# Lecture 2

## 2.4 Finding the right rescaling

1. Systematic approach. First pose a general rescaling with scaling factor $\delta(\epsilon)$:

$$x = \delta X,$$

in which $X$ is strictly of order one as $\epsilon \to 0$. This gives

$$\epsilon \delta^2 X^2 + \delta X - 1 = 0.$$

Then consider the dominant balance in the equation as $\delta$ varies from very small to very large.

(i) $\delta \ll 1$. Then

$$\epsilon \delta^2 X^2 + \delta X - 1 = \text{small} + \text{small} - 1$$

which cannot possibly balance the zero on the right-hand side of the equation. As $\delta$ is gradually increased the first term to break the domination of the $-1$ term is $\delta X$, which comes into play when $\delta = 1$.

(ii) $\delta = 1$. Now the left-hand side is

$$\epsilon \delta^2 X^2 + \delta X - 1 = \text{small} + X - 1$$

This can balance the zero on the right-hand side, and produces the regular root $X = -1+$ small.

(iii) $1 \ll \delta \ll \epsilon^{-1}$. Now the term $\delta X$ dominates the left-hand side

$$\frac{\epsilon \delta^2 X^2 + \delta X - 1}{\delta} = \text{small} + X + \text{small}$$

This can only balance the zero on the right-hand side if $X = 0$, but that violates the restriction that $X$ is strictly of order one. As $\delta$ is further increased the dominance of $\delta X$ is broken when the first term comes into play at $\delta = \epsilon^{-1}$.

(iv) $\delta = \epsilon^{-1}$. Now the left-hand side divided by $\epsilon \delta^2$ is

$$\frac{\epsilon \delta^2 X^2 + \delta X - 1}{\epsilon \delta^2} = X^2 + X + \text{small}$$

This can balance the zero on the right-hand side and gives the singular root $X = -1+$ small. (Note that the solution $X = 0$ is not permitted since $X$ has to be strictly order one).

(v) $\delta \gg \epsilon^{-1}$. Finally, if $\delta$ is larger still then the left-hand side divided by $\epsilon\delta^2$ is dominated by the first term

$$\frac{\epsilon\delta^2 X^2 + \delta X - 1}{\epsilon\delta^2} = X^2 + \text{small} + \text{small}$$

which cannot balance the zero on the right-hand side with $X$ strictly of order one.

2. An alternative method is to compare terms pairwise, which is quicker when there are a small number of terms. To get a sensible answer from equating the left-hand side to zero we need at least two terms to be in balance (sometimes known as a **distinguished limit**). The possible combinations are the first and second terms, the first and third terms, or the second and third terms.

(i) First and second terms in balance. To have $\epsilon x^2$ and $x$ in balance requires $x$ to be of size $\epsilon^{-1}$. Then these terms are both of size $\epsilon^{-1}$, and dominate the remaining term $-1$, which is of size one. This leads to the singular root.

(ii) First and third terms in balance. To have $\epsilon x^2$ and $-1$ in balance requires $x$ to be of size $\epsilon^{-1/2}$. Then these terms are both of size one, but the remaining term $x$ is of size $\epsilon^{-1/2}$, so that this single term dominates and there is no sensible balance.

(iii) Second and third terms in balance. To have $x$ and $-1$ in balance requires $x$ to be of size one. Then these terms are both of size one, and dominate the remaining term which is size $\epsilon$. This leads to the regular root.

## 2.5 Non-integral powers

Consider the quadratic equation

$$(1-\epsilon)x^2 - 2x + 1 = 0.$$

Setting $\epsilon = 0$ gives $x = 1$ as the double root (a sign of the danger to come). Proceeding as usual we pose the expansion

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots.$$

Substituting into the equation

$$(1-\epsilon)\left(1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right)^2 - 2\left(1 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) + 1 = 0.$$

Expanding gives

$$1 + 2x_1\epsilon + (2x_2 + x_1^2)\epsilon^2 + \cdots - \epsilon - 2x_1\epsilon^2 + \cdots - 2 - 2x_1\epsilon - 2x_2\epsilon^2 + \cdots + 1 = 0.$$

Comparing coefficients of $\epsilon$ gives

At $\epsilon^0$: $\qquad 1 - 2 + 1 = 0,$

which is automatically satisfied because we started with the correct value $x = 1$ at $\epsilon = 0$.

At $\epsilon^1$: $\qquad 2x_1 - 1 - 2x_1 = 0,$

which cannot be satisfied by any value of $x_1$ (except $x_1 = \infty$ in some sense).

The cause of the difficulty is illustrated by looking at the exact solution

$$x = \frac{1 \pm \sqrt{\epsilon}}{1-\epsilon}.$$

Expanding the positive root for small $\epsilon$ gives

$$x = 1 + \epsilon^{1/2} + \epsilon + \epsilon^{3/2} + \cdots.$$

We should have expanded in powers of $\epsilon^{1/2}$ instead of $\epsilon$. This is what $x_1 = \infty$ is hinting at: the scaling on $x_1$ is too small. (In retrospect we could have guessed that an order $\epsilon^{1/2}$ change in $x$ would be required to produce and order $\epsilon$ change in a function at its minimum.)

If we pose the expansion

$$x = 1 + \epsilon^{1/2} x_{1/2} + \epsilon x_1 + \cdots,$$

and substitute into the equation

$$(1-\epsilon)\left(1 + \epsilon^{1/2} x_{1/2} + \epsilon x_1 + \cdots\right)^2 - 2\left(1 + \epsilon^{1/2} x_{1/2} + \epsilon x_1 + \cdots\right) + 1 = 0.$$

Expanding gives

$$\begin{aligned} 1 + 2x_{1/2}\epsilon^{1/2} + (2x_1 + x_{1/2}^2)\epsilon + (2x_{3/2} + 2x_{1/2}x_1)\epsilon^{3/2} + \cdots - \epsilon - 2x_{1/2}\epsilon^{3/2} + \cdots \\ -2 - 2x_{1/2}\epsilon^{1/2} - 2x_1\epsilon - 2x_{3/2}\epsilon^{3/2} + \cdots + 1 = 0. \end{aligned}$$

Comparing coefficients of $\epsilon$ we find that

At $\epsilon^0$: $\qquad 1 - 2 + 1 = 0,$

is automatically satisfied as usual and

At $\epsilon^{1/2}$: $\qquad 2x_{1/2} - 2x_{1/2} = 0,$

is satisfied for all values of $x_{1/2}$. Slightly disturbing that $x_{1/2}$ is not determined but at least the expansion is consistent so far.

At $\epsilon^1$: $\qquad 2x_1 + x_{1/2}^2 - 1 - 2x_1 = 0,$

so that $x_{1/2} = \pm 1$ and $x_1$ is not determined at this level.

At $\epsilon^{3/2}$: $\qquad 2x_{3/2} + 2x_{1/2}x_1 - 2x_{1/2} - 2x_{3/2} = 0,$

so that $x_1 = 1$ for both roots $x_{1/2}$, while $x_{3/2}$ is not determined.

## 2.6 Finding the right expansion sequence

How would we determine the expansion sequence if we did not have the exact solution to compare with? First pose a general expansion

$$x = 1 + \delta_1 x_1$$

and substitute this into the equation to get

$$(1-\epsilon)(1+\delta_1 x_1)^2 - 2(1+\delta_1 x_1) + 1 = 0.$$

Expanding

$$1 + 2\delta_1 x_1 + \delta_1^2 x_1^2 - \epsilon + 2\epsilon\delta_1 x_1 + \delta_1^2 \epsilon x_1^2 - 2 - 2\delta_1 x_1 + 1 = 0.$$

Simplifying leaves

$$\delta_1^2 x_1^2 - \epsilon + 2\epsilon\delta_1 x_1 + \delta_1^2 \epsilon x_1^2 = 0.$$

Now we play the dominant balance game again. Since $\epsilon\delta_1 \ll \epsilon$ the leading terms are $\delta_1^2 x_1^2$ and $\epsilon$. Thus to get a sensible balance we need $\delta_1 = \epsilon^{1/2}$. With this value for $\delta_1$ we equate coefficients of $\epsilon$ to get

$$x_1^2 - 1 = 0, \quad \text{i.e. } x_1 = \pm 1.$$

To proceed to higher order we play the game again. Choosing $x_1 = 1$ for example, we now have

$$x = 1 + \epsilon^{1/2} + \delta_2 x_2, \qquad \delta_2 \ll \epsilon^{1/2}.$$

Substituting into the equation

$$(1-\epsilon)\left(1+\epsilon^{1/2}+\delta_2 x_2\right)^2 - 2\left(1+\epsilon^{1/2}+\delta_2 x_2\right) + 1 = 0.$$

Expanding

$$\begin{aligned} 1 &+ 2\epsilon^{1/2} + \epsilon + 2\delta_2 x_2 + 2\epsilon^{1/2}\delta_2 x_2 + \delta_2^2 x_2^2 \\ &- \epsilon - 2\epsilon^{3/2} - \epsilon^2 - 2\epsilon\delta_2 x_2 - 2\epsilon^{3/2}\delta_2 x_2 - \epsilon\delta_2^2 x_2^2 - 2 - 2\epsilon^{1/2} - 2\delta_2 x_2 + 1 = 0. \end{aligned}$$

Simplifying leaves

$$2\epsilon^{1/2}\delta_2 x_2 + \delta_2^2 x_2^2 - 2\epsilon^{3/2} - \epsilon^2 - 2\epsilon\delta_2 x_2 - 2\epsilon^{3/2}\delta_2 x_2 - \epsilon\delta_2^2 x_2^2 = 0.$$

Since $\delta_2 \ll \epsilon^{1/2}$ the dominant term involving $\delta_2$ is $2\epsilon^{1/2}\delta_2 x_2$. This must balance with $-2\epsilon^{3/2}$, giving $\delta_2 = \epsilon$ and $x_2 = 1$.

## 2.7 Iterative method

This is often very useful in cases where the expansion sequence is not known. Writing the original quadratic as

$$(x-1)^2 = \epsilon x^2$$

we are led to the iterative process

$$x_{n+1} = 1 \pm \epsilon^{1/2} x_n.$$

Starting with $x_0 = 1$ the positive root gives

$$x_1 = 1 + \epsilon^{1/2}$$

and

$$x_2 = 1 + \epsilon^{1/2} + \epsilon.$$