

I. Perturbation Theory (Asymptotic Approximations)

INTRODUCTORY DISCUSSION

Contrary to the impression given in freshman-sophomore classes, most mathematical problems can't be solved exactly. In practical work, approximation methods are a necessity. You're probably aware of various *numerical* methods for evaluating definite integrals, solving differential equations, etc. The first part of this course concerns a different category of approximation methods, which are called *analytical* or *symbolic*. These are often *complementary* to numerical methods, in the sense that (1) one type of method works well in a situation where the other works badly, or (2) the two methods give different information or insight about the true solution of the problem.

A second reason for studying approximate solutions is that sometimes they are actually *more useful* than exact solutions. For example, consider the function

$$y = \sqrt{1 + x^2 - \frac{\sin x}{x}}.$$

(Let's imagine that it arose as the solution to some differential equation.) Could you sketch the graph of this function if you were given 5 seconds for the task? How about 15 minutes (without a calculator)? The problem suddenly looks easier if we ask a less ambitious question: "How does y behave (approximately) when x is very large or very small?"

- (1) When x is large ($x \rightarrow +\infty$), we see that the sine term is much less than 1, hence negligible compared to the others. Also, the x^2 term is larger than the constant. So, "to lowest order", y is approximately equal to x when x is very large. (What could be simpler?) To get a slightly better approximation we can neglect the sine but keep the 1, and we can expand the square root in a Taylor series:

$$\begin{aligned} y &\approx \sqrt{1 + x^2} \\ &= \sqrt{x^2 \left(\frac{1}{x^2} + 1 \right)} \\ &= x \sqrt{1 + \frac{1}{x^2}} \\ &= x \left(1 + \frac{1}{2} \frac{1}{x^2} + \dots \right) \\ &\approx x + \frac{1}{2x}. \end{aligned}$$

(2) When x is small ($x \rightarrow 0^+$), we can use the Taylor series for the sine function:

$$\sin x = x - \frac{1}{6}x^3 + \dots$$

Thus

$$\begin{aligned} y &\approx \sqrt{1 + x^2 - 1 + \frac{1}{6}x^2} \\ &= \sqrt{\frac{7}{6}x^2} \\ &= \sqrt{\frac{7}{6}}x. \end{aligned}$$

We could get better approximations (“higher-order terms”) by keeping more terms in the Taylor series of sine and then using the Taylor series of the square root to expand the result. (The result is simply the Taylor series of $y(x)$ about the point $x = 0$. Note, by the way, that this method of calculating the Taylor series of the compound function is easier than the textbook method, which requires calculating the derivatives $d^n y/dx^n$.) Notice that because of an accidental cancellation of two coefficients, to get a nonzero result we had to carry the expansion of sine to one order higher than the leading order. There is no general rule stating how many terms in a power series you should keep; it depends on context, including how the final result of your calculation will be used.

In summary, we get a clearer picture of how the function behaves in certain limits by looking at approximate *asymptotic* formulas for it, instead of the more complicated exact formula. (Of course, there’s a vast interval between the two end regions, where our approximations give no information.)

The problems we’ll be working with will be similar to this introductory example, except that there will be other variables in the problem besides the one we’re taking to be large or small. We’re primarily interested in *ordinary differential equations*, but we’ll start out with some *algebraic equations* to demonstrate the principles in a more elementary context. (Polynomial algebraic equations are important in their own right: Two places where they arise are (1) finding eigenvalues of matrices, and (2) solving differential equations with constant coefficients.)

Perturbation Theory for Algebraic Equations: A “Regular” Example

Let’s consider the cubic equation

$$x^3 + \frac{1}{10}x + 8 = 0.$$

Now, there is a formula, or rather a set of formulas, for solving any cubic equation (analogous to the famous quadratic formula for equations whose highest power is x^2), and also one for solving the general quartic (4th-degree) equation. But those formulas are so complicated and hard to use that hardly anybody bothers with them nowadays.* Except in those few cases where one root is obvious (and hence the cubic can be reduced to a quadratic), cubic equations are nearly always solved in practice by a numerical or approximate method. *For equations of degree 5 or higher, there is no formula for the general solution.*

In our case, we notice that the coefficient $\frac{1}{10}$ is smaller than the others. This suggests that we study the equation

$$x^3 + \epsilon x + 8 = 0,$$

find an approximation to the solutions that’s good when ϵ is small, and set ϵ equal to $\frac{1}{10}$ at the end. Let’s assume that

$$x \approx x_0 + \epsilon x_1$$

and find the numbers x_0 and x_1 .

“*Why* should we assume x has that form?” you ask. Well, let’s pause for a small dose of theory. The solution x is a *function of* ϵ . Most of the functions we encounter in real life are differentiable. If x is differentiable at $\epsilon = 0$, then we know that

$$x(\epsilon) \approx x(0) + \left. \frac{dx}{d\epsilon} \right|_0 \epsilon$$

(or, in the notation of differentials,

$$dx = \frac{dx}{d\epsilon} d\epsilon,$$

where $d\epsilon$ is just ϵ in this case, since the starting value of ϵ is 0). So if $x(\epsilon)$ turns out to be differentiable at 0, it will indeed have such a linear approximation, with

* The formulas are in the National Bureau of Standards *Handbook of Mathematical Functions* (ed. by M. Abramowitz and I. A. Stegun), Sec. 3.8. Some of the tribulations of using them are described in Sec. 9.5 of the book *muMATH: A Microcomputer Algebra System*, by C. Woolf and D. Hodgkinson.

x_0 equal to $x(0)$ and x_1 equal to the first derivative at 0. If it happens that x is *not* differentiable, it will be back to the drawing board for a new type of approximation.

The key idea, of course, is that if ϵ is small, then ϵ^2 is even smaller, and terms involving ϵ^2 or higher powers can probably be ignored. However, it isn't always true that functions can be expanded in power series, involving only nonnegative integer exponents (because not all functions are differentiable at all points). We'll see examples of some more complicated situations later.

Incidentally, when (as here) we guess the general form of the solution to an equation and then substitute the guess into the equation to determine some unknown constants or unknown functions in it, the guessed formula is called an *ansatz*.

Turning back to the problem, we calculate

$$x^3 = x_0^3 + 3\epsilon x_0^2 x_1 + 3\epsilon^2 x_0 x_1^2 + \epsilon^3 x_1^3.$$

[In general,

$$(x_0 + \epsilon x_1)^n = x_0^n + \epsilon n x_0^{n-1} x_1 + \epsilon^2 \frac{n(n-1)}{2} x_0^{n-2} x_1^2 + \dots.$$

This *binomial expansion* is valid (as an infinite Taylor series) even if n is not a positive integer. When n is a positive integer, the coefficients are the numbers in the famous *Pascal triangle*.] From now on, I'll write the relation between an exact quantity and its asymptotic approximation in the style

$$x^3 \sim \dots,$$

rather than

$$x^3 \approx \dots,$$

so that \approx can return to its more general meaning of "approximately equal to".

Now the equation becomes

$$\begin{aligned} 0 &\sim x_0^3 + 3\epsilon x_0^2 x_1 + 3\epsilon^2 x_0 x_1^2 + \epsilon^3 x_1^3 \\ &\quad + \epsilon x_0 \quad + \epsilon^2 x_1 \\ &\quad + 8 \\ &= (x_0^3 + 8) + \epsilon(3x_0^2 x_1 + x_0) + \epsilon^2(3x_0 x_1^2 + x_1) + \epsilon^3(x_1^3). \end{aligned}$$

As usual in power-series calculations, we will try to make the coefficient of each power of ϵ separately equal to 0, so that the equation is satisfied for all values of ϵ . The lowest-order equation is

$$\epsilon^0 : \quad 0 = x_0^3 + 8.$$

One solution of this is $x_0 = -2$. (We'll return to the question of additional roots later.) Substitute this result into the next equation:

$$\epsilon^1 : \quad 0 = 12x_1 - 2 \Rightarrow x_1 = \frac{1}{6}.$$

So we have found a *first-order solution*,

$$x \sim -2 + \frac{\epsilon}{6}.$$

Let's take a quick look at the remaining equations.

$$\epsilon^2 : \quad 0 = 3x_0x_1^2 + x_1 = -\frac{6}{36} + \frac{1}{6} = 0 \quad \checkmark$$

$$\epsilon^3 : \quad 0 = x_1^3 = \frac{1}{216} \quad (!)$$

At the final step we seem to have a contradiction. The reason is that we have no right to assume that the problem has a solution of the form $x_0 + \epsilon x_1$ that is accurate through order ϵ^3 . If we wanted an approximation that good, we should have started with a *third-order ansatz*,

$$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3.$$

It happens in this particular problem that $x_2 = 0$, so our first-order solution is actually valid through second order by accident. Indeed, if you plug our solution into the original equation, you will get

$$x^3 + \epsilon x + 8 = \frac{\epsilon^3}{216},$$

where we wanted the right-hand side to be zero, and we had no right to expect anything smaller than ϵ^2 . (Checking the answer in this way is a good habit to adopt, since it will catch many calculational errors, even though in a sense the checking calculation is just a repetition of the one you already did to find the answer.)

COMPLEX ROOTS

Now let's return to the zeroth-order equation,

$$x_0^3 + 8 = 0,$$

whose solution is formally

$$x_0 = \sqrt[3]{-8}.$$

As a cubic equation, it should have three roots. Some of them may be complex, and some may coincide. Since we already know that -2 is one root, we could factor $(x_0 + 2)$ out of the equation and use the quadratic formula on the result to find two more roots. However, the problem of finding all n th roots of a complex number comes up so often that I want to review how to solve it directly.

Recall that any complex number z can be written as $x + iy$, where x and y are real, and it can also be written

$$z = re^{i\theta} \quad (e^{i\theta} = \cos \theta + i \sin \theta),$$

where $r \equiv |z|$ is real and nonnegative, and θ is real and is determined only up to a multiple of 2π (that is, the angles $\theta + 2M\pi$ describe the same complex number for all integers M). Clearly, (r, θ) are just the polar coordinates in the (x, y) plane.

Suppose that $z = w^n$. We can solve for w using one of the laws of exponents:

$$w = r^{\frac{1}{n}} e^{i(\theta+2M\pi)/n}.$$

Any value of M in this formula will give a w which yields z when raised to the n th power. The subtlety is that M 's which give the same z may give different w 's, all of which are valid n roots of z . For example, if $n = 3$ and

$$z = -8 = 8e^{i\pi} = 8e^{i(\pi+2M\pi)},$$

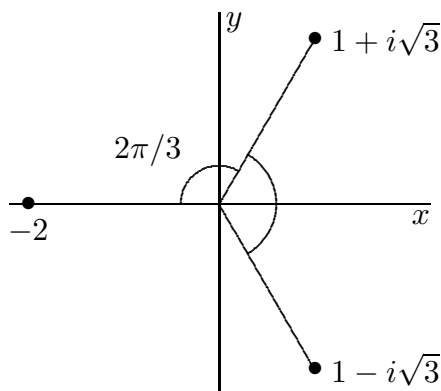
we have

$$w = 2e^{i(1+2M)\pi/3}.$$

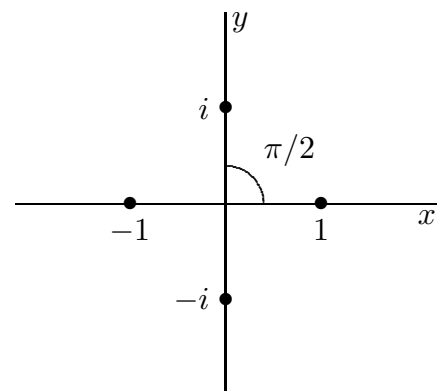
$M = 1$ gives $w = -2$, the solution we already know. Taking $M = 0$ and $M = -1$ gives two new roots,

$$w = 2e^{\pm i\pi/3} = 1 \pm i\sqrt{3}.$$

If we take M still smaller or larger, the roots start to repeat.



Cube roots of -8



Fourth roots of 1

In general, a complex number z has n distinct n th roots. They all have modulus (length) $\sqrt[n]{|z|}$ and are separated in the complex plane by equal angles ($2\pi/n$). At least one of the roots will be real if z is real and n is odd, or if z is positive.

Back to the example. We have found two new solutions for the possible leading behavior of x ,

$$x_0 = 1 \pm i\sqrt{3}.$$

For these cases the equation of order ϵ^1 becomes

$$\begin{aligned} 0 &= 3x_0^2 x_1 + x_0 \\ &= 3(-2 \pm 2i\sqrt{3})x_1 + 1 \pm i\sqrt{3}, \end{aligned}$$

so

$$x_1 = -\frac{1}{3} \frac{1 \pm i\sqrt{3}}{-2 \pm 2i\sqrt{3}}.$$

This requires us to review something else about complex numbers — how to divide them. Answer: Rationalize the denominator.

$$\begin{aligned} x_1 &= -\frac{1}{3} \frac{1 \pm i\sqrt{3}}{-2 \pm 2i\sqrt{3}} \frac{-2 \mp 2i\sqrt{3}}{-2 \mp 2i\sqrt{3}} \\ &= -\frac{1}{3} \frac{-2 + 6 \pm i\sqrt{3}(-2 - 2)}{4 + 12} \\ &= \frac{-1 \pm i\sqrt{3}}{12}. \end{aligned}$$

This completes the calculation of the linear approximation to the two complex roots of $x^3 + \epsilon x + 8 = 0$. [This calculation could have been simplified by dividing the equation by x_0 at the start, but since the point of the example was to demonstrate complex arithmetic, I didn't do that.]

In many problems, of course, one knows in advance, for physical reasons, that the solution is real, and hence complex solutions can be ignored. But there are situations where complex solutions are meaningful and important; one should not forget the possibility of their existence.

HAVE WE FOUND GENUINE SOLUTIONS?

