi(c)+\frac{1}{2}\phi''(c)\cdot(t-c)^2\right)}\,\mathrm{d}t \\ &\sim f(c)\mathrm{e}^{x\phi(c)}\int_{-\infty}^{\infty}\mathrm{e}^{x\cdot\frac{\phi''(c)}{2}(t-c)^2}\,\mathrm{d}t \\ &= \frac{\sqrt{2}f(c)\mathrm{e}^{x\phi(c)}}{\sqrt{-x\phi''(c)}}\int_{-\infty}^{\infty}\mathrm{e}^{-s^2}\,\mathrm{d}s \end{aligned} \tag{4.6}$$

where in the last step we made the substitution $s^2 = -x\cdot\frac{\phi''(c)}{2}(t-c)^2$. Since $\int_{-\infty}^{\infty}\mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}$, we get

$$I(x;\epsilon) \sim \frac{\sqrt{2\pi}f(c)\mathrm{e}^{x\phi(c)}}{\sqrt{-x\phi''(c)}} \quad \text{as} \quad x \to +\infty. \tag{4.7}$$

If $c = a$ or $c = b$, then the leading order behaviour for $I(x)$ is the same as that in (4.7), except multiplied by a factor $\frac{1}{2}$—when we extend the limits of integration, we only do so in one direction, so that the integral in (4.6) only extends over a semi-infinite range.

- If $c = a$ and $\phi'(c) \neq 0$, we must have $\phi'(c) < 0$, and as $x \to +\infty$,

$$\begin{aligned} I(x;\epsilon) &\sim \int_{a}^{a+\epsilon} f(a)\mathrm{e}^{x(\phi(a)+\phi'(a)\cdot(t-a))}\,\mathrm{d}t \\ &\sim f(a)\mathrm{e}^{x\phi(a)}\int_{0}^{\infty}\mathrm{e}^{x\cdot\phi'(c)(t-c)}\,\mathrm{d}t \\ &= -\frac{f(a)\mathrm{e}^{x\phi(a)}}{x\phi'(a)} \end{aligned}$$

A similar argument for the case $c = b$, for which $\phi'(b) > 0$, gives

$$I(x) \sim \frac{f(b)\mathrm{e}^{x\phi(b)}}{x\phi'(b)} \quad \text{as} \quad x \to +\infty.$$

Note. If $\phi(t)$ achieves its global maximum at several points in $[a, b]$, decompose the integral $I(x)$ into several intervals, each containing a single maximum. Perform the analysis above and compare the contributions to the asymptotic behaviour of $I(x)$ (which will be additive) from each subinterval. The final ordering of the asymptotic expansion will then depend on the behaviour of $f(t)$ at the maximal values of $\phi(t)$. If the maximum is such that $\phi'(c) = \phi''(c) = \cdots = \phi^{(m-1)}(c) = 0$ and $\phi^{(m)}(c) \neq 0$ then use the approximation $\phi(t) \approx \phi(c) + \frac{1}{m!}\phi^{(m)}(c)\cdot(t-c)^m$. In (4.5), we assumed $f(c) \neq 0$—see the beginning of this section—the case when $f(c) = 0$ is more delicate and treated in many books (see the bibliography).

Example (Stirling's formula). We shall try to find the leading order behaviour of the (complete) Gamma function

$$\Gamma(x+1) := \int_0^\infty \mathrm{e}^{-t}\, t^x \,\mathrm{d}t \equiv \int_0^\infty \mathrm{e}^{-t+x\ln t}\,\mathrm{d}t \quad \text{as} \quad x \to +\infty.$$

First we try to convert it more readily to the standard Laplace integral form by making the substitution, $t = xr$, (this really has the effect of creating the maximum of ϕ away from the origin)

$$\Gamma(x+1) = \int_0^\infty \mathrm{e}^{-xr+x\ln x+x\ln r}\, x \,\mathrm{d}r = x^{x+1}\int_0^\infty \mathrm{e}^{x(-r+\ln r)}\,\mathrm{d}r.$$

Hence $f(r) \equiv 1$ and $\phi(r) = -r + \ln r$. Since $\phi'(r) = -1 + \frac{1}{r}$ and $\phi''(r) = -\frac{1}{r^2}$, for all $r > 0$, we conclude that ϕ has a global (& local) maximum at $r = 1$. Hence after collapsing the range of integration to a narrow region surrounding $r = 1$, we approximate

$$\phi(r) \approx \phi(1) + \frac{\phi''(1)}{2}\cdot(r-1)^2 = -1 - \frac{1}{2}\cdot(r-1)^2$$

Subsequently extending the range of integration out to infinity again we see that

$$\Gamma(x+1) \sim x^{x+1}\int_{-\infty}^\infty \mathrm{e}^{x}\cdot \mathrm{e}^{-x(r-1)^2}\,\mathrm{d}r.$$

Making the subsitution $s^2 = \frac{x}{2}\cdot(r-1)^2$ and using that $\int_{-\infty}^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}$, then reveals

$$\Gamma(x+1) \sim \sqrt{2\pi x}\cdot x^x \cdot \mathrm{e}^{-x} \quad \text{as} \quad x \to +\infty.$$

When $x \in \mathbb{N}$, this is *Stirling's formula* for the asymptotic behaviour of the factorial function for large integers.

