

The Method of Averaging

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1 Introduction

Remarks: The method leads generally to asymptotic series as opposed to convergent series. It is not restricted to periodic solutions.

Averaging Method. Put the equation

$$\ddot{x} + x = \epsilon f(x, \dot{x})$$

into Lagrange standard form and do the averaging.

Example 11.1

$$\ddot{x} + x = \epsilon(-\dot{x} + x^2).$$

2 The Lagrange standard form

Unperturbed Equation is Linear.

$$\dot{x} = A(t)x + \epsilon g(t, x), \quad x(0) = x_0.$$

Remarks: (1) The procedure is called “introducing co-moving coordinates”. (2) If the unperturbed equation is nonlinear, the variation of parameters techniques still applies. In practice, however, there are usually many technical obstructions while carrying out the procedure

3 Averaging in the Periodic Case

Asymptotic Validity of Averaging Method. Consider equation (11.17)

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon), \quad x(0) = x_0.$$

We assume that $f(t, x)$ is T -periodic in t and we introduce the average

$$f^0(y) = \frac{1}{T} \int_0^T f(t, y) dt.$$

Consider now equation (11.18)

$$\dot{y} = \epsilon f^0(y), \quad y(0) = x_0.$$

Theorem 11.1 Consider the initial value problem 11.7 and 11.8 with $x, y, x_0 \in D \subset \mathbb{R}^n, t \geq 0$. Suppose that

1. f, g and $\partial f / \partial x$ are defined, continuous and bounded by a constant M in $[0, \infty) \times D$;
2. g is Lipschitz-continuous in x for $x \in D$;
3. $f(t, x)$ is T -periodic in t with average $f^0(x)$ where T is a constant independent of ϵ ;
4. $y(t)$ is contained in the interior of D .

Then we have $x(t) - y(t) = O(\epsilon)$ on the time-scale $1/\epsilon$.

Remark on Example 11.1: The estimates are not valid if we start near the saddle point $x = 1/\epsilon, \dot{x} = 0$.

Example 11.3 Consider

$$\ddot{x} + x = \epsilon f(x, \dot{x})$$

and the van der Pol equation

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}.$$

4 Averaging in the General Case

Theorem 11.2 Consider the initial value problem

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon), \quad x(0) = x_0.$$

with $x, x_0 \in D \subset \mathbb{R}^n, t \geq 0$. Assume that

1. f, g and $\partial f/\partial x$ are defined, continuous and bounded by a constant in $[0, \infty) \times D$;
2. g is Lipschitz-continuous in x for $x \in D$;
3. $f(t, x) = \sum_{i=1}^N f_i(t, x)$ with $f_i(t, x)$ being T_i -periodic in t where T_i constants independent of ϵ ;
4. $y(t)$ is the solution of the initial value problem

$$\dot{y} = \epsilon \sum_{i=1}^N \frac{1}{T_i} \int_0^{T_i} f_i(t, y) dt, \quad y(0) = x_0.$$

and $y(t)$ is contained in the interior of D .

Then we have $x(t) - y(t) = O(\epsilon)$ on the time-scale $1/\epsilon$.

Theorem 11.3 Consider the initial value problem

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon), \quad x(0) = x_0.$$

with $x, x_0 \in D \subset R^n, t \geq 0$. Assume that

1. f, g and $\partial f/\partial x$ are defined, continuous and bounded by a constant in $[0, \infty) \times D$;
2. g is Lipschitz-continuous in x for $x \in D$;
3. the average $f^0(x)$ of $f(t, x)$ exists where

$$f^0(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t, x) dt;$$

4. $y(t)$ is the solution of the initial value problem

$$\dot{y} = \epsilon f^0(y), \quad y(0) = x_0.$$

and $y(t)$ is contained in the interior of D .

Then we have $x(t) - y(t) = O(\delta(\epsilon))$ on the time-scale $1/\epsilon$ with

$$\delta(\epsilon) = \sup_{x \in D} \sup_{0 \leq \epsilon t \leq C} \left\| \int_0^t [f(s, x) - f^0(x)] ds \right\|.$$

5 Periodic Solutions

Theorem 11.5 Consider equation (11.48)

$$\dot{x} = \epsilon f(t, x) + \epsilon^2 g(t, x, \epsilon)$$

with $x \in D \subset R^n, t \geq 0$. Suppose that

1. $f, g, \partial f/\partial x, \partial^2 f/\partial x^2$ and $\partial g/\partial x$ are defined, continuous and bounded by a constant M in $[0, \infty) \times D, 0 \leq \epsilon \leq \epsilon_0$;
2. f and g are T -periodic in t .

If p is critical point of the averaged equation

$$\dot{y} = \epsilon f^0(y),$$

with $|\partial f^0(y)/\partial y|_{y=p} \neq 0$, then there exists a T -periodic solution $\phi(t, \epsilon)$ of equation (11.48) which is close to p such that

$$\lim_{\epsilon \rightarrow 0} \phi(t, \epsilon) = p.$$

Theorem 11.6 Consider equation (11.48) and suppose that the conditions of theorem 11.5 have been satisfied. If the eigenvalues of the critical point $y = p$ of the averaged equation have all negative real parts, the corresponding periodic solution $\phi(t, \epsilon)$ of equation (11.48) is asymptotically stable for ϵ sufficiently small. If one of the eigenvalues has positive real part, $\phi(t, \epsilon)$ is unstable.

Example 11.9 (autonomous equations) Van der Pol Equation.