There is a loose end in all of our discussion so far. Recall our first example,

$$x^3 + \epsilon x + 8 = 0. \quad (\dagger)$$

What we did was to find numbers which *satisfy the given equation approximately*. Specifically, we found the second-order approximation

$$x_{\text{approx}} \equiv -2 + \frac{\epsilon}{6} \quad (+ 0 \times \epsilon^2)$$

and verified that it satisfies

$$x^3 + \epsilon x + 8 = \frac{\epsilon^3}{216}. \quad (\ddagger)$$

But what we would really like are numbers which *are good approximations to the numbers that satisfy the given equation exactly*. This is not quite the same thing.

Let x_{exact} be a true solution of (\ddagger) . The question I'm posing is whether the *error*,

$$E \equiv |x_{\text{exact}} - x_{\text{approx}}|,$$

is *small* in some sense. (That's what it means to have a good approximation to x_{exact} .) Now (\ddagger) suggests that probably E is "of the order of ϵ^3 "; that is,

$$E \leq \text{const.} \times \epsilon^3,$$

which is small because ϵ is small. (Moreover, with each term we keep in the series we would expect to gain a power of ϵ in the error estimate.) However, that may be what (\ddagger) *suggests*, but it is not what (\ddagger) *says*. Instead, it says (or implies) this: Let

$$F(x) \equiv x^3 + \epsilon x + 8.$$

Then

$$|F(x_{\text{exact}}) - F(x_{\text{approx}})| = \text{const.} \times \epsilon^3.$$

How can we get from this to a similar conclusion with the F 's missing?

Well, somewhere in your calculus textbook is something called the *inverse function theorem*. [For example, in the 6th edition of Thomas and Finney it's on p. 169.] Translated into the special notation of our problem, it says this:

If a function $F(x)$ is one-to-one and differentiable on an interval around some point $x = a$, and if $F'(x)$ never takes on the value 0 on that interval, then the inverse function $x = F^{-1}(y)$ exists (with domain an appropriate interval of the y variable), it is differentiable, and the value of its derivative at the point $y = F(a)$ in its domain is

$$F^{-1}'(y) = \frac{1}{F'(a)} = \frac{1}{F'(F^{-1}(y))}.$$

In calculations we usually abbreviate all this by the simple formula

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

We can often use this theorem to show that $|x_{\text{exact}} - x_{\text{approx}}|$ is small if we know that $|F(x_{\text{exact}}) - F(x_{\text{approx}})|$ is small in the same sense (for instance, of the order of ϵ^3). In our example,

$$F'(x) = 3x^2 + \epsilon.$$

If ϵ is positive, then $F'(x)$ is positive for all x . This implies that F is increasing and therefore one-to-one; also that F' is never zero. Thus all the hypotheses of the inverse function theorem are satisfied (if $\epsilon > 0$). Let the a in the theorem be our x_{approx} . Then

$$F(x_{\text{approx}}) = \frac{\epsilon^3}{216}; \quad x_{\text{approx}} = F^{-1}\left(\frac{\epsilon^3}{216}\right).$$

Also,

$$F(x_{\text{exact}}) = 0; \quad x_{\text{exact}} = F^{-1}(0).$$

Thus

$$x_{\text{exact}} - x_{\text{approx}} = F^{-1}(0) - F^{-1}\left(\frac{\epsilon^3}{216}\right).$$

Since F^{-1} is differentiable, we can apply the first-order (or differential) approximation,

$$F^{-1}(y+h) - F^{-1}(y) = F^{-1'}(y)h + o(h),$$

with

$$y = \frac{\epsilon^3}{216}, \quad h = -\frac{\epsilon^3}{216}.$$

Here $o(h)$ stands for a “remainder” term, which is negligible compared to the term $F^{-1'}(y)h$ when h is small. By the formula in the theorem,

$$F^{-1'}\left(\frac{\epsilon^3}{216}\right) = \frac{1}{F'(x_{\text{approx}})}.$$

The numerical value of this quantity is not crucial to the argument, but let’s calculate it for clarity:

$$F'(x_{\text{approx}}) = 3\left(-2 + \frac{\epsilon}{6}\right)^2 + \epsilon = 12 + O(\epsilon).$$

So we have

$$x_{\text{exact}} - x_{\text{approx}} = -\frac{\epsilon^3}{12 \times 216} + o(\epsilon^3) = O(\epsilon^3).$$

(The notation $O(\epsilon^3)$ stands for some term of order ϵ^3 , while $o(\epsilon^3)$ stands for a term that is even smaller than a cubic term. The O and o notation will be discussed more thoroughly later on.)

This last is exactly the kind of inequality we wanted to prove. In fact, we not only proved existence of a C such that $E < C\epsilon^3$; we even found a *numerical value* for C , so we have a very good idea of how accurate the approximation is.

This example should give some idea of what is involved in proving that the approximations we construct really are asymptotic to (unknown) exact solutions. It also makes clear why in this short course we normally will *not* prove such theorems. Not only did the argument take quite some time to explain (although the basic idea behind it is actually quite simple); it also works, in its simplest form, only in special cases. For instance, there is already a complication in this example if ϵ is negative.

A more serious complication would arise if x_{approx} were a *double root* of the equation determining it. (See “A more complicated example” elsewhere in the notes.) Then $F'(x_{\text{approx}})$ would be proportional to ϵ , this would put an extra ϵ in the denominator of $F^{-1'}$, and the argument would break down. Indeed, we will see from the “more complicated example” that the corrections to a zeroth-order approximation, x_0 , which is a double root typically are proportional to the *square root* of ϵ rather than to ϵ itself, although the error in $F(x_{\text{approx}})$ is of order ϵ . So the kind of inequality one would expect to get from the inverse function theorem is actually *false* in such a case.

In summary, proving these error estimates is a complicated subject, especially for “singular” perturbation problems, and in this course we will almost always have to take for granted that the approximations we construct by “formal” calculations are indeed asymptotic to the true solutions. In other words, we will spend most of our time constructing *exact solutions to approximate problems*, while hoping that these are *approximate solutions to the exact problems*.

Perturbation Theory for Algebraic Equations: “Singular” Examples

Next, let’s consider the equation

$$\epsilon x^2 + 2x + 1 = 0.$$

What makes this different from the previous example is the fact that the small parameter multiplies the leading term (highest power) in the polynomial.

As an example of the example, look at

$$\frac{x^2}{1,000,000,000} + 2x + 1 = 0.$$

Since the equation is only quadratic, you might try to solve it by the quadratic formula:

$$x = \frac{-2 \pm \sqrt{4 - 4 \times 10^{-9}}}{2 \times 10^{-9}} = \frac{-2 \pm \sqrt{3.999999996}}{2 \times 10^{-9}}.$$

The root with the minus sign is clearly very close to -2×10^9 . However, we have a problem in proceeding further with the one containing the plus sign: our calculator or computer may not carry enough significant figures to produce an accurate result for the numerator. (The main problem is not extracting the square root, but subtracting two nearly equal quantities.) The situation is especially embarrassing if the 2 in the equation is a measured experimental quantity, known to only 3 significant figures. Nor will it do just to say that the numerator is essentially 0, because it eventually gets multiplied by 10^9 ; the unknown root is not tiny.

Our second try might be to say: Since the first term is so small, why not ignore it entirely? We should get a good approximation. Well, the equation then becomes

$$2x + 1 = 0,$$

so that

$$x = -\frac{1}{2}.$$

(What happened to the second root, the one that was so big? We’ll have to come back to that question later.)

Let’s try to improve this answer. Replace the small constant 10^{-9} by the variable ϵ . (Sometimes making a problem more general or abstract makes it easier to solve!) We’ll now apply the perturbation technique that worked on the cubic equation earlier. We make the ansatz

$$x \sim -\frac{1}{2} + x_1 \epsilon.$$

(Since we already know that the leading term is $-\frac{1}{2}$, there is no need to write it as x_0 and solve for it — though we certainly could do that.) We calculate

$$x^2 \sim \frac{1}{4} - x_1\epsilon + x_1^2\epsilon^2.$$

After multiplying by ϵ we should ignore all but the first term of x^2 , because the others are of the same order as terms in $2x$ that we're neglecting. (Remember that in the cubic example we reached an incorrect (in fact, inconsistent) equation by taking incomplete high-order terms seriously.) So we get

$$0 \sim \frac{\epsilon}{4} + (-1 + 2x_1\epsilon) + 1 = \epsilon \left(\frac{1}{4} + 2x_1 \right)$$

Therefore

$$x_1 = -\frac{1}{8}$$

and hence

$$x \sim -\frac{1}{2} - \frac{1}{8}\epsilon.$$

For the numerical example problem we thus get

$$x \approx -\frac{1}{2} - \frac{1}{8} \times 10^{-9}.$$

This calculation was much easier than wrestling with the square root in the exact formula. It is very accurate — better than you would get with a calculator that didn't keep enough digits. (One could use it as the starting point in an iterative calculation, such as Newton's method, to get something even better.)

Finally, note that this form of the answer shows clearly that if b (coefficient of the x^1 term) is known only to 3 significant figures, there is no point in keeping the order- ϵ term at all.

But what about that other root? Let's look again closely at the exact answer. The second root was very big, so the perturbation assumption that ϵx^2 is small becomes false. Thus this root is nowhere near a root of the approximating first-degree equation. That's the explanation for its disappearance.

Maybe near the other root we can neglect a different term in the equation. Then the other two terms must "balance" each other, and that condition will tell us how big x must be.

- (1) Can we neglect the middle term? Then the other two terms must approximately cancel, so

$$x \approx \text{constant} \times \epsilon^{-\frac{1}{2}}.$$

But then $2x$ would be *large* compared to the other two terms, so this assumption is inconsistent!

(2) Can we neglect the last term? Then

$$\epsilon x^2 \approx -2x.$$

Cancel an x — we agreed we are looking for a *large* x , so we don't waste time worrying about whether we're dividing by 0. We get

$$x \approx -\frac{2}{\epsilon}.$$

In the numerical problem this is -2×10^9 , so it is a good approximation to the root we got from the quadratic formula.

To get higher-order corrections it is convenient to *rescale* the variable so that it is of order unity (ϵ^0) rather than of order ϵ^{-1} . Define \bar{x} by

$$x \equiv \frac{\bar{x}}{\epsilon}.$$

Then the equation becomes

$$\epsilon^{-1}\bar{x}^2 + 2\epsilon^{-1}\bar{x} + 1 = 0.$$

Multiply through by ϵ :

$$\bar{x}^2 + 2\bar{x} + \epsilon = 0.$$

Now try a power series

$$\bar{x} = \bar{x}_0 + \epsilon\bar{x}_1 + \epsilon^2\bar{x}_2 + \dots.$$

Plug in:

$$\begin{aligned} 0 \sim & \bar{x}_0^2 + 2\epsilon\bar{x}_0\bar{x}_1 + \epsilon^2\bar{x}_1^2 + 2\epsilon^2\bar{x}_0\bar{x}_2 \\ & + 2\bar{x}_0 + 2\epsilon\bar{x}_1 + 2\epsilon^2\bar{x}_2 \\ & + \epsilon. \end{aligned}$$

Pick out the terms independent of ϵ :

$$0 = \bar{x}_0^2 + 2\bar{x}_0.$$

This says that either $\bar{x}_0 = 0$ (the vanishing root we rejected earlier), or $\bar{x}_0 = -2$. The latter gives $x_0 = -2/\epsilon$, which is the leading term we expect.

Next look at the terms proportional to ϵ :

$$0 = 2\bar{x}_0\bar{x}_1 + 2\bar{x}_1 + 1.$$

We know \bar{x}_0 , so we plug it in and solve for \bar{x}_1 , getting $\frac{1}{2}$.

Finally, the ϵ^2 terms:

$$0 = \bar{x}_1^2 + 2\bar{x}_0\bar{x}_2 + 2\bar{x}_2.$$

This yields $\bar{x}_2 = \frac{1}{8}$.

So we have

$$\bar{x} \sim -2 + \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2$$

and hence

$$x \sim -\frac{2}{\epsilon} + \frac{1}{2} + \frac{1}{8}\epsilon.$$

By going up through order ϵ^2 in the series for \bar{x} , we have obtained an answer that is good through order ϵ^1 , the same order to which we carried the solution for the other root.

What do you think would happen if we took the root $\bar{x}_0 = 0$ seriously? Try it.

A MORE COMPLICATED EXAMPLE

Consider

$$\epsilon x^4 + \epsilon x^3 - x^2 + 2x - 1 = 0.$$

As always, we assume ϵ is small.

First, let's just neglect the ϵ terms:

$$0 = x^2 - 2x + 1 = (x - 1)^2.$$

So $x = 1$ is a double root.

Before looking for the other two roots, let's extend the solution we've just found to the next higher order.

First we try the standard ansatz,

$$x \sim 1 + \epsilon x_1.$$

Then for each positive integer n ,

$$x^n \sim 1 + n\epsilon x_1,$$

by the binomial formula. Substituting into the equation, we get

$$0 \sim \epsilon(1 + \dots) + \epsilon(1 + \dots) - (1 + 2\epsilon x_1) + (2 + 2\epsilon x_1) - 1.$$

The terms of order ϵ^0 cancel, verifying that we chose the right x_0 . The terms of order ϵ^1 are

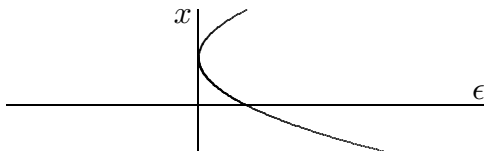
$$0 = 1 + 1 - 2x_1 + 2x_1 = 2 \quad (!) \quad (*)$$

Our assumed form for the answer is inconsistent!