4.2. Watson's lemma

Based on the ideas above, we can prove a simpler result.

WATSON'S LEMMA. *Consider the following example of a Laplace integral*

$$I(x) = \int_0^b f(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t \quad (b > 0).$$

Suppose $f(t)$ *is continuous on* $[0, b]$ *and has the asymptotic expansion*

$$f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n} \quad \text{as} \quad t \to 0+\,.$$

We assume that $\alpha > -1$ *and* $\beta > 0$ *so that the integral is bounded near* $t = 0$*; if* $b = \infty$*, we also require that* $f(t) = \mathrm{o}(\mathrm{e}^{ct})$ *as* $t \to +\infty$ *for some* $c > 0$*, to guarantee the integral is bounded for large* t*. Then*

$$I(x) \sim \sum_{n=0}^\infty \frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad \text{as} \quad x \to +\infty. \tag{4.8}$$

PROOF. Following some of the basic ideas of Laplace's method outlined above:

Step 1. Replace $I(x)$ by $I(x;\epsilon)$ where

$$I(x;\epsilon) = \int_0^\epsilon f(t)\mathrm{e}^{-xt}\,\mathrm{d}t. \tag{4.9}$$

This approximation only introduces exponentially small errors for any $\epsilon > 0$.

Step 2. We can now choose $\epsilon > 0$ small enough so that the first N terms in the asymptotic series for $f(t)$ are a good approximation to $f(t)$, i.e.

$$\left| f(t) - t^\alpha \sum_{n=0}^{N} a_n t^{\beta n} \right| \leq K \cdot t^{\alpha+\beta(N+1)}, \tag{4.10}$$

for $0 \leq t \leq \epsilon$ and $K > 0$ is a constant. Substituting the first N terms in the series for $f(t)$ into (4.9) we see that, using (4.10),

$$\begin{aligned}\left| I(x;\epsilon) - \sum_{n=0}^{N} a_n \int_0^\epsilon t^{\alpha+\beta n}\mathrm{e}^{-xt}\,\mathrm{d}t \right| &\leq K \cdot \int_0^\epsilon t^{\alpha+\beta(N+1)}\mathrm{e}^{-xt}\,\mathrm{d}t \\ &\leq K \cdot \int_0^\infty t^{\alpha+\beta(N+1)}\mathrm{e}^{-xt}\,\mathrm{d}t \\ &= K \cdot \frac{\Gamma(\alpha+\beta+\beta N+1)}{x^{\alpha+\beta+\beta N+1}}.\end{aligned}$$

Step 3. Replace ϵ by ∞ and use the identity

$$\int_0^\infty t^{\alpha+\beta n}\mathrm{e}^{-xt}\,\mathrm{d}t \equiv \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}$$

so that

$$I(x) - \sum_{n=0}^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} = \mathrm{o}(x^{-\alpha-\beta N-1}) \quad \text{as} \quad x \to +\infty.$$

Since this is true for every N, we have proved (4.8) and thus the Lemma. □

Example. To apply Watson's lemma to the modified Bessel function

$$K_0(x) := \int_1^\infty (s^2-1)^{-\frac{1}{2}}\,\mathrm{e}^{-xs}\,\mathrm{d}s,$$

we first substitute $s = t+1$, so the lower endpoint of integration is $t = 0$:

$$K_0(x) = \mathrm{e}^{-x}\int_0^\infty (t^2+2t)^{-\frac{1}{2}}\,\mathrm{e}^{-xt}\,\mathrm{d}t.$$

For $|t| < 2$ the binomial theorem implies

$$(t^2+2t)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}}\left(1+\frac{t}{2}\right)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\left(-\frac{t}{2}\right)^n \cdot \frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}.$$

Watson's lemma then immediately tells us that

$$K_0(x) \sim \mathrm{e}^{-x}\sum_{n=0}^{\infty}(-1)^n \cdot \frac{\left(\Gamma(n+\frac{1}{2})\right)^2}{2^{n+\frac{1}{2}}n!\,\Gamma(\frac{1}{2})x^{n+\frac{1}{2}}} \quad \text{as} \quad x \to +\infty.$$

Note. We can use Watson's lemma to determine the leading order behaviour of more general Laplace integrals such as (4.1), as $x \to +\infty$. Simply make the change of variables $s = -\phi(t)$ in (4.1) so that

$$I(x) = -\int_{-\phi(a)}^{-\phi(b)} F(s)\mathrm{e}^{-xs}\,\mathrm{d}s \quad \text{where} \quad F(s) = -\frac{f\big(\phi^{-1}(-s)\big)}{\phi'\big(\phi^{-1}(-s)\big)}\,.$$

However, if $t = \phi^{-1}(-s)$ is intricately *multi-valued*, then use the more direct version of Laplace's method—for example if $\phi(t)$ has a global maximum at $t = c$.