DIMENSIONAL ANALYSIS – Is that all there is?

The Secrets of Physics Revealed

Outline

A. What's the secret of being a Scientist or an Engineer?

B. What are Units and Dimensions anyway?

C. What is Dimensional Analysis and why should I care?

D. Why aren't there any mice in the Polar Regions?

E. Why was Gulliver driven out of Lillipute?

F. What if Pythagorus had known Dimensional Analysis?

G. But what do I really need to know about Dimensional Analysis so that I can pass the test?

H. Can I get into trouble with Dimensional Analysis? The ballad of G.I. Taylor.

I. But can it be used in the Lab ?

How to be a Scientist or Engineer

The steps in understanding and/or control any physical phenomena is to

1. **Identify the relevant physical variables.**
2. **Relate these variables using the known physical laws.**
3. **Solve the resulting equations.**

Secret #1: Usually not all of these are possible. Sometimes none are.

ALL IS NOT LOST BECAUSE OF

Secret #2: Dimensional Analysis

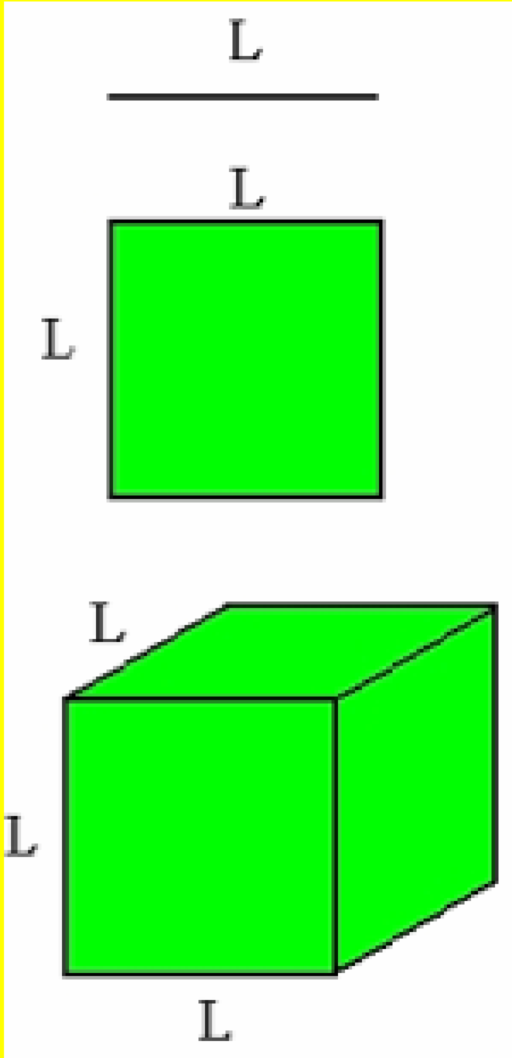
Rationale

- Physical laws must be **independent** of arbitrarily chosen units of measure. **Nature does not care if we measure lengths in centimeters or inches or light-years or …**
- Check your units! **All natural/physical relations must be dimensionally correct.**

Dimensional Analysis

Dimensional Analysis refers to the physical nature of the quantity and the type of unit (**Dimension**) used to specify it.

- Distance has dimension L.
- Area has dimension L^2.
- Volume has dimension L^3.
- Time has dimension T.
- Speed has dimension L/T



Why are there no small animals in the polar regions?

- Heat Loss $\propto$ Surface Area (L^2)
- Mass $\propto$ Volume (L^3)
- Heat Loss/Mass $\propto$ Area/Volume

$$= L^2 / L^3$$

$$= L^{-1}$$

Heat Loss/Mass $\propto$ Area/Volume

$$= L^2 / L^3$$

$$= L^{-1}$$

Mouse (L = 5 cm)

$$1/L = 1/(0.05\ m)$$

$$= 20\ m^{-1}$$

Polar Bear (L = 2 m)

