Lecture 1

1 Introduction

Making precise approximations to solve equations is what distinguishes applied mathematicians from pure mathematicians, physicists and engineers. There are two methods for obtaining precise approximations: numerical methods and analytical (asymptotic) methods. These are not in competition but complement each other. Perturbation methods work when some parameter is large or small. Numerical methods work best when all parameters are order one. Agreement between the two methods is reassuring when doing research. Perturbation methods often give more physical insight.

Finding perturbation approximations is more of an art than a science. It is difficult to give rules, only guidelines. Experience is valuable.

2 Algebraic equations

Suppose we want to solve

$$x^2 + \epsilon x - 1 = 0$$

for x, where ϵ is a small parameter. The exact solutions are

$$x = -\frac{\epsilon}{2} \pm \sqrt{1 + \frac{\epsilon^2}{4}},$$

which we can expand using the binomial theorem

$$x = \begin{cases} +1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} - \frac{\epsilon^4}{128} + \cdots \\ -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^4}{128} + \cdots \end{cases}$$

These expansions converge if $|\epsilon| < 2$. More important is that the truncated expansions give a good approximation to the roots when ϵ is small. For example, when $\epsilon = 0.1$

$$\begin{array}{rcll} x & \sim & 1.0 & \text{1 term} \\ & & 0.95 & \text{2 terms} \\ & & 0.95125 & \text{3 terms} \\ & & 0.951249 & \text{4 terms} \\ \text{exact} & = & 0.95124922... & \end{array}$$

Here, we first found the exact solution, then approximated. Usually we need to make the approximation first, and then solve.

2.1 Iterative method

First rearrange the equation so that it is an a form which can form the basis of an iterative process:

$$x = \pm\sqrt{1 - \epsilon x}.$$

Now, if we have an approximation to the root, x_n, say, a better approximation is given by

$$x_{n+1} = \sqrt{1 - \epsilon x_n}.$$

We need a starting point for the iteration: the solution when $\epsilon = 0$, $x_0 = 1$. After one iteration (on the positive root) we have

$$x_1 = \sqrt{1 - \epsilon}.$$

If we expand this as a binomial series we find

$$x_1 = 1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} - \frac{\epsilon^3}{16} + \cdots.$$

We see that this is correct up to ϵ, but the ϵ^2 terms and higher are wrong. Hence we only need keep the first two terms

$$x_1 = 1 - \frac{\epsilon}{2} + \cdots.$$

Using this in the next iteration we have

$$x_2 = \sqrt{1 - \epsilon\left(1 - \frac{\epsilon}{2}\right)},$$

which can again be expanded to give

$$\begin{aligned} x_2 &= 1 - \frac{\epsilon}{2}\left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon^2}{8}\left(1 - \frac{\epsilon}{2}\right)^2 - \frac{\epsilon^3}{16}\left(1 - \frac{\epsilon}{2}\right)^3 + \cdots \\ &= 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \frac{\epsilon^3}{16} + \cdots. \end{aligned}$$

Now the ϵ^2 term is right, but the ϵ^3 term is still wrong. At each iteration more terms are correct, but more and more work is required. We can only check that a term is correct (without the exact solution) by proceeding to one more iteration and seeing if it changes.

2.2 Expansion method

First set $\epsilon = 0$ and find the unperturbed roots $x = \pm 1$ as in the iterative method. Now pose an expansion about on of these roots:

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots.$$

Substitute the expansion into the equation:

$$(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots)^2 + \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots) - 1 = 0.$$

Expanding the first term

$$1 + 2x_1\epsilon + (x_1^2 + 2x_2)\epsilon^2 + (2x_1x_2 + 2x_3)\epsilon^3 + \cdots + \epsilon + \epsilon^2 x_1 + \epsilon^3 x_2 + \cdots - 1 = 0.$$

Now we equate coefficients of powers of ϵ.

At ϵ^0: $\quad 1 - 1 = 0.$

This level is automatically satisfied because we started the expansion with the correct value $x = 1$ at $\epsilon = 0$.

At ϵ^1: $\quad 2x_1 + 1 = 0,\quad$ i.e. $x_1 = -\dfrac{1}{2}$.

At ϵ^2: $\quad x_1^2 + 2x_2 + x_1 = 0,\quad$ i.e. $x_2 = \dfrac{1}{8}$,

where the previously determined value of x_1 is used.

At ϵ^3: $\quad 2x_1x_2 + 2x_3 + x_2 = 0,\quad$ i.e. $x_3 = 0$.

The expansion method is much easier that the iterative method when working to higher orders. However, it is necessary to assume the form of the expansion (in powers of ϵ).

2.3 Singular perturbations

Consider the problem:

$$\epsilon x^2 + x - 1 = 0.$$

When $\epsilon = 0$ there is just one root $x = 1$, but when $\epsilon \neq 0$ there are two roots. This is an example of a **singular perturbation** problem, in which the limit problem $\epsilon = 0$ differs in an important way from the limit $\epsilon \to 0$.

The most interesting problems are often singular. Problems which are not singular are said to be **regular**.

To see what is happening let us look at the exact solutions

$$x = \frac{-1 \pm \sqrt{1 + 4\epsilon}}{2\epsilon},$$

and expand them for small ϵ (convergent if $|\epsilon| < 1/4$). The expansions of the two roots are

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^4 + \cdots \\ -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^4 + \cdots \end{cases}$$

Thus the second root disappears to $x = \infty$ as $\epsilon \to 0$.

We see that to capture the second root we need to start the expansion not with ϵ^0 but with ϵ^{-1}:

$$x = \frac{x_{-1}}{\epsilon} + x_0 + \epsilon x_1 + \cdots.$$

Substituting into the equation gives

$$\epsilon\left(\frac{x_{-1}}{\epsilon}+x_0+\epsilon x_1+\cdots\right)^2+\left(\frac{x_{-1}}{\epsilon}+x_0+\epsilon x_1+\cdots\right)-1=0.$$

Expanding the first term gives

$$\frac{1}{\epsilon}x_{-1}^2+2x_{-1}x_0+\epsilon(2x_{-1}x_1+x_0^2)+\cdots+\frac{1}{\epsilon}x_{-1}+x_0+\epsilon x_1+\cdots-1=0.$$

Comparing coefficients of ϵ^n gives

At ϵ^{-1}: $\quad x_{-1}^2+x_{-1}=0, \quad$ i.e. $x_{-1}=-1$ or 0.

The root $x_{-1}=0$ leads to the regular root, so we consider the singular root $x_{-1}=-1$.

At ϵ^0: $\quad 2x_{-1}x_0+x_0-1=0, \quad$ i.e. $x_0=-1$.

At ϵ^1: $\quad 2x_{-1}x_1+x_0^2+x_1=0, \quad$ i.e. $x_1=1$.

2.3.1 Rescaling the equation

Instead of starting the expansion with ϵ^{-1}, a very useful idea for singular problems is to rescale the variables before making the expansion. If we introduce the rescaling

$$x=\frac{X}{\epsilon}$$

into the originally singular equation we find that the equation for X,

$$X^2+X-\epsilon=0,$$

is regular. Thus the problem of finding the correct starting point for the expansion can be viewed as the problem of finding a suitable rescaling to regularise the singular problem.