Laplace Method Formulas

Laplace's method is used to approximate integrals of the form

$$I(x) \;=\; \int_a^b f(t)e^{x\phi(t)}\,dt \quad \text{as} \quad x \to \infty.$$

The leading approximation is determined at the point(s) where ϕ attains its absolute maximum on $[a, b]$.

Case: $\phi'(t) \neq 0$ in $[a, b]$

In this case, ϕ has no local maximums in $[a, b]$ so the absolute maximum must occur at an endpoint.

max at t = a: $\boxed{I(x) \;\sim\; -\frac{f(a)}{x\,\phi'(a)}\,e^{x\phi(a)}}$ provided $f(a) \neq 0$

max at t = b: $\boxed{I(x) \;\sim\; \frac{f(b)}{x\,\phi'(b)}\,e^{x\phi(b)}}$ provided $f(b) \neq 0$

Case: $\phi'(t) = 0$ at a unique interior point, say at $t = c \in (a, b)$

In this case $\phi(c)$ is either a local maximum or a local minimum (or an inflection point).

local min: If $\phi(c)$ is a local minimum (or an inflection point), then the absolute maximum of $\phi(t)$ occurs at an endpoint and the above formulas can be used.

- There is one *exception*: If $\phi(t)$ has a local minimum at an interior point, then it may be that $\phi(a) = \phi(b)$, in which case there is a significant contribution to the value of the integral at each endpoint. The contributions given by the above formulas must be added.

local max: If $\phi''(c) < 0$, then $\phi(c)$ must be the absolute maximum of ϕ on $[a, b]$. In this case, the dominant contribution to the integral occurs at $t = c$.

$$\boxed{I(x) \;\sim\; \sqrt{\frac{2\pi}{-x\,\phi''(c)}}\; f(c)e^{x\phi(c)}} \quad \text{provided } f(c) \neq 0$$

Case: $\phi'(d) = 0$ where d is an endpoint and the absolute maximum of ϕ

$$\boxed{I(x) \;\sim\; \sqrt{\frac{\pi}{-2x\,\phi''(d)}}\; f(d)e^{x\phi(d)}} \quad \text{provided } f(d) \neq 0$$

Case: Interior absolute max at $t = c$ with $\phi'(c)=\phi''(c)=\cdots=\phi^{(p-1)}(c)=0$

$$\boxed{I(x) \;\sim\; \frac{2\,\Gamma\!\left(\frac{1}{p}\right)(p!)^{1/p}}{p\left[-x\,\phi^{(p)}(c)\right]^{1/p}}\; f(c)e^{x\phi(c)}} \quad \text{provided } f(c) \neq 0$$

Case: The absolute maximum occurs at multiple points: $t = c_1, c_2, \ldots, c_n$

$$I(x) \;\sim\; I_1(x) + I_2(x) + \cdots + I_n(x),$$

where $I_k(x)$ denotes the contribution to the value of the integral at $t = c_k$. $I_k(x)$ can be determined from the above formulas. One example of this case is when ϕ is periodic.