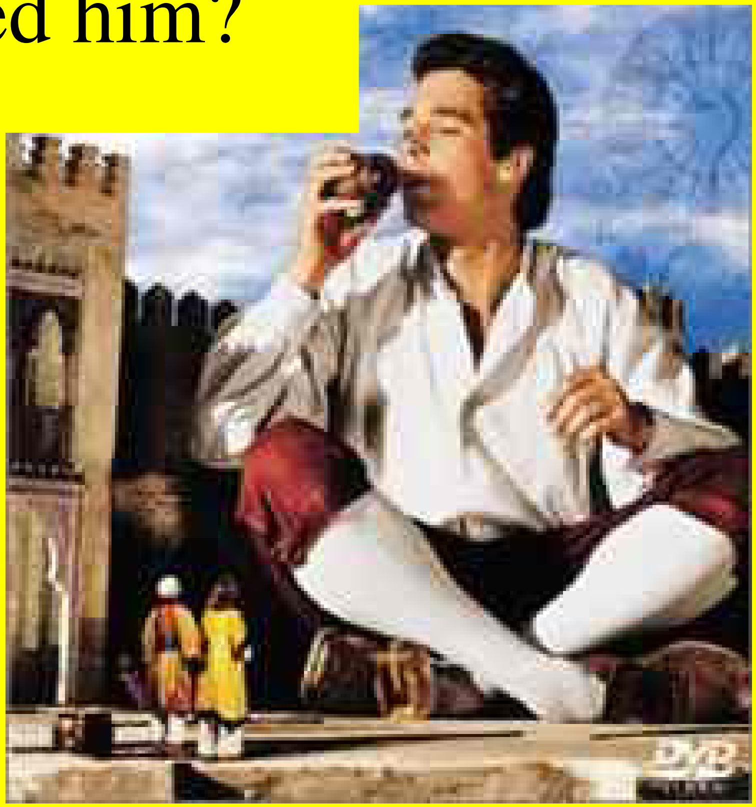
$$1/L = 1/(2\ m)$$

$$= 0.5\ m^{-1}$$

20 : 0.5 or 40 : 1

Gulliver's Travels: Dimensional Analysis

- Gulliver was 12x the Lilliputians
- How much should they feed him? 12x their food ration?
- A persons food needs are related to their mass (volume) – This depends on the cube of the linear dimension.



Let $\mathbf{L_G}$ and $\mathbf{V_G}$ denote Gulliver's linear and volume dimensions.
Let $\mathbf{L_L}$ and $\mathbf{V_L}$ denote the Lilliputian's linear and volume dimensions.

Gulliver is 12x taller than the Lilliputians, $\mathbf{L_G} = 12\ \mathbf{L_L}$

Now $V_G \propto (L_G)^3$ and $V_L \propto (L_L)^3$, so

$$
\begin{aligned}
V_G / V_L &= (L_G)^3 / (L_L)^3 \\
&= (12\ L_L)^3 / (L_L)^3 \\
&= 12^3 \\
&= 1728
\end{aligned}
$$

Gulliver needs to be fed 1728 times the amount of food each day as the Lilliputians.

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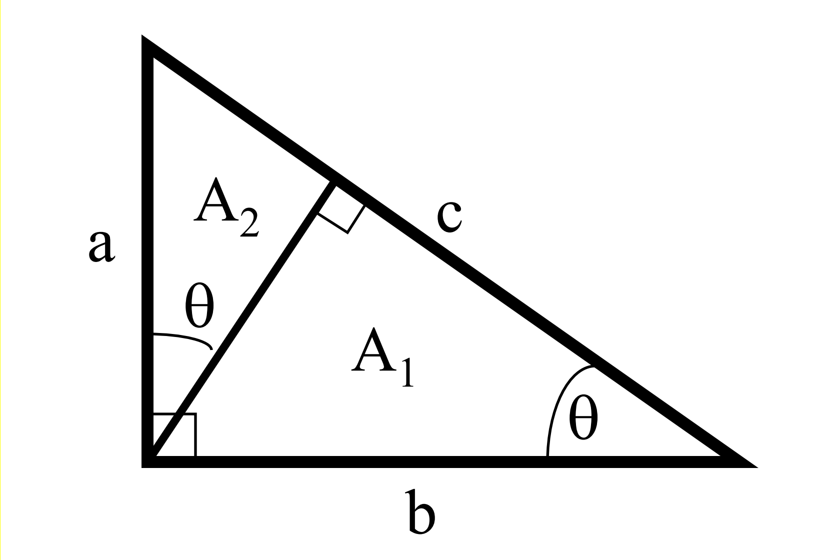
"I found my arms and legs were strongly fastened on each side to the ground"

This problem has direct relevance to drug dosages in humans

Pythagorean Theorem

- Area = $F(\theta)\ c^2$
- $A_1 = F(\theta)\ b^2$
- $A_2 = F(\theta)\ a^2$



- Area = $A_1 + A_2$

$$\cancel{F(\theta)}\ c^2 = \cancel{F(\theta)}\ a^2 + \cancel{F(\theta)}\ b^2$$

$$c^2 = a^2 + b^2$$

Dimensions of Some Common Physical Quantities

[x], Length – L

[m], Mass – M

[t], Time – T

[v], Velocity – LT^{-1}

[a], Acceleration – LT^{-2}

[F], Force – MLT^{-2}

[ρ], Mass Density – ML^{-3}

[P], Pressure – $ML^{-1}T^{-2}$

[*E*], Energy – $ML^{2}T^{-2}$

[*I*], Electric Current – QT^{-1}

[q], Electric Change – Q

[E], Electric Field - $MLQT^{-2}$

All are powers of the fundamental dimensions:

[Any Physical Quantity] = $M^{a}L^{b}T^{c}Q^{d}$

Dimensional Analysis Theorems

- **Dimensional Homogeneity Theorem**: Any physical quantity is dimensionally a power law monomial - **[Any Physical Quantity] = $M^aL^bT^cQ^d$**

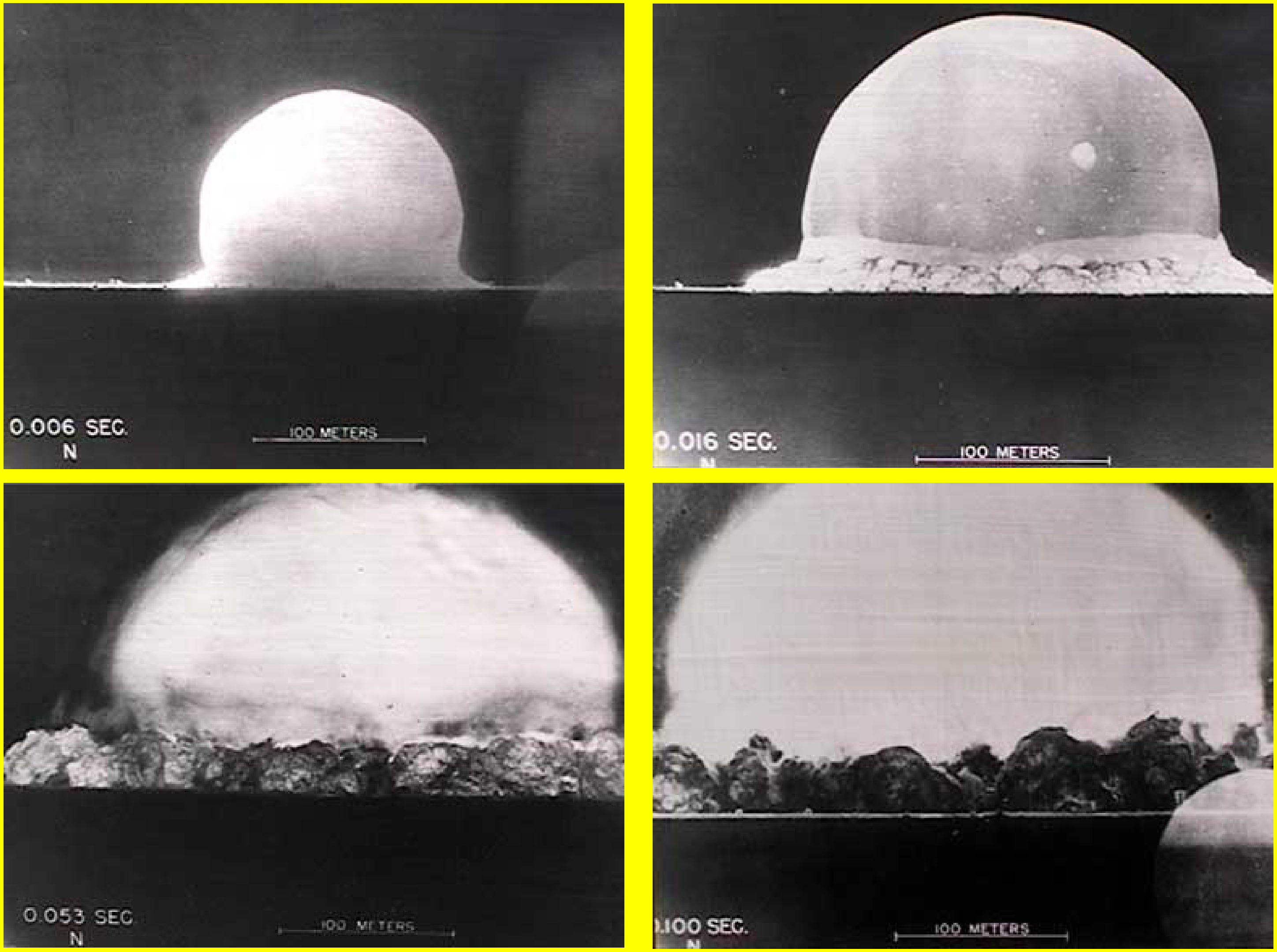
- **Buckingham Pi Theorem