Before giving up, we try a more general type of series:

$$x = 1 + \sqrt{\epsilon} x_1 + \epsilon x_2 + \dots$$

There are several reasons why one might guess this form for the answer. (1) The presence of the double root suggests that the dependence on ϵ near $\epsilon = 0$ might be *nonanalytic* (not representable as a Taylor series) and in fact might be of a square-root type. Compare the behavior of the quadratic formula as a function of one of the coefficients, near the point where the discriminant vanishes. (For instance, consider $c = b^2/4a + \epsilon$. This graph is for $a < 0$, $b > 0$.)



(2) Since in (*) the first-order contributions from the “big” terms cancelled among themselves ($-2x_1 + 2x_1$) without help from the “small” terms, we might “promote” the former to order $\epsilon^{\frac{1}{2}}$, a larger order than the leading contribution of the small terms ($1 + 1$).

Let’s substitute the $\sqrt{\epsilon}$ series into the equation, and keep terms up through order ϵ^1 (but drop those of order $\epsilon^{\frac{3}{2}}$). We get

$$0 \sim \epsilon(1) + \epsilon(1) - (1 + 2\sqrt{\epsilon} x_1 + 2\epsilon x_2 + \epsilon x_1^2) + (2 + 2\sqrt{\epsilon} x_1 + 2\epsilon x_2) - 1.$$

[To get the x^2 term, apply the binomial formula

$$(1 + b)^2 = 1 + 2b + b^2$$

with $b = \sqrt{\epsilon} x_1 + \epsilon x_2$, then collect terms with like powers of $\sqrt{\epsilon}$ together. Higher powers, when needed, can be treated similarly. If x_0 is not 1, remember that its powers also appear in the terms of the binomial series.] Now collect the terms of each order:

$$\epsilon^0 : \quad -1 + 2 - 1 = 0 \quad \checkmark$$

$$\epsilon^{\frac{1}{2}} : \quad -2x_1 + 2x_1 = 0 \quad \checkmark$$

$$\epsilon^1 : \quad 1 + 1 - 2x_2 - x_1^2 + 2x_2 = 0$$

Thus $x_1 = \pm\sqrt{2}$. Thus

$$x \sim 1 \pm \sqrt{2\epsilon}.$$

The double root has split into two distinct roots as ϵ moves away from 0. This is the usual behavior: If you “wiggle” a curve in some arbitrary manner, the probability is very high that a tangent line will become a secant — or not touch the curve at all. (If ϵ is negative, the roots become complex.)

Now let’s try the “balancing” technique to find the other two roots of the fourth-degree equation. If one of the ϵx^n terms is to be of order unity (the same order as the other terms), then x must be large. Therefore, ϵx^4 will dominate ϵx^3 . Also, x^2 will dominate $2x - 1$ in this regime. Therefore, we expect the zeroth-order approximation to be determined by the equation

$$\epsilon x^4 \approx x^2,$$

which leads to

$$x \approx \pm \epsilon^{-\frac{1}{2}}.$$

This will be our x_0 .

Also, it suggests the rescaling

$$x \equiv \epsilon^{-\frac{1}{2}} \bar{x}.$$

Then $x^n = \epsilon^{-\frac{n}{2}} \bar{x}^n$, so the equation becomes

$$0 = \epsilon^{-1} \bar{x}^4 + \epsilon^{-\frac{1}{2}} \bar{x}^3 - \epsilon^{-1} \bar{x}^2 + 2\epsilon^{-\frac{1}{2}} \bar{x} - 1,$$

or

$$0 = \bar{x}^4 + \sqrt{\epsilon} \bar{x}^3 - \bar{x}^2 + 2\sqrt{\epsilon} \bar{x} - \epsilon.$$

Now we perform the perturbation. We try

$$\bar{x} = \bar{x}_0 + \sqrt{\epsilon} \bar{x}_1 + \dots,$$

so that

$$\bar{x}^n \sim \bar{x}_0^n + n\bar{x}_0^{n-1}\bar{x}_1\sqrt{\epsilon}$$

and the equation becomes

$$0 \sim \bar{x}_0^4 + 4\bar{x}_0^3\bar{x}_1\sqrt{\epsilon} + \bar{x}_0^3\sqrt{\epsilon} - (\bar{x}_0^2 + 2\bar{x}_0\bar{x}_1\sqrt{\epsilon}) + 2\bar{x}_0\sqrt{\epsilon}.$$

The ϵ^0 equation is

$$0 = \bar{x}_0^4 - \bar{x}_0^2.$$

Its roots are $\bar{x}_0 = 0$ (which we ignore, as before) and $\bar{x}_0 = \pm 1$ (which is what we predicted).

In order $\epsilon^{\frac{1}{2}}$ we get

$$0 = 4\bar{x}_0^3\bar{x}_1 + \bar{x}_0^3 - 2\bar{x}_0\bar{x}_1 + 2\bar{x}_0.$$

Note that $\bar{x}_0 = \bar{x}_0^3 = \pm 1$. Factor this out:

$$0 = 4\bar{x}_1 + 1 - 2\bar{x}_1 + 2 = 2\bar{x}_1 + 3.$$

Therefore, $\bar{x}_1 = -\frac{3}{2}$.

So, for the two elusive roots we get $\bar{x} = \pm 1 - \frac{3}{2}\sqrt{\epsilon} + \text{terms of order } \epsilon$, or

$$x = \pm \frac{1}{\sqrt{\epsilon}} - \frac{3}{2} + \text{terms of order } \sqrt{\epsilon}.$$

To carry the expansion up to a term of positive order, we still have some work to do: Go back and include the term $\epsilon\bar{x}_2$ in the calculation.

Regular Perturbation Theory for Differential Equations

Now we'll take an equation like

$$\frac{dy}{dt} + y - \epsilon y^2 = 0$$

or

$$\frac{d^2y}{dt^2} + 2\epsilon \frac{dy}{dt} + y = 0$$

and hunt for a solution as a power series in the parameter ϵ :

$$y = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$$

These two examples (one nonlinear and one linear) are useful because they have known exact solutions against which the perturbative solutions can be compared. Of course, in practice one would usually want to use the method on a problem that *can't* be solved exactly.

One kind of power-series solution of linear differential equations, the *Frobenius series* in the *independent variable*, is traditionally studied in courses on ordinary differential equations. Here, that would be a series in t , not ϵ . Let me tell, or remind, you of a few features of those series, to compare and contrast them with the series in ϵ we are going to construct.

Power series in the independent variable:

1. Around an “ordinary point” t_0 you get a standard power series (Taylor series). Around a “regular singular point” you get fractional powers and sometimes logarithms of $t - t_0$. (Irregular singular points were left as unexplored territory.)
2. A *truncation* of the series (the sum of the first N terms) is a very good approximation very near t_0 , but a horrible approximation far away.
3. The full series converges in some disk around t_0 .

Power series in a parameter: We'll find that ...

1. Sometimes you get a standard series; this is called a *regular* perturbation problem. But often you must guess a more general form for the series — say

$$\sqrt{\epsilon} e^{t/\epsilon} (y_0 + \sqrt{\epsilon} y_1 + \dots).$$

This is one kind of *singular* perturbation.

2. Typically the series will be good over a broad interval. However,

- (a) It may not be good everywhere. In some regions a separate kind of expansion may be needed. Such a region is sometimes called a *boundary layer*. Some authors reserve the term “singular” specifically for this situation.
- (b) Often you find that the ϵ expansion turns out to be an expansion in t as well. That is, each term in the series depends on a higher power of t , so we’re back to the old problem of the approximation not being good except near $t = 0$. This is the problem of *secular terms*. [Challenge: Go to an etymological dictionary and figure out what connection this has with “secular humanism”.] However, it is sometimes possible to get rid of the secular terms by making a change of independent variable (*distorting the time scale*, *Poincaré–Lindstedt method*, “two-timing”, etc.)
- (3) The series may not converge. This issue is irrelevant to whether the first few terms give a good approximation to the exact solution! We are most interested in whether the series is *asymptotic*, not whether it’s convergent. (A definition of “asymptotic” will be forthcoming soon; for now, just think of it as meaning “the first few terms give a good approximation.”)

EXAMPLE: THE DAMPED HARMONIC OSCILLATOR

Let’s consider

$$\frac{d^2y}{dt^2} + 2\epsilon \frac{dy}{dt} + y = 0.$$

For definiteness, consider the initial conditions

$$y(0) = 0, \quad y'(0) = 1.$$

Try

$$y = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$$

We get (to second order)

$$0 \sim (y_0'' + \epsilon y_1'' + \epsilon^2 y_2'') + (2\epsilon y_0' + 2\epsilon^2 y_1') + (y_0 + \epsilon y_1 + \epsilon^2 y_2);$$

$$0 \sim (y_0'' + y_0) + \epsilon(y_1'' + 2y_0' + y_1) + \epsilon^2(y_2'' + 2y_1' + y_2).$$

The initial conditions don’t depend on ϵ , so they break into $y_n(0) = 0$ and

$$y_0'(0) = 1, \quad y_1'(0) = 0, \quad y_2'(0) = 0.$$

The solution for y_0 is

$$y_0(t) = \sin t.$$

Substitute this into the equation for y_1 :

$$y_1'' + y_1 = -2y_0' = -2 \cos t.$$

Now remember the *method of undetermined coefficients* for an inhomogeneous linear equation with the forcing term “on resonance”:

$$y_1 = A t \cos t + B t \sin t + \text{homogeneous solution.}$$

After algebra, you find that the solution satisfying the null initial conditions is

$$y_1(t) = -t \sin t.$$

[Check it.] This is called a *secular term*, because it grows with t .

For the second-order term we get the equation

$$y_2'' + y_2 = -2y_1' = 2 \sin t + 2 t \cos t.$$

We know that the solution will involve t^2 times a trig function. And so on to higher orders. (The secular terms are getting worse!) So we have constructed

$$y(t; \epsilon) = \sin t - \epsilon t \sin t + \text{term involving } \epsilon^2 t^2 + \dots.$$

To judge this approximation, let's look at the exact solution. It is

$$y(t; \epsilon) = \frac{e^{-\epsilon t}}{\sqrt{1 - \epsilon^2}} \sin(\sqrt{1 - \epsilon^2} t).$$

Expanding this in a Taylor series in ϵ (with t fixed), we get agreement with our perturbative solution, as far as we've carried it. [If this isn't obvious to you, work it out!] For any given t , our approximation is good if ϵ is sufficiently small. But for a fixed ϵ , there eventually comes a t for which the error is large. Our method has led us to expand $e^{-\epsilon t}$ as a power series, but that is *a bad thing to do*, clearly. Later we'll see how reformulating the problem can help us to avoid such self-defeating, overenthusiastic use of power series!

A NONLINEAR EXAMPLE

Let's try

$$\frac{dy}{dt} = 1 + (1 + \epsilon)y^2, \quad y(0) = 1.$$

Try the usual substitution $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$. Then

$$y^2 = y_0^2 + 2\epsilon y_0 y_1 + \epsilon^2 (y_1^2 + 2y_0 y_2) + \dots.$$

So the equation is (through second order)

$$y'_0 + \epsilon y'_1 + \epsilon^2 y'_2 \sim 1 + y_0^2 + 2\epsilon y_0 y_1 + \epsilon^2 (y_1^2 + 2y_0 y_2) + \epsilon y_0^2 + 2\epsilon^2 y_0 y_1.$$

The zeroth-order equation is

$$y'_0 = 1 + y_0^2, \quad y_0(0) = 1.$$

This is a separable equation:

$$\int \frac{dy_0}{1 + y_0^2} = \int dt$$

$$\tan^{-1} y_0 = t + C$$

$$\tan^{-1} 1 = C \Rightarrow C = \frac{\pi}{4}$$

So

$$y_0 = \tan\left(t + \frac{\pi}{4}\right).$$

The first-order equation is therefore

$$y'_1 = 2 \tan\left(t + \frac{\pi}{4}\right) y_1 + \left[\tan\left(t + \frac{\pi}{4}\right)\right]^2.$$

General observations:

- (1) Every equation after the first (order 0) will be linear: Because $(\epsilon^n y_n)^2$ is of higher order than $\epsilon^n y'_n$, it does not appear in the equation that determines y_n , but only in some later equation when y_n is already known. This is a marvelous property of the method, which enables it to convert unsolvable problems into solvable ones!
- (2) Just because the equation is linear doesn't necessarily mean that it is trivial to solve! I leave it to you to finish this problem, by a standard method but a fair amount of work.

Asymptotics and Uniformity

So far we have looked at a variety of examples, without any general theory to hold them together. It's time now to frame some precise definitions to describe the type of phenomena we're encountering.

ORDER SYMBOLS (Big-Oh and Little-Oh)

Definition: The expression

$$f(\epsilon) = O(\epsilon^n) \quad \text{as } \epsilon \rightarrow 0$$

means that there is some constant C such that

$$|f(\epsilon)| \leq C|\epsilon|^n$$

at least whenever ϵ is sufficiently small.

Examples:

- (1) $\sin \epsilon = O(\epsilon)$.
- (2) $\sin \epsilon - \epsilon = O(\epsilon^3)$.

Note:

1. The symbol O is a capital 0, not a zero.
2. The notation does not refer to a particular function named “ O ”. Thus the equations

$$f(\epsilon) = O(\epsilon) \quad \text{and} \quad 5f(\epsilon) = O(\epsilon)$$

can both be true.

3. The specification “as $\epsilon \rightarrow 0$ ” may be left tacit if there is no chance of misunderstanding.

Here is another way of stating the definition:

$$f(\epsilon) = O(\epsilon^n) \quad \text{as } \epsilon \rightarrow 0$$

means that

$$\frac{f(\epsilon)}{\epsilon^n} \quad \text{is bounded}$$

at least on some neighborhood of $\epsilon = 0$.

[The terminology here may itself need explaining. Let I stand for an interval in the range of the independent variable of a function, such as $0 \leq x \leq 1$ or $-\infty < x < 0$.

Definition: The function $g(x)$ is *bounded* on the interval I if there is a constant C such that

$$|g(x)| \leq C \quad \text{for all } x \text{ in } I.$$

In other words, a function is *not* bounded if it becomes arbitrarily large somewhere. $g(x) = 1/x$ is unbounded on any interval containing 0 or having 0 as an endpoint. $g(x) = x^2$ is unbounded on an interval that extends to ∞ or $-\infty$. A continuous function that approaches a limit as x approaches an endpoint of an interval is certainly bounded in the vicinity of that endpoint, and this is often the easiest way to verify the boundedness that we need to apply the definition of “ O ”; but a function can be bounded without having a limit. For instance, $\sin z$ is bounded on $[0, \infty)$ although $\lim_{z \rightarrow +\infty} \sin z$ does not exist.]

Definition: The expression

$$f(\epsilon) = o(\epsilon^n) \quad \text{as } \epsilon \rightarrow 0$$

means that

$$\frac{f(\epsilon)}{\epsilon^n} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Example: $\sqrt{\epsilon} = o(\epsilon^0) = o(1) \quad \text{--- NOT } O(\epsilon).$

Note that the symbol o is a lower-case o , not a sigma or a zero. The other numbered remarks above apply here too.

A function which is $o(\epsilon^n)$ is certainly also $O(\epsilon^n)$, but the converse is not necessarily true. A function which is $O(\epsilon^{n+1})$ is certainly also $o(\epsilon^n)$, but the converse is not necessarily true, as we just saw in an example. Thus the two sequences of order symbols interlace to provide an increasingly stringent classification of the behavior of functions near 0:

$$\dots \subset O(\epsilon^3) \subset o(\epsilon^2) \subset O(\epsilon^2) \subset o(\epsilon) \subset O(\epsilon) \subset o(1) \subset O(1).$$

We could also consider the negative powers extending in the opposite direction, describing more and more lenient types of possible blowup of the function as ϵ approaches 0.

When we expand a function in a Taylor series and stop at the n th term,

$$f(\epsilon) = f(0) + f'(0)\epsilon + \frac{1}{2}f''(0)\epsilon^2 + \dots + \frac{1}{n!}f^{(n)}(0)\epsilon^n + R_n(\epsilon),$$

the remainder R_n is guaranteed to be $o(\epsilon^n)$ if f is n times differentiable. If f is actually $n + 1$ times differentiable (so that the *next* term in the series makes sense), then R_n will actually be $O(\epsilon^{n+1})$ — it “looks like” the first neglected term in the series.

There are lots of generalizations of these definitions, not all of which I’ll write down here. See Chapter 1 of Cole’s book for a more general formulation. In particular (besides the obvious fact that the variable isn’t always called ϵ):

- (1) We may be interested in the behavior near some point other than 0. For example, Taylor series around a point ϵ_0 are naturally associated with estimates of the form

$$f(\epsilon) = O((\epsilon - \epsilon_0)^n) \quad \text{as } \epsilon \rightarrow \epsilon_0.$$

- (2) We can play the same game at infinity:

$$f(z) = O(z^n) \quad \text{as } z \rightarrow +\infty$$

means that $|f(z)| < Cz^n$, at least when z is sufficiently large.

- (3) We can use other functions besides integral powers. This is especially likely to be useful at infinity. For example, functions which are $O(e^{-z})$ fall off faster than any (negative) power of z ; functions $O(e^{-3z})$ fall off faster still, and those that are $O(e^{-z^2})$ even faster. On the other hand, functions $O(\ln z)$ grow more slowly than any (positive) power, $O((\ln z)^2)$ just a bit faster than those, and those that are $O(\ln(\ln(\ln z)))$ slower still.

[You might go back now and reread the subsection titled “Have we found genuine solutions?” to see if it is easier to understand now.]

ASYMPTOTIC APPROXIMATIONS AND ASYMPTOTIC SERIES

Suppose that $f(\epsilon)$ is a function which is difficult to calculate exactly. For example, $f(t; \epsilon)$ may be the solution of a differential equation (containing a parameter ϵ) that we don’t know how to solve. Suppose that $f_0(\epsilon)$ is a simpler (easier to calculate) function, and that

$$f(\epsilon) - f_0(\epsilon) = O(\epsilon^n) \tag{†}$$

for some n . Then f_0 is a good approximation to f when ϵ is sufficiently small. (Here I am assuming that f and f_0 themselves are *not* $O(\epsilon^n)$; if that were true, (†) would not have much content.) One writes

$$f(\epsilon) \sim f_0(\epsilon)$$

and says that f is *asymptotic* to f_0 , or that f_0 is an *asymptotic approximation* to f in the limit $\epsilon \rightarrow 0$, or in the “regime” of small ϵ .

Often f_0 is just the first of a sequence of approximations, each of which is asymptotic to f to one higher order. Normally each of these approximations is written as the sum of the previous approximation plus a correction term, so that it appears as a truncation of an *asymptotic series*:

$$f(\epsilon) \sim \sum_{j=0}^{\infty} f_j(\epsilon). \quad (\ddagger)$$

To elucidate this, let’s first consider the case where $f_j(\epsilon) = a_j \epsilon^j$ ($a_j = \text{constant}$). Then (\ddagger) means that for each N ,

$$f(\epsilon) - \sum_{j=0}^N a_j \epsilon^j = O(\epsilon^{N+1}),$$

as usual for Taylor series. [In place of $O(\epsilon^{N+1})$ we could equally well write $o(\epsilon^N)$. Why would this *not make any difference*? (Answer follows shortly.)]

Note that *an asymptotic infinite series need not converge*. The definition says that a *finite* number of terms of the series gives a *good approximation* in the limit that ϵ is small. It says nothing about an *exact* representation of $f(\epsilon)$ for any fixed ϵ .

More generally, we have to assume that the f_j in (\ddagger) satisfy

$$f_{j+1}(\epsilon) = o(f_j(\epsilon))$$

(for all j). Then (\ddagger) means

$$f(\epsilon) - \sum_{j=0}^N f_j(\epsilon) = o(f_N(\epsilon)).$$

In words, the error in the approximation is of a higher order than any term f_j kept in the approximation.

This definition takes care of the situation where the terms in the series are not just integral powers of ϵ , but it is still not general enough to cover all the cases we might need to work with. For example, consider the series

$$f(t; \epsilon) \sim \sum_{j=0}^{\infty} e^{t/\epsilon} \sin(jt/\epsilon) \epsilon^{j/2}.$$

If we try to apply the definition just given, we find ourselves dividing by 0 at certain values of t . (And we can't go back to the first definition, even with fractional exponents, because the exponential factor prevents the error terms from being $O(\epsilon^N)$.) The cure is to choose a set of *gauge functions*, $\phi_j(\epsilon)$, such that

$$\phi_{j+1}(\epsilon) = o(\phi_j(\epsilon)) \quad \text{and} \quad f_j(\epsilon) = O(\phi_j(\epsilon)).$$

(In the example, $\phi_j(\epsilon) \equiv e^{t/\epsilon} \epsilon^{j/2}$.) Then we can say that (\ddagger) is a valid asymptotic expansion (with respect to those gauge functions) if

$$f(\epsilon) - \sum_{j=0}^N f_j(\epsilon) = o(\phi_N(\epsilon)).$$

Equivalently,

$$\lim_{\epsilon \rightarrow 0} \frac{f(\epsilon) - \sum_{j=0}^N f_j(\epsilon)}{\phi_N(\epsilon)} = 0.$$

Clearly, just as for the order symbols, we can define asymptotic series at some point other than 0, and also at infinity. In fact, asymptotic series at infinity are quite commonly used (and the need for nontrivial gauge functions is more common there than at zero). See Chapter 4 of the Schaum's Outline book for some examples.

Here is one famous example, the *Stirling approximation* for $z!$:

$$z! \equiv \Gamma(z+1) \sim \sqrt{2\pi z} z^z e^{-z} \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots \right\}.$$

This is most familiar in the form where only the first term is kept and the logarithm is taken:

$$\ln z! \approx z \ln z - z + \frac{1}{2} \ln z + \frac{1}{2} \ln(2\pi).$$

(Notice that the terms have been arranged here in an order so that each is "o" of the one preceding it.)

This is the stock example of a series that *does not converge*, but nevertheless is extremely useful because its first term is an excellent approximation to the exact answer (for reasonably large z). (This means, in particular, that for a fixed z you may get a better approximation from just the first term of the series than from three terms!) At the opposite extreme, the Taylor series for e^z ,

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!},$$

converges for all z , but it is utterly useless for computing, say, e^{-100} . [Why? Compare the 99th and 100th terms with the size of the desired answer.]

O VS. o IN THE DEFINITION OF “ASYMPTOTIC”

Some books define

$$f(\epsilon) \sim \sum_{j=0}^{\infty} a_j \epsilon^j$$

to mean

$$f(\epsilon) - \sum_{j=0}^N a_j \epsilon^j = O(\epsilon^{N+1}) \quad \text{for all } N, \tag{1}$$

while others define it as

$$f(\epsilon) - \sum_{j=0}^N a_j \epsilon^j = o(\epsilon^N) \quad \text{for all } N. \tag{2}$$

I want to show that these two statements are equivalent, to make good on a challenge raised earlier.

It's obvious that (1) implies (2), because any function that is $O(\epsilon^{N+1})$ is also $o(\epsilon^N)$.

To show that (2) implies (1), we have to exploit the fact that (2) holds for *all* N . Replace the N in (2) by $N + 1$:

$$f(\epsilon) - \sum_{j=0}^{N+1} a_j \epsilon^j = o(\epsilon^{N+1}).$$

Move the highest-order term to the other side:

$$f(\epsilon) - \sum_{j=0}^N a_j \epsilon^j = a_{N+1} \epsilon^{N+1} + o(\epsilon^{N+1}).$$

The right-hand side of this equation is $O(\epsilon^{N+1})$. So (1) holds.

The point of this discussion is that when we are working with *infinite* asymptotic series, we don't need to worry about the distinction between $O(\epsilon^{N+1})$ and $o(\epsilon^N)$. That distinction becomes important when the asymptotic approximations for some reason stop after finitely many terms. For example, consider

$$f(\epsilon) \equiv e^\epsilon + \epsilon^{\frac{3}{2}} \sin \frac{1}{\epsilon}.$$

Then

$$f(\epsilon) \sim 1 + \epsilon,$$

since

$$f(\epsilon) = 1 + \epsilon + o(\epsilon).$$

However, it is not true that

$$f(\epsilon) = 1 + \epsilon + O(\epsilon^2), \quad (**\mathbf{WRONG}**)$$

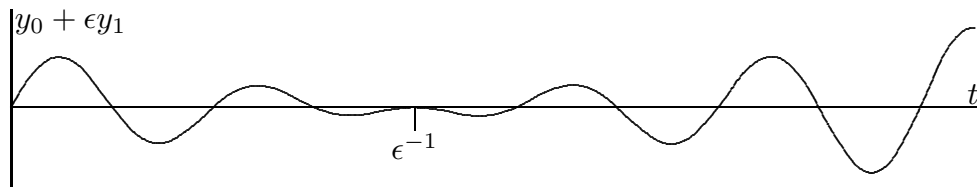
and there does not exist any number a_2 such that

$$f(\epsilon) = 1 + \epsilon + a_2\epsilon^2 + o(\epsilon^2). \quad (**\mathbf{WRONG}**)$$

POINTWISE AND UNIFORM CONVERGENCE

We are talking about functions, say $f(t)$, which also depend on a parameter, ϵ . (We could write $f(\epsilon; t)$ (a function of two variables) or $f_\epsilon(t)$ (a family of functions, indexed by ϵ .) In this situation, it becomes ambiguous to say that the difference between two f 's is "small", so that one is a good approximation to the other. The problem is that $f_\epsilon(t)$ (the difference function) may be close to 0 for some values of t but very large for other t 's.

We saw an example of this in the damped-oscillator problem. The first-order solution $(1 - \epsilon t) \sin t$ was a good approximation to the exact solution for very small t ($t \ll 1/\epsilon$, to be precise), and it correctly reproduced the damping (decay) of the oscillations during the initial stage of the motion. However, as t increases, the first-order correction term actually *grows* and becomes larger than the unperturbed solution, although we know that the exact solution should continue to show damping. For any fixed t , there are ϵ 's so small that the first-order solution is a good approximation on the interval $[0, t]$. However, for a fixed ϵ , there are always t 's so big that the first-order solution becomes very bad. Unfortunately, the second point of view is the one we usually want to take in practice: We have a *given* physical system, and we want to know how it behaves for all time.



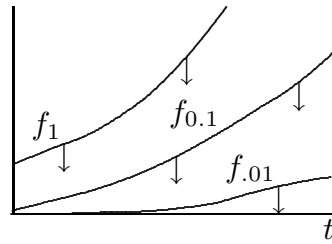
To talk about this type of trouble clearly, we need a precise mathematical vocabulary. This may sound like abstruse theoretical mathematics, but it concerns a very real practical problem of finding and validating a good approximate solution. We see that there are (at least) two different ways of defining *limits* of functions of two variables:

Definition:

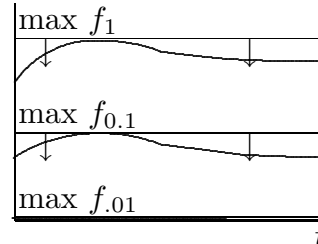
$$f(\epsilon; t) \rightarrow 0 \text{ pointwise as } \epsilon \rightarrow 0$$

means: For any given t , $f(\epsilon; t)$ approaches 0 (in the usual sense of a limit of a function of a single variable ϵ). In other words, for every $\eta > 0$ there is an ϵ_0 (which may depend on t as well as η) such that $|f(\epsilon; t)| < \eta$ whenever $|\epsilon| < \epsilon_0$.

(Note, by the way, that we don't need to discuss any number or function other than 0 as the value of the limit: $f(\epsilon; t) \rightarrow L(t)$ is synonymous with $f(\epsilon; t) - L(t) \rightarrow 0$, which is defined by the foregoing.)



pointwise convergence



uniform convergence

Definition:

$$f(\epsilon; t) \rightarrow 0 \text{ uniformly on the interval } [a, b] \text{ as } \epsilon \rightarrow 0$$

means: For every $\eta > 0$ there is an ϵ_0 (independent of t) such that $|f(\epsilon; t)| < \eta$ for all t in $[a, b]$ whenever $|\epsilon| < \epsilon_0$. In other words, the *maximum value* of $f(\epsilon; t)$ for $a \leq t \leq b$ approaches 0 as $\epsilon \rightarrow 0$. [The same sort of definition applies when one of the endpoints is infinite, for instance $a \leq t < \infty$.]

Warning: We have to assume that the functions are continuous and that the interval is closed and finite in order to guarantee that the functions *have* maximum values. (What would be the maximum value of the function $1 - e^{-t}$ on $[0, \infty)$?) When these conditions are violated, we should talk about the *supremum* or *least upper bound* of the functions instead of the maximum. (This is the smallest number that the function does not go beyond — the same thing as the maximum when there is one, and 1 in the example I just gave.) I have included the maximum definition in spite of this technical complication, because it is easier to visualize and think about than the η - ϵ_0 definition.

So, to restate our problem and bring the two sets of definitions together, regular perturbation theory often gives approximations or expansions that are *pointwise asymptotic* to the desired exact answers. We would like to modify the method to produce approximations that are *uniformly asymptotic* on some reasonably large interval of the independent variable.

This is a good time to clear the air on another technical matter. So far I have given you a rigorous definition of “asymptotic”, but I have not actually *proved* that any of our results are asymptotic to the exact solutions — except for the introductory example of a cubic algebraic equation. At most, we have relied on some common-sense arguments to make it *plausible* that our approximations are numerically good. In particular, we have often looked at the *first neglected term* in a series to judge whether the terms we’ve kept are a good approximation. If the neglected term is small, we expect that the approximation is good; if the neglected term grows as a power of t , we denounce the approximation as nonuniform. Such reasoning is dangerous: The next term, or all the remaining terms in a series, could be small, but the difference between the truncated series and the exact function might still be large. Proving that a given approximation really is within $O(\epsilon^n)$ of a genuine solution is often very hard; this brings us right up into topics of current research. (In other words, often I don’t know the answer, and sometimes nobody does.) Treating such matters is beyond the scope of this one-month introduction to the subject. In practice, engineers often rely on experimental evidence and a common-sense intuition for the physics of their problems to judge the correctness of an approximation for which the mathematical theory is underdeveloped.

Futhermore, in the process of finding asymptotic approximations it is common to write “ $+O(\epsilon^n)$ ” at the end of a truncated series to mean, “If we were to carry this expansion to still higher order, the next term would be proportional to ϵ^n .” (This notation conveys more information than the alternative, “ $+\dots$ ”.) Thus, despite the definitions earlier in this section, the $O(\epsilon^n)$ notation in this informal usage does not necessarily amount to a claim that the approximation has been proved asymptotic to the order ϵ^n . This ambiguity, while regrettable, should not cause discomfort to readers who are under no professional obligation to construct such proofs.

Searching for Uniform Approximations by Modifying the Time Scale

In our example of the damped harmonic oscillator, the solution of the unperturbed equation was

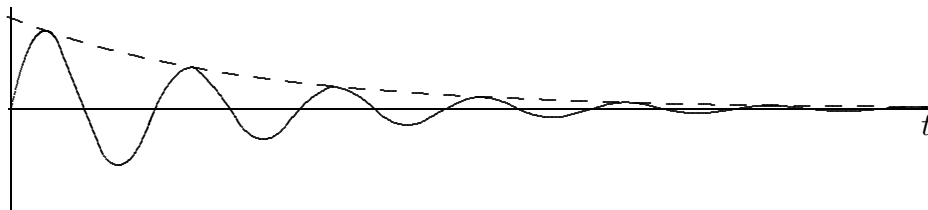
$$y_0(t) = \sin t$$

and the exact solution of the perturbed equation was

$$y(t) = \frac{e^{-\epsilon t}}{\sqrt{1-\epsilon^2}} \sin(\sqrt{1-\epsilon^2} t)$$

Comparing them, we see that the perturbing force has two effects:

- (1) The motion is *damped* by the factor $e^{-\epsilon t}$. This factor varies on a time scale of the order $1/\epsilon$, in contrast to the oscillations of the other factor, whose characteristic time scale is of the order unity (ϵ^0). More generally, when a rapidly varying function is multiplied by a slowly varying one, one says that the fast function is *modulated* by the slow one (in the sense of AM radio). The slow function defines an *envelope*, a curve within which the oscillations of the fast function are confined.



We have seen how expanding the exponential factor in a power series gives rise to secular terms that eventually become large and ruin the approximation.

- (2) The oscillations are *slowed down*, since the frequency changes from 1 to $\sqrt{1-\epsilon^2}$. Note that this is a *second-order effect* ($\sqrt{1-\epsilon^2} \sim 1 - \frac{1}{2}\epsilon^2$), while the damping is first-order. It is easy to see that such a frequency change will also give rise to secular terms: When t is small, the effect of the frequency change is very small, but as t increases, the two functions ($\sin t$ and $\sin(\sqrt{1-\epsilon^2} t)$) get farther and farther out of phase, so that eventually the difference between them is as large as $\sin t$ itself.

These two features of the true solution suggest two maneuvers that might lead to better approximate solutions: changing to a different time variable (from feature (2)), and working with more than one time variable at once (from feature (1)). We'll consider them in turn.

Be forewarned that there are no clear rules for telling in advance precisely what method will work on a particular problem. An effective method is often found by trial and error. There are many possible variations of these two general methods, and sometimes both are used together.

THE POINCARÉ DISTORTED-TIME METHOD

The first method is variously named after Poincaré, Lindstedt, Lighthill, or Stokes; it's also called the method of *distorted time*, or, more generally, the method of *strained coordinates*. The idea is to expand the independent variable, as well as the dependent one, as a power series in the perturbation parameter: Introduce a new variable τ by

$$\tau = t + \epsilon\omega_1 t + \epsilon^2\omega_2 t + \dots,$$

where the ω 's are unknown constants or functions to be determined later. In this way one hopes to discover the “true” time variable of the perturbed motion, such as $\sqrt{1 - \epsilon^2} t \equiv \tau$ in the example.

Dealing with two power series gives us twice as many unknown coefficients to solve for, and therefore some additional input is needed to determine unique solutions. The second key idea of the method is that these additional conditions can be obtained from the requirement that there be *no resonant forcing terms* in the equations for the higher-order corrections, and hence no secular terms in their solutions.

In many textbooks this method is applied to the cubic oscillator equation. For the sake of another example, I once wanted to apply it to the damped oscillator,

$$\frac{d^2y}{dt^2} + 2\epsilon\frac{dy}{dt} + y = 0, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1,$$

where we know the exact answer. However, a straightforward application of the method in this case didn't work; the algebraic condition I needed to solve, in order to kill off the secular terms, had no solution. (More on this later.) Looking at the exact solution, one suspects that the trouble stems from the damping factor. Therefore, I redefined the *dependent* variable so as to remove that factor: Let

$$y(t) \equiv e^{-\epsilon t} u(t).$$

Then a short calculation shows that the problem becomes

$$\frac{d^2u}{dt^2} + (1 - \epsilon^2)u = 0, \quad u(0) = 0, \quad \frac{du}{dt}(0) = 1,$$

and we know that the exact solution must be

$$u(t) = \frac{\sin(\sqrt{1 - \epsilon^2} t)}{\sqrt{1 - \epsilon^2}}.$$

Now we are ready to apply the method. Since ϵ appears in the equation only as ϵ^2 , we first try to get away with power series just in ϵ^2 (i.e., leave out the odd

powers of ϵ ; if you put in the odd powers, you find out that their coefficients are zero, after a lot of extra algebra.)

$$u = u_0 + \epsilon^2 u_2 + O(\epsilon^4).$$

The new time variable is

$$\tau = t + \epsilon^2 \omega_2 t + O(\epsilon^4),$$

where ω_2 is independent of t . Thus

$$\frac{du}{dt} \approx (1 + \epsilon^2 \omega_2) \frac{du}{d\tau},$$

so to order ϵ^2 the initial condition is

$$\begin{aligned} 1 = \frac{du}{dt}(0) &\sim (1 + \epsilon^2 \omega_2) \left(\frac{du_0}{d\tau}(0) + \epsilon^2 \frac{du_2}{d\tau}(0) \right) \\ &\sim \frac{du_0}{d\tau}(0) + \epsilon^2 \left(\omega_2 \frac{du_0}{d\tau}(0) + \frac{du_2}{d\tau}(0) \right) \end{aligned}$$

Therefore,

$$\frac{du_0}{d\tau}(0) = 1, \quad \frac{du_2}{d\tau}(0) = -\omega_2.$$

Similarly,

$$\frac{d^2 u}{dt^2} = (1 + \epsilon^2 \omega_2)^2 \frac{d^2 u}{d\tau^2} \sim (1 + 2\epsilon^2 \omega_2) \left(\frac{d^2 u_0}{d\tau^2} + \epsilon^2 \frac{d^2 u_2}{d\tau^2} \right),$$

and after another step we get (for the differential equation)

$$\begin{aligned} 0 &= \frac{d^2 u}{dt^2} + (1 - \epsilon^2)u \\ &\sim \frac{d^2 u_0}{d\tau^2} + u_0 + \epsilon^2 \left(2\omega_2 \frac{d^2 u_0}{d\tau^2} + \frac{d^2 u_2}{d\tau^2} + u_2 - u_0 \right). \end{aligned}$$

The lowest-order equation is

$$\epsilon^0 : \quad \frac{d^2 u_0}{d\tau^2} + u_0 = 0$$

whose solution (with the given initial data) is

$$u_0 = \sin \tau.$$

This is not yet a usable approximation, since we don't know what ω_2 is, hence what τ is in terms of the true time variable t .

The next equation is

$$\begin{aligned} \epsilon^2 : \quad \frac{d^2 u_2}{d\tau^2} + u_2 &= -2\omega_2 \frac{d^2 u_0}{d\tau^2} + u_0 \\ &= (2\omega_2 + 1) \sin \tau \end{aligned}$$

We now make the crucial argument: Since the forcing term is resonant, the solution for u_2 will grow with τ , making u_0 a *bad* approximation. The only way out is to choose ω_2 so that the resonant forcing term vanishes:

$$\omega_2 = -\frac{1}{2}.$$

This kills both birds with one stone; or perhaps it would be better to say that we let the two birds kill each other!

We solve the u_2 equation, which is now homogeneous:

$$u_2 = C_2 \cos \tau + D_2 \sin \tau.$$

Recalling the initial conditions on u_2 , we get

$$C_2 = 0, \quad D_2 = -\omega_2 = \frac{1}{2}.$$

Thus

$$\tau \sim \left(1 - \frac{\epsilon^2}{2}\right) t,$$

and hence

$$u \sim \left(1 + \frac{\epsilon^2}{2}\right) \sin \left[\left(1 - \frac{\epsilon^2}{2}\right) t \right].$$

This agrees with the exact solution up through order ϵ^2 . It is a more nearly uniform approximation than the solution of the same order given by regular perturbation theory,

$$\sin t + \frac{\epsilon^2}{2} \sin t - \frac{\epsilon^2}{2} t \cos t,$$

which has a secular term. Eventually our new approximation will get out of phase with the exact solution, but only on the time scale ϵ^{-4} , not ϵ^{-2} . We could get a better approximation by carrying the distorted-time calculation out through order ϵ^4 , and so on.

Finally, a report on the attempt to apply this method directly to the damped oscillator. [I invite you to try this yourself, but I refrain from adding it to the homework assignment.] Expanding y and τ as power series in ϵ , I arrived at

$$\epsilon^1 : \quad y_1'' + y_1 = -2\omega_1 y_0'' - 2y_0' = 2\omega_1 \sin \tau - 2 \cos \tau.$$

Since the two terms here are linearly independent, and both are resonant, there is no constant ω_1 that will eliminate the secular terms.

The method can be redeemed, however, if you temporarily ignore the initial condition and look for a basis of solutions with zeroth-order terms $y_0 = e^{\pm i\tau}$ instead of $\sin \tau$. Then the first-order equation becomes

$$y_1'' + y_1 = (2\omega_1 \mp 2i)e^{\pm i\tau}.$$

So, if we're willing to swallow a *complex* time coordinate, we can get rid of the resonant term by taking $\omega_1 = \pm i$. The y_1 equation is now homogeneous, and since we don't care at the moment about initial data, we can choose y_1 to be 0. (This point is worth noting, since it applies in many other problems!) Pressing on, you find that $\omega_2 = -\frac{1}{2}$ and y_2 can be taken to be 0 for the same reason. Thus

$$\tau \sim \left(1 - \frac{\epsilon^2}{2}\right) t \pm i\epsilon t$$

and you have found two linearly independent solutions

$$y_{\pm} = e^{\pm i\tau} \sim e^{-\epsilon t} e^{\pm i(1 - \frac{1}{2}\epsilon^2)t}.$$

Finding the linear combination that satisfies the original initial conditions, you come up with $e^{-\epsilon t}$ times the solution for u that we arrived at earlier:

$$y(t) \sim \left(1 + \frac{\epsilon^2}{2}\right) e^{-\epsilon t} \sin \left[\left(1 - \frac{\epsilon^2}{2}\right) t \right],$$

which is a good second-order approximation to our familiar exact solution.

THE METHOD OF TWO TIME SCALES

We return to the observation that the parts of the exact solution of the damped-oscillator problem seem to move on two different time scales. A method which seeks to reveal this structure in a problem where the answer is not known beforehand is called *two-timing*, or the method of *multiple scales*. I'll apply it to the damped oscillator, seeking on this first try a solution through order ϵ only.

Our problem is

$$\frac{d^2y}{dt^2} + 2\epsilon \frac{dy}{dt} + y = 0, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1.$$

We assume that $y(t)$ can be expressed in terms of a function of two variables, $y(u, v)$, where u and v are some functions of t . In seeking a solution to higher order, we might take u and v to be unknown functions and find them by assuming a power-series form for them; this would be a generalization of the distorted-time method. For a first-order solution, however, we can simply take v to be t itself and u to be ϵt :

$$u \equiv \epsilon t, \quad v \equiv t.$$

Then by the multivariable chain rule,

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial y}{\partial u} \frac{du}{dt} + \frac{\partial y}{\partial v} \frac{dv}{dt} \\ &= \epsilon \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}. \end{aligned}$$

Since v is just t , this is often written

$$\frac{dy}{dt} = \frac{\partial y}{\partial t} + \epsilon \frac{\partial y}{\partial u},$$

the two different types of derivative with respect to t being distinguished by the shapes of their “dees”. (I’ve also switched the order of the terms so that the lower-order one comes first.)

[Perhaps it is advisable to pause here to review the multivariable chain rule. If the temperature in the air is $T(\mathbf{x}) = T(x, y, z)$ and you are flying an airplane along the path $\mathbf{x}(t)$, hence with velocity $\mathbf{v}(t) \equiv d\mathbf{x}/dt \equiv \mathbf{x}'(t)$, then the time derivative of the temperature recorded on a thermometer carried by the plane is

$$\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{\partial T}{\partial z} \frac{dz}{dt} = \nabla T \cdot \mathbf{v}.$$

The situation in the singular-perturbation calculation is the same, except that the intermediate variables u and v do not have an independent physical significance — they are just a mathematical device.]

Let’s make this substitution into the original differential equation. We must differentiate with respect to t once more:

$$\frac{d^2y}{dt^2} = \frac{\partial^2 y}{\partial t^2} + 2\epsilon \frac{\partial^2 y}{\partial t \partial u} + \epsilon^2 \frac{\partial^2 y}{\partial u^2}.$$

[Apply the multivariable chain rule to each term of our formula for $\frac{dy}{dt}$. Two of the four terms combine.] Then the equation is

$$0 = \frac{\partial^2 y}{\partial t^2} + 2\epsilon \frac{\partial^2 y}{\partial t \partial u} + \epsilon^2 \frac{\partial^2 y}{\partial u^2} + 2\epsilon \frac{\partial y}{\partial t} + 2\epsilon^2 \frac{\partial y}{\partial u} + y.$$

The nontrivial initial condition is

$$1 = \frac{dy}{dt}(0) = \frac{\partial y}{\partial t}(0, 0) + \epsilon \frac{\partial y}{\partial u}(0, 0).$$

Now we expand y in a power series as usual:

$$y(u, t) = y_0(u, t) + \epsilon y_1(u, t) + O(\epsilon^2).$$

Separate out the terms of order ϵ^0 :

$$0 = \frac{\partial^2 y_0}{\partial t^2} + y_0, \quad y_0(0, 0) = 0, \quad \frac{\partial y_0}{\partial t}(0, 0) = 1. \quad (0)$$

And the terms of order ϵ^1 :

$$0 = \frac{\partial^2 y_1}{\partial t^2} + 2 \frac{\partial^2 y_0}{\partial t \partial u} + 2 \frac{\partial y_0}{\partial t} + y_1, \quad (1)$$

$$y_1(0, 0) = 0, \quad \frac{\partial y_1}{\partial t}(0, 0) + \frac{\partial y_0}{\partial u}(0, 0) = 0.$$

Solve (0):

$$y_0(u, t) = C_0(u) \sin t + D_0(u) \cos t.$$

Although (0) is written as a partial differential equation, it is really only an easy ordinary differential equation as far as solution technique is concerned, since derivatives with respect to u do not appear. However, it is important to remember that the unknown constants of integration may depend on u . In this case, the initial condition $y_0(0, 0) = 0$ tells us that

$$D_0(0) = 0.$$

It does not guarantee that $D_0(u) = 0$ for all u ! Similarly, the other initial condition says that $C_0(0) = 1$. To determine the functions for $u \neq 0$ we need to look ahead at the problem of finding y_1 .

We can now rewrite (1) as

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = -2(C_0'(u) + C_0(u)) \cos t + 2(D_0'(u) + D_0(u)) \sin t.$$

All the terms here are resonant, so to avoid secular terms in the solution we must require that the coefficients equal zero — the same idea as in the Poincaré method, but this time implemented differently. We get ordinary differential equations

$$\begin{aligned} C_0' &= -C_0, & C_0(0) &= 1, \\ D_0' &= -D_0, & D_0(0) &= 0. \end{aligned}$$

Thus

$$C_0(u) = e^{-u}, \quad D_0(u) = 0.$$

Therefore,

$$y_0 = e^{-u} \sin t = e^{-\epsilon t} \sin t. \quad (*)$$

This agrees with the exact solution when terms of order ϵ^2 are neglected!

The equation for y_1 is now homogeneous, with solution

$$y_1(u, t) = C_1(u) \sin t + D_1(u) \cos t.$$

Recall that the initial data are

$$y_1(0, 0) = 0, \quad \frac{\partial y_1}{\partial t}(0, 0) = -\frac{\partial y_0}{\partial u}(0, 0) = 0.$$

Thus $C_1(0) = D_1(0) = 0$. Thus y_1 starts off small; in this example (*) is already the whole solution to first order in ϵ , at least initially. To determine $C_1(u)$ and $D_1(u)$ we need to consider a second-order approximation and demand that it contain no secular terms.

How would we do that? The method recommended in Sec. 3.2 of J. D. Cole, *Perturbation Methods in Applied Mathematics*, is to make the second variable a power series in ϵ

$$v \equiv t + \epsilon^2 \omega_2 t + \dots$$

as in the Poincaré method, but with the ϵ^1 term omitted since its effect is presumed already contained in the factor e^{-u} . See Cole's book for the details of the ensuing calculation. Looking back at the exact solution, which contains $\sqrt{1 - \epsilon^2} t$, we see that it is quite reasonable to expect this maneuver to succeed. But how does one know what kind of power series to substitute, and where, when one does not know the exact answer beforehand? This is a legitimate question to ask, but you will not get an answer. As Cole says, "A disadvantage of the [two-time] method is that the proper choice of fast and slow variables is not always obvious."

Another approach is to construct a second-order approximation by introducing *three* time variables,

$$t, \quad u \equiv \epsilon t, \quad w \equiv \epsilon^2 t.$$

I have carried this out for the damped oscillator; it works, and by virtue of a trig identity gives the same answer as Cole's mixture of the two-variable and distorted-time methods. However, the calculations are more complex and convoluted than in Cole's method. (The constants of integration in y_0 depend on both u and w ; the removal of secular terms from the first-order approximation yields a differential equation in u that determines the constants up to new constants of integration that are functions of w ; and then you need to look at the second-order approximation to determine those.)

The Liouville–Green or WKB Approximation

Now we’re going to consider differential equations of one of the following forms:

$$\frac{d^2y}{dt^2} + [\omega^2 - V(t)]y = 0 \tag{1}$$

or

$$\frac{d^2y}{dt^2} + \omega^2 V(t)y = 0, \tag{2}$$

where V is some given function and ω is a *large* constant. (Thus $\epsilon \equiv 1/\omega$ will be our small parameter.) These equations are linear; what makes them nontrivial is that the coefficient function V is nonconstant. The perturbation theory associated with them has even more names attached to it than the distorted-time method has. I will not list them all; extensive historical remarks appear in Chapter 1 of W. Wasow, *Linear Turning Point Theory*. Having been raised as an American physicist, I tend to call it the *WKB approximation*. (Wentzel, Kramers, and Brillouin were theoretical physicists of the 1920s. Actually, Liouville and Green got the subject started 100 years earlier.)

This type of equation is often encountered in studying the propagation of waves through an inhomogeneous medium — for example, the earth or the atmosphere, modeled by layers of material of different densities. (In such an application the independent variable is usually a *spatial* coordinate, but I have called it t to match the notation in the rest of this part of the course.) More generally, given any second-order homogeneous linear ODE

$$a(t)y'' + b(t)y' + c(t)y = 0,$$

it is possible in principle to make a change of variables that will concentrate the t dependence in the y term and eliminate the y' term entirely; thus (1) and (2) cover a wide territory!

For the time being we will assume that $V(t)$ in (2) and $\omega^2 - V(t)$ in (1) are everywhere *positive*.

If $V = 0$ in (1), then the solutions are linear combinations of $\sin \omega t$ and $\cos \omega t$ — or, equivalently, $e^{i\omega t}$ and $e^{-i\omega t}$. Consequently, we expect those functions to be the zeroth-order terms in a perturbation expansion in $1/\omega$ (appropriate to the limit where V is small compared to ω^2).

Sure enough, a first-order approximation can be found by the distorted-time method: Let

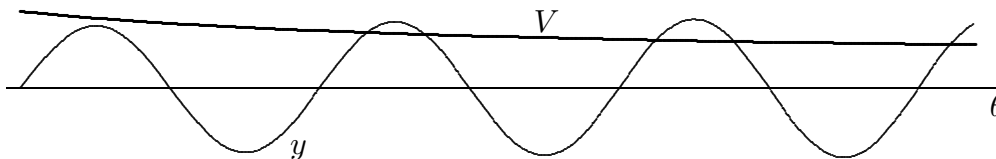
$$y = y_0 + \epsilon^2 y_2 + \dots, \\ \tau = \frac{t}{\epsilon} + \epsilon \tau_1(t) + \dots.$$

There are two new wrinkles in this ansatz for τ . First, we start out with a term of order ϵ^{-1} ($= \omega$), as suggested by the zeroth-order solution already noted. Second, and more significantly, we must be prepared for $\tau_1(t)$ to be a nonlinear function of t . (In applying the Poincaré method to (linear or nonlinear) ODEs with *constant coefficients*, there was no reason to expect the stretching of the time scale to be different for different times, hence the only possibility was a constant scaling, $\tau_1(t) = \omega_1 t$.) I leave you to finish the calculation as a homework problem.

Now consider (2). This time the solutions would be trigonometric functions of $\omega\sqrt{V}t$ if V were constant. If ω is large, these functions oscillate rapidly, and the fixed function $V(t)$ really is almost constant over one period of the oscillation. It is reasonable to expect then that the constant- V solutions provide a zeroth-order approximation to the true solutions; V is varying so slowly (compared to y) that the oscillator y hardly notices. However, it turns out that we can get a better approximation by taking the argument of the trig functions to be

$$\omega \int_0^t \sqrt{V(\tilde{t})} d\tilde{t}$$

instead of $\omega\sqrt{V(t)}t$. (If V is constant, these are the same thing.)



This argument may be more convincing if we make the variable change $\theta \equiv \omega t$ in (2). We get

$$\frac{d^2 y}{d\theta^2} + V\left(\frac{\theta}{\omega}\right) y = 0. \quad (3)$$

From this perspective, the limit $\omega \rightarrow +\infty$ is clearly the limit where V is *slowly varying*. On the other hand, (2) is a better starting point for a systematic perturbation expansion, because the small parameter is isolated as a numerical factor instead of hidden inside the argument of the coefficient function. (Working from (3), you might be tempted to expand V as a power series in θ/ω . This is virtually guaranteed to produce a nonuniform approximation!)

A TWO-TIME SOLUTION

As I said, the natural time coordinate in the solutions of (2) turns out to be

$$\tau \equiv \omega \int_0^t \sqrt{V(\tilde{t})} d\tilde{t}.$$

We'll do a multiple-scale calculation with this as the fast variable and t itself as the slow variable. (One way in which this choice might have been discovered is indicated in one of the homework problems. I'll show another way later.)

Remark: The lower limit 0 in integrals like this one is arbitrary. *Any* antiderivative of $\omega\sqrt{V}$ would do. In fact, there are situations where it is helpful to choose the lower limit to be something other than 0 — for example, to choose it to be the point where the initial data are prescribed.

So, let $y = y(t, \tau)$. Then

$$\begin{aligned}\frac{dy}{dt} &= \frac{\partial y}{\partial t} + \omega\sqrt{V} \frac{\partial y}{\partial \tau}, \\ \frac{d^2y}{dt^2} &= \frac{\partial^2 y}{\partial t^2} + 2\omega\sqrt{V} \frac{\partial^2 y}{\partial t \partial \tau} + \omega^2 V \frac{\partial^2 y}{\partial \tau^2} + \frac{\omega V'}{2\sqrt{V}} \frac{\partial y}{\partial \tau}.\end{aligned}$$

The ODE is

$$\begin{aligned}0 &= \frac{d^2y}{dt^2} + \omega^2 V y \\ &= \frac{\partial^2 y}{\partial t^2} + 2\omega\sqrt{V} \frac{\partial^2 y}{\partial t \partial \tau} + \omega^2 V \frac{\partial^2 y}{\partial \tau^2} + \frac{\omega V'}{2\sqrt{V}} \frac{\partial y}{\partial \tau} + \omega^2 V y.\end{aligned}$$

We expand

$$y = y_0 + \frac{1}{\omega} y_1 + O(\omega^{-2})$$

and keep only the lowest two orders in the ODE; these are the terms associated with positive powers of ω .

$$0 \sim 2\omega\sqrt{V} \frac{\partial^2 y_0}{\partial t \partial \tau} + \omega^2 V \frac{\partial^2 y_0}{\partial \tau^2} + \omega V \frac{\partial^2 y_1}{\partial \tau^2} + \frac{\omega V'}{2\sqrt{V}} \frac{\partial y_0}{\partial \tau} + \omega^2 V y_0 + \omega V y_1.$$

The equation of order ω^2 is (after cancelling a V)

$$\frac{\partial^2 y_0}{\partial \tau^2} + y_0 = 0,$$

whence

$$y_0(t, \tau) = C(t)e^{\pm i\tau}.$$

(We have not prescribed any initial conditions, so we're looking for any convenient basis, consisting of two independent solutions.) The ω^1 equation is (after dividing by V and separating the known terms from the unknown)

$$\begin{aligned}\frac{\partial^2 y_1}{\partial \tau^2} + y_1 &= -2V^{-\frac{1}{2}} \frac{\partial^2 y_0}{\partial t \partial \tau} - \frac{1}{2} V^{-\frac{3}{2}} V' \frac{\partial y_0}{\partial \tau} \\ &= -2V^{-\frac{1}{2}} (\pm i C'(t)) e^{\pm i\tau} - \frac{1}{2} V^{-\frac{3}{2}} V' (\pm i C(t)) e^{\pm i\tau}.\end{aligned}$$

To kill off the secular terms we take the right-hand side to be 0:

$$-2V^{-\frac{1}{2}}C'(t) - \frac{1}{2}V^{-\frac{3}{2}}V'C(t) = 0;$$

$$\frac{C'}{C} = -\frac{V'}{4V};$$

$$\ln |C| = -\frac{1}{4} \ln V + \text{constant};$$

$$C(t) = \text{Constant} \times V(t)^{-\frac{1}{4}}.$$

Since the normalization of our basis solutions is arbitrary, we take the constant to be 1. We arrive at the famous *WKB approximation*,

$$\boxed{y_0 = V(t)^{-\frac{1}{4}} e^{\pm i\omega \int_0^t \sqrt{V(\tilde{t})} d\tilde{t}}.} \quad (***)$$

The equation for y_1 is now the same as that for y_0 , so adding the y_1 term will simply change the normalization of the basis solutions again. In the absence of ϵ -dependent initial data, therefore, we may ignore y_1 .

I'll give a proof of the asymptotic validity of this approximation much later in the course, after we've studied Green functions.

THE AMPLITUDE-PHASE DECOMPOSITION; HIGHER-ORDER APPROXIMATIONS

The best way to extend the WKB approximation to higher orders is suggested by writing the sought-for solution as the product of an *amplitude* and a *phase*:

$$y(t) = A(t)e^{i\omega S(t)},$$

$$A \text{ and } S \text{ real,} \quad A > 0.$$

(We assume that $y(t)$ is never zero. This will be true of the solution we will construct.)

We calculate

$$y' = (i\omega S' A + A')e^{i\omega S},$$

$$y'' = (-\omega^2(S')^2 A + 2i\omega S' A' + i\omega S'' A + A'')e^{i\omega S}.$$

So the ODE is

$$\begin{aligned} 0 &= (y'' + \omega^2 V y) / e^{i\omega S} \\ &= -\omega^2((S')^2 - V)A + i\omega(2S' A' + S'' A) + A''. \end{aligned}$$

Since A and S are real, the only imaginary numbers here are the ones we see, and they must add to 0 by themselves:

$$\frac{S''}{S'} = -2 \frac{A'}{A}.$$

Therefore,

$$\begin{aligned} \ln |S'| &= -2 \ln A + \text{constant}; \\ S' &= CA^{-2} \quad \text{or} \quad A \propto \frac{1}{\sqrt{|S'|}}. \end{aligned} \quad (**)$$

This is a rather remarkable result: It says that no matter what the function V is, we know *exactly* how the amplitude and the derivative of the phase function are related. (In this regard it is somewhat like Abel's identity for the Wronskian of two solutions, which gives a calculable formula even when the solutions themselves can't be written down exactly.)

Now let's go back to the real part of the ODE:

$$0 = -\omega^2((S')^2 - V)A + A''. \quad (\dagger)$$

To lowest order, this says that $S' = \pm\sqrt{V}$, or (if we neglect two constants of integration)

$$S = \pm \int_0^t \sqrt{V(\tilde{t})} d\tilde{t}, \quad A = V(t)^{-\frac{1}{4}}.$$

Putting these together, we see that we have rediscovered (***):

$$y(t) \sim V^{-\frac{1}{4}} e^{\pm i\omega \int \sqrt{V}}.$$

It is reasonable to expect that the next-order correction to S will be of order $\epsilon^2 = \omega^{-2}$. In fact, an asymptotic approximation to y of arbitrarily high order (the *Fröman phase-integral approximation*) can be obtained by substituting

$$S' = \pm\sqrt{V} + \epsilon^2 S'_2(t) + \epsilon^4 S'_4(t) + \dots.$$

The calculations can be substantially simplified by observing that

$$\ln y = i\omega S + \ln A = i\omega(S - i\epsilon \ln A),$$

and thus

$$-i\epsilon(\ln y)' = S' - i\epsilon(\ln A)' \equiv N(t).$$

The quantity N that we have just defined possesses a power series in ϵ , wherein the even powers are provided by the S' term and the odd ones by the $(\ln A)'$ term:

$$N = \pm\sqrt{V} + \frac{i}{4}\epsilon\frac{V'}{V} - \epsilon^2 N_2(t) - i\epsilon^3 N_3(t) + \cdots + (i\epsilon)^n N_n(t) + \cdots.$$

Note that

$$y' = i\omega N y,$$

so N is essentially the logarithmic derivative of y . Now go back to the original differential equation to get a first-order differential equation for N :

$$-\omega^2 V y = y'' = (i\omega N y)' = -\omega^2 N^2 y + i\omega N' y,$$

or (divide by $\omega^2 y$)

$$i\epsilon N' = N^2 - V. \quad (*)$$

We substitute $N \sim \sum_{n=0}^{\infty} (i\epsilon)^n N_n$ into $(*)$ and get

$$\begin{aligned} N' &\sim \sum_{n=0}^{\infty} (i\epsilon)^n N'_n; \\ N^2 &\sim \left(\sum_{j=0}^{\infty} (i\epsilon)^j N_j \right) \left(\sum_{k=0}^{\infty} (i\epsilon)^k N_k \right) \\ &= \sum_{n=0}^{\infty} (i\epsilon)^n \sum_{j+k=n} N_j N_k \\ &= \sum_{n=0}^{\infty} (i\epsilon)^n \sum_{j=0}^n N_j N_{n-j}; \end{aligned}$$

hence

$$\sum_{n=1}^{\infty} (i\epsilon)^n N'_{n-1} = \sum_{n=0}^{\infty} (i\epsilon)^n \sum_{j=0}^n N_j N_{n-j} - V.$$

Now separate this into orders:

$$\epsilon^0 : \quad 0 = N_0^2 - V$$

— that is,

$$N_0 = \pm\sqrt{V},$$

as predicted. (From now on I will simplify things by considering only the positive square root.)

$$\epsilon^n, n > 0 : \quad \sum_{j=0}^n N_j N_{n-j} = N'_{n-1}.$$

This equation is to be solved for N_n .

In particular,

$$\epsilon^1 : \quad 2N_0N_1 = N_0',$$

or

$$N_1 = \frac{1}{2}(\ln N_0)' = \frac{1}{4}(\ln V)' = \frac{1}{4} \frac{V'}{V},$$

which we also knew already. Now to get something new:

$$\epsilon^2 : \quad 2N_0N_2 + N_1^2 = N_1',$$

or

$$\begin{aligned} N_2 &= \frac{N_1' - N_1^2}{2N_0} \\ &= \frac{\frac{1}{4} \frac{V''}{V} - \frac{1}{4} \left(\frac{V'}{V}\right)^2 - \frac{1}{16} \left(\frac{V'}{V}\right)^2}{2\sqrt{V}} \\ &= \frac{1}{8} V'' V^{-\frac{3}{2}} - \frac{5}{32} (V')^2 V^{-\frac{5}{2}}. \end{aligned}$$

This can also be written

$$N_2 = -\frac{1}{2} V^{-\frac{1}{4}} \frac{d^2}{dt^2} V^{-\frac{1}{4}}.$$

Remembering that S' is built out of the even-order terms of N , we get

$$S(t) \sim \int_0^t \left[\sqrt{V} - \epsilon^2 \left(\frac{1}{8} V'' V^{-\frac{3}{2}} - \frac{5}{32} (V')^2 V^{-\frac{5}{2}} \right) \right] d\tilde{t}.$$

When we substitute this into

$$y \sim \frac{1}{\sqrt{S'}} e^{i\omega S},$$

we get an expression valid through third order. (That is, the correction to A coming from the S'_2 term agrees with what you would obtain by calculating N_3 and using

$$(\ln A)' \sim i\omega [i\epsilon N_1 + (i\epsilon)^3 N_3].$$

We need to find the odd-order N 's to calculate the even-order ones, but the information they give about A is then redundant.)

Remark: In a homework problem I asked you to show that the equation (2) is actually equivalent to an equation of the form (1),

$$\frac{d^2 z}{d\tau^2} + [\omega^2 - U(\tau)]z = 0,$$

where $z = V^{\frac{1}{4}}y$, $\tau = \int \sqrt{V} dt$, and U is built out of derivatives of V . (U is most easily expressed in terms of t , but that makes it implicitly a function of τ .) Up to a factor $\frac{1}{2}\sqrt{V}$, the answer you found for U is identical to our result for N_2 above. This is consistent with the result of another homework problem, where I asked you to replace V by $1 - \epsilon^2 U$ in the first-order WKB approximation (**), getting the second-order correction (in today's notation)

$$\frac{dS}{d\tau} \sim 1 - \frac{1}{2}\epsilon^2 U.$$

(The disappearance of a factor \sqrt{V} is due to the chain-rule factor $\frac{dt}{d\tau}$.) Higher-order approximations for (1) can be found exactly as for (2), except that the coefficient function enters the recursion relation for N_2 instead of that for N_0 .

In a paper by Campbell, *Journal of Computational Physics* **10**, 308 (1972), you can find the even-order N_n 's calculated up to $n = 20$. (That last one fills a whole page!) The calculation was done by a computer; it's an excellent (though now somewhat outdated) example of how a computer can be used to do massive algebra as well as massive arithmetic. Actually, Campbell substituted the Fröman series for S' into the original second-order ODE, rather than taking the (easier) approach of substituting the series for N into the first-order equation (*). Also, he (in effect) applied the transformation described in the foregoing Remark, so that his answers are given in terms of τ -derivatives of U , rather than t -derivatives of V ; this keeps the expressions simpler than they would otherwise be.

WHAT HAPPENS WHEN V IS NOT POSITIVE?

If V is strictly negative, \sqrt{V} is imaginary. All our previous results then continue to apply, provided $i\epsilon$ is replaced by ϵ throughout. (This means ϵ^2 becomes $-\epsilon^2$, etc.) In the lowest order, the basis solutions are approximated by

$$|V|^{-\frac{1}{4}} e^{\pm \omega \int \sqrt{|V|}}.$$

Of course, the interpretation of the two factors as amplitude and phase does not apply now.

Major complications arise if $V(t)$ passes through 0 at some value of t (say at $t = 0$, to simplify the notation). Obviously, the factor $V^{-\frac{1}{4}}$ in the approximation blows up there. The true solution does not behave that way; the approximation is just no good near such a point. This is a kind of *nonuniformity* more vicious than those we have met before. Bad points like this are called *turning points*, for an interesting physical reason which I can't explain here.

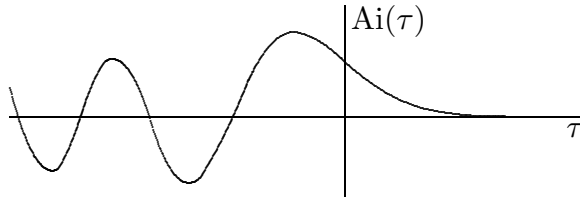
In the case where $V(t) = tf(t)$ with $f(0) \neq 0$, a complete theory has been worked out. (V 's that behave like t^2 or t^3 , or that have two turning points very

close together, are much harder.) The method is a case of the theory of *boundary layers* which we are to study next.

The trick is to approximate f by a constant in the immediate vicinity of $t = 0$; thus (after some rescaling of the time variable) the ODE becomes, to lowest order in t ,

$$\frac{d^2y}{d\tau^2} - \tau y = 0.$$

(This equation is exact if f is constant, and the zeroth-order equation in a perturbation expansion of the scaled equation otherwise.) This deceptively simple-looking equation can't be solved in terms of "elementary" functions. However, the power-series expansions of its solutions are easy to construct; this is probably an example in your old ODE textbook. The solutions are called *Airy functions*. What the elementary ODE books don't tell you — but the information can be found in standard handbooks — is that the behavior of the Airy functions as $\tau \rightarrow \pm\infty$ is also known. One of them looks like this:



It resembles an oscillatory WKB-type function for large negative τ and an exponential-type function (decaying) for large positive τ . Another (linearly independent) Airy function would be quasiexponentially *growing* at $+\infty$.

By matching these Airy solutions (which are good approximations in the boundary layer near the turning point) to the WKB solutions (which are good away from the turning point) one finds solutions which are good everywhere. In particular, this construction tells one how a given decaying exponential solution continues into the oscillatory region — that is, what linear combination of the two oscillatory solutions it becomes. Namely [Reference: A. Messiah, *Quantum Mechanics*, Sec. VI.9; the formulas are for the case $f(0) > 0$, so that negative τ corresponds to positive t], the solution

$$C(-V)^{-\frac{1}{4}} e^{-\int_t^0 \omega(-V)^{\frac{1}{2}} d\tilde{t}}$$

in the region of negative V becomes a certain Airy function, which in turn passes into

$$2CV^{-\frac{1}{4}} \cos\left(\int_0^t \omega V^{\frac{1}{2}} d\tilde{t} - \frac{\pi}{4}\right).$$

Similarly, a given oscillatory solution in the positive region,

$$CV^{-\frac{1}{4}} \cos\left(\int_0^t \omega V^{\frac{1}{2}} d\tilde{t} - \frac{\pi}{4} + \phi\right),$$

where ϕ is an arbitrary (nonzero) phase angle, continues into the negative region as

$$C \sin \phi (-V)^{-\frac{1}{4}} e^{+\int_t^0 \omega(-V)^{\frac{1}{2}} d\tilde{t}},$$

a growing exponential (as $t \rightarrow -\infty$).

Using these two rules in the opposite directions to those stated is notoriously dangerous, because the decaying component of a general exponential function is swamped by the growing component as soon as $-t$ becomes large. An unknown decaying component that is completely lost in the noise of the approximation in the exponential region will make it impossible to determine the phase shift ϕ in the adjacent oscillatory region.

Many authors insist on historical grounds that only this turning-point theory should be called the WKB approximation; the phase-integral formula (***) , they say, should be called the *Liouville–Green approximation*.

Boundary Layers

In treating algebraic equations we noticed that a perturbation term of *higher degree* than the terms in the unperturbed equation

$$\text{e.g.,} \quad \epsilon x^3 + 5x^2 - 2x + 3 = 0$$

changed the character of the solutions of the problem. In particular, the number of solutions increased, since the number of roots of a polynomial is equal to its degree. To find the “extra” roots we had to resort to an ansatz more general than a simple Taylor series in ϵ . We rescaled the variable and ended up with expansions starting out with negative powers of ϵ .

In differential equations, similar things happen when the perturbing term is of *higher order* than the unperturbed terms (in the sense of involving higher derivatives). Again, the perturbed equation has more solutions than the unperturbed one: Roughly speaking, the number of arbitrary constants in the general solution is equal to the order of the equation. Therefore, adding a second-order term to a first-order equation changes a one-dimensional solution space into a two-dimensional one. Not surprisingly, the “extra” solutions can’t all be obtained by adding small correction terms to solutions of the first-order equation; so something more general than a Taylor series in ϵ is required.

Perturbation problems of this variety are called *singular*. This term is not terribly well-defined, but it is usually applied when one or more of the following conditions hold:

- (1) The perturbing term is the “leading”, or most important, term in the equation. This usually means that that term is of higher order (for a differential equation) or higher degree (for an algebraic equation) than the unperturbed ones.
- (2) The perturbation fundamentally changes the character of the problem or its solutions.
- (3) The solutions cannot be found as regular perturbation series (Taylor series) in the perturbation parameter ϵ (or in a positive power of it, such as $\epsilon^{\frac{1}{2}}$).

Notice that the WKB problem can be regarded as an extreme case of this situation. If we write equation (2) of the previous section as

$$\epsilon^2 y'' + V(t)y = 0$$

and try to solve by the naive substitution $y = y_0(t) + \epsilon y_1(t) + \dots$, we immediately get the nonsensical results $y_0(t) = 0$. The unperturbed equation is of *zeroth order* and has no nontrivial solutions! As we know, the correct response here is to involve the ϵ more intimately in the structure of the solution (in the denominator of the argument of an exponential function).

In boundary-layer theory, on the other hand, one deals with unperturbed equations which are still differential equations — that is, they have order ≥ 1 . Typically the exact solution is well approximated by some unperturbed solution over most of the domain of interest, but the approximation becomes bad in some narrow region where the exact solution must satisfy some boundary condition which is inconsistent with the approximation — because the general solution of the lower-order approximate equation does not have enough arbitrary constants (free parameters) to accommodate that boundary condition. A practical situation where such a phenomenon occurs is fluid flow past a solid body, such as air past an aircraft. The thin layer of fluid in direct contact with the airplane has to be treated separately from the bulk of the fluid; the physical situation there is different, and this difference is reflected in the mathematical behavior of the solutions of equations modeling the flow.

Don't confuse “boundary-layer problem” with “boundary-value problem”. A second-order ODE problem is a *boundary-value problem* if, instead of the usual type of *initial conditions*,

$$y(0) = A, \quad y'(0) = B,$$

we study conditions at two different points (*boundary conditions*):

$$y(0) = A, \quad y(\pi) = 0.$$

More generally, *partial* differential equations are normally accompanied, for both physical and mathematical reasons, by conditions to be satisfied by the solution along parts of the boundary of the spatial or space-time region concerned. (This is what most of the rest of the course is about.) Solutions of such a problem may or may not involve *boundary layers*, the topic of this section.

EXAMPLE

Consider

$$\epsilon y'' + y' + y^2 = 0; \quad y(0) = \frac{1}{4}, \quad y(1) = \frac{1}{2}.$$

This is a nonlinear problem. (Assume $\epsilon > 0$.)

First let's neglect the ϵ term:

$$\begin{aligned} y' + y^2 &= 0; \\ \int \frac{dy}{y^2} &= \int (-1) dt; \\ -\frac{1}{y} &= -t + C; \end{aligned}$$

$$y = \frac{1}{t - C}.$$

Now let's try to apply the boundary conditions. At $t = 0$ we get $\frac{1}{4} = -\frac{1}{C}$, which would imply $C = -4$. At $t = 1$ we get $\frac{1}{2} = \frac{1}{1-C}$, which would imply $C = -1$. They can't both be right.

The resolution of this dilemma is that somewhere the true solution develops a second derivative so large that the term $\epsilon y''$ is no longer small compared to the others. Since both our solutions are curving up toward the left, we suspect that the trouble occurs near the left endpoint. Therefore, we tentatively accept the solution

$$y_o \equiv \frac{1}{t + 1}$$

as the valid approximation everywhere on the interval except near $t = 0$ (the *outer solution*), and we perform a scaling of t to "magnify" the region near 0 so that we can look at it better:

$$\begin{aligned} \tau &\equiv \frac{t}{\epsilon}; & \frac{d\tau}{dt} &= \frac{1}{\epsilon}; \\ \frac{d^2y}{d\tau^2} + \frac{dy}{d\tau} + \epsilon y^2 &= 0. \end{aligned}$$

To lowest order we must solve

$$\begin{aligned} \frac{d}{d\tau} \frac{dy}{d\tau} + \frac{dy}{d\tau} &= 0; \\ \frac{dy}{d\tau} &= Ae^{-\tau}; \\ y &= -Ae^{-\tau} + B. \end{aligned}$$

Thus $\frac{1}{4} = y(0) = -A + B$, so $B = A + \frac{1}{4}$. So the *inner solution* is

$$y_i \equiv A(1 - e^{-\tau}) + \frac{1}{4}.$$

We have no way of determining A without relating the inner solution to the boundary data at the other endpoint. We must somehow *match* y_i with y_o . The crudest way to do this would be to choose some arbitrary value of t between 0 and 1 and require that the two solutions exactly agree there. The result would depend on which point you chose. The preferred method is to make an intermediate scaling of the time variable in hopes of focusing attention on a regime of "overlap", where both approximations are accurate.

Accordingly, we let

$$\eta \equiv \frac{t}{\sqrt{\epsilon}} = \sqrt{\epsilon} \tau.$$

As $\epsilon \rightarrow 0$ with η fixed, we have $t \rightarrow 0$ in y_o and $\tau \rightarrow \infty$ in y_i . Looking at the solutions, we see

$$y_o = \frac{1}{\sqrt{\epsilon}\eta + 1} \rightarrow 1 \quad \text{as } \epsilon \rightarrow 0.$$

$$y_i = A \left(1 - e^{-\eta/\sqrt{\epsilon}}\right) + \frac{1}{4} \rightarrow A + \frac{1}{4}.$$

Since these limits should agree, $A = \frac{3}{4}$. Thus we get

$$y_i = \frac{3}{4} \left(1 - e^{-t/\epsilon}\right) + \frac{1}{4},$$

and we already knew

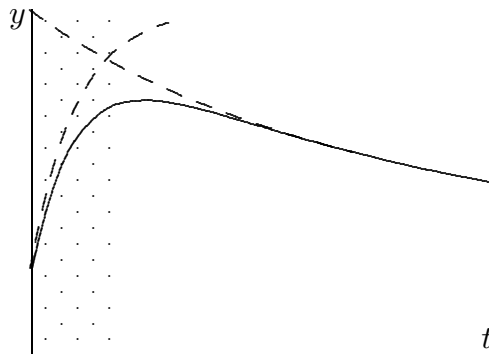
$$y_o = \frac{1}{t + 1}.$$

We still have the practical problem of deciding how big t should become before we switch from using y_i over to y_o ; if we did so suddenly, there would be an unwelcome jump or kink in our approximate solution. These problems can be finessed by constructing a single expression valid over the whole interval:

$$y \equiv y_i + y_o - 1.$$

(The 1 is the common limit we just calculated.) Since the common limit cancels out each approximation at its “bad” end, we are left with the “good” term at each end and a smooth interpolation between them in the interior! The expression simplifies to

$$y = -\frac{3}{4} e^{-t/\epsilon} + \frac{1}{t + 1}.$$



Let’s check how well the approximation satisfies the ODE and the boundary conditions:

$$y(0) = -\frac{3}{4} + 1 = \frac{1}{4}. \quad \checkmark$$

$$y(1) = -\frac{3}{4} e^{-1/\epsilon} + \frac{1}{2} \rightarrow \frac{1}{2} \quad \text{as } \epsilon \rightarrow 0.$$

This condition is not satisfied exactly, because y_i does not exactly equal the common limit 1 when $t = 1$; but it is satisfied very well if ϵ is small.

$$y' = \frac{3}{4\epsilon} e^{-t/\epsilon} - \frac{1}{(t+1)^2},$$

$$\epsilon y'' = -\frac{3}{4\epsilon} e^{-t/\epsilon} + \frac{2\epsilon}{(t+1)^3},$$

$$y^2 = \frac{9}{16} e^{-2t/\epsilon} - \frac{3}{2} \frac{e^{-t/\epsilon}}{t+1} + \frac{1}{(t+1)^2}.$$

Adding, we get

$$\frac{2\epsilon}{(t+1)^3} + \frac{9}{16} e^{-2t/\epsilon} - \frac{3}{2} \frac{e^{-t/\epsilon}}{t+1}.$$

This as good as we have a right to expect, since our approximation was done only to lowest order in ϵ ; notice that terms of the orders ϵ^0 and $\epsilon^{-1} e^{-t/\epsilon}$ have cancelled.

ADDITIONAL ISSUES

(1) There is no guarantee that the boundary layer will always be at $t = 0$. Indeed, if we made the variable transformation $u \equiv 1 - t$, the boundary layer would move to the right end of the interval, while the ODE would remain the same except for a sign. A nonlinear equation may even have boundary layers in the interior of the interval, the precise location depending on the numerical values of the boundary data! (Our book points out that this can't happen for a linear equation with damping coefficient of a fixed sign.)

Let's see what happens if we try to put a boundary layer at the right end in our example. Introduce the scaled coordinate

$$\tau \equiv \frac{t-1}{\epsilon}.$$

(Take good note of this, since this is the sort of thing you need to do when the boundary layer really is at $t = 1$!) Note that τ takes on only *negative* values in the interval of interest.

The transformed equation is again

$$\frac{d^2 y}{d\tau^2} + \frac{dy}{d\tau} + \epsilon y^2 = 0,$$

this time with the boundary condition

$$y(0) = \frac{1}{2} \quad (\tau = 0 \iff t = 1).$$

Solving the equation as before, we get

$$y_i = A(1 - e^{-\tau}) + \frac{1}{2}.$$

And this time we take our other old solution as the outer solution:

$$y_o = \frac{1}{t+4} \quad (y_o(0) = \frac{1}{4}).$$

To match these two solutions we need to consider

$$\eta \equiv \frac{t-1}{\sqrt{\epsilon}} = \sqrt{\epsilon}\tau.$$

$$t = 1 + \sqrt{\epsilon}\eta, \quad \tau = \frac{\eta}{\sqrt{\epsilon}}.$$

Then $y_o \rightarrow \frac{1}{5}$ as $\epsilon \rightarrow 0$ with fixed η . But since τ is negative, we see that y_i does not approach a limit consistent with that one; instead, the exponential goes rapidly to ∞ .

So there is no way to match the inner and outer solutions. This is reassuring, since otherwise we would have succeeded in producing a solution with a boundary layer compressed against the right end of the interval, in contradiction to our previous conclusion that the layer is at the other end.

(2) Is there a way to see in advance where the boundary layer will be? Yes, for a broad class of linear equations of the form

$$\epsilon \frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = 0$$

with boundary conditions at, say, $t = 0$ and $t = 1$.

If $p(t)$ has the same sign as ϵ throughout the interval, then the boundary layer is at the left end of the interval. If $p(t)$ and ϵ have opposite sign, then the boundary layer is at the right end.

Note that this theorem tells us nothing about what happens if $p(t)$ changes sign, or $p(t)$ equals 0 over an entire subinterval of t , or the equation is nonlinear. Recall also that if the boundary condition on the boundary-layer side just happens to coincide with the value of the outer solution there, then no boundary layer is present.

(3) In the example problem, how did I know that the correct scaling is $\tau = t/\epsilon$? In general, one may have to search for a correct power of ϵ . One technique is to write $\delta(\epsilon)$ instead of ϵ , and then explore the consequences of “balancing” the newly enlarged leading term against each other term of the ODE until a consistent ansatz is found. This is similar to how we did the scaling in solving algebraic equations with singular perturbing terms.