Asymptotic Expansions of Special Functions

Error Function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$\operatorname{erf}(x) \sim \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right), \quad \text{for} \quad x \ll 1$

$\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}\, x} \left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \cdots \right), \quad \text{for} \quad x \gg 1$

Complementary Error Function:

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

$\operatorname{erfc}(x) \sim 1 - \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right), \quad \text{for} \quad x \ll 1$

$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}\, x} \left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2x^2)^3} + \cdots \right), \quad \text{for} \quad x \gg 1$

Exponential Integral:

$$\operatorname{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t} dt$$

$\operatorname{Ei}(x) = -\gamma - \ln x + \int_0^x \frac{1 - e^{-t}}{t} dt \sim -\gamma - \ln x + \left(\frac{x}{1 \cdot 1!} - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} - \cdots \right), \quad \text{for} \quad x \ll 1$

$\operatorname{Ei}(x) \sim \frac{e^{-x}}{x} \left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots \right), \quad \text{for} \quad x \gg 1$

where γ is Euler's constant:

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right) = 0.577215664902$$

Sine Integral:

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt$$

$\operatorname{Si}(x) \sim \frac{x}{1 \cdot 1!} - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots, \quad \text{for} \quad x \ll 1$

$\operatorname{Si}(x) \sim \frac{\pi}{2} - \frac{\sin x}{x} \left(\frac{1!}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots \right) - \frac{\cos x}{x} \left(1! - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots \right), \quad \text{for} \quad x \gg 1$

Cosine Integral:

$$\operatorname{Ci}(x) = \int_x^\infty \frac{\cos t}{t} dt$$

$\operatorname{Ci}(x) = -\gamma - \ln x + \int_0^x \frac{1 - \cos t}{t} dt \sim -\gamma - \ln x + \frac{x^2}{2 \cdot 2!} - \frac{x^4}{4 \cdot 4!} + \frac{x^6}{6 \cdot 6!} - \frac{x^8}{8 \cdot 8!} + \cdots, \quad \text{for} \quad x \ll 1$

$\operatorname{Ci}(x) \sim \frac{\cos x}{x} \left(\frac{1!}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots \right) - \frac{\sin x}{x} \left(1! - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots \right), \quad \text{for} \quad x \gg 1$

where γ is Euler's constant as defined above

Gamma Function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0$$

$$\Gamma(x) \sim \sqrt{\tfrac{2\pi}{x}}\, x^x e^{-x} \left(1 + \tfrac{1}{12x} + \tfrac{1}{288x^2} - \tfrac{139}{51{,}840x^3} + \cdots\right), \quad \text{for} \ \ x \gg 1$$

If x is a positive integer, say $x = n$, then we have $n! = n\Gamma(n) \sim \sqrt{2\pi n}\, n^n e^{-n} + \cdots$

Bessel Function of the First Kind of Order p:

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{p+2k}}{k!\,\Gamma(p+k+1)}$$

$$J_p(x) \sim \tfrac{x^n}{2^n \Gamma(n+1)} \left(1 - \tfrac{x^2}{2\cdot(2n+2)} + \tfrac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \cdots\right), \quad \text{for} \ \ x \ll 1$$

$$J_p(x) \sim \sqrt{\tfrac{2}{\pi x}} \cos\left(x - \tfrac{p\pi}{2} - \tfrac{\pi}{4}\right) + \cdots, \quad \text{for} \ \ x \gg 1$$

$$J_p(x) \sim \tfrac{1}{\sqrt{2\pi p}} \left(\tfrac{ex}{2p}\right)^p + \cdots, \quad \text{for} \ \ p \gg 1$$

Bessel Function of the Second Kind of Order p:

$$Y_p(x) = \lim_{q \to p} \frac{J_q(x)\cos(q\pi) - J_{-q}(x)}{\sin(q\pi)}$$

$$Y_p(x) \sim \sqrt{\tfrac{2}{\pi x}} \sin\left(x - \tfrac{p\pi}{2} - \tfrac{\pi}{4}\right) + \cdots, \quad \text{for} \ \ x \gg 1$$

$$Y_p(x) \sim -\tfrac{2}{\sqrt{\pi p}} \left(\tfrac{ex}{2p}\right)^{-p} + \cdots, \quad \text{for} \ \ p \gg 1$$

Modified Bessel Function of the First Kind of Order p:

$$I_p(x) = i^{-p} J_p(i\,x) = \sum_{k=0}^{\infty} \frac{(x/2)^{p+2k}}{k!\,\Gamma(p+k+1)}$$

$$I_p(x) \sim \tfrac{x^p}{2^p \Gamma(p+1)} \left(1 + \tfrac{x^2}{2\cdot(2p+2)} + \tfrac{x^4}{2\cdot 4\cdot(2p+2)(2p+4)} + \cdots\right), \quad \text{for} \ \ x \ll 1$$

$$I_p(x) \sim \tfrac{e^x}{\sqrt{2\pi x}} \left[1 - \tfrac{1}{2x}\left(p^2 - \tfrac{1}{4}\right) + \cdots\right], \quad \text{for} \ \ x \gg 1$$

Modified Bessel Function of the Second Kind of Order p:

$$K_p(x) = \frac{\pi}{2} \lim_{q \to p} \frac{I_{-q}(x) - I_q(x)}{\sin(q\pi)}$$

$$K_p(x) \sim e^{-x} \sqrt{\tfrac{\pi}{2x}} \left[1 - \tfrac{1}{8x} + \cdots\right], \quad \text{for} \ \ x \gg 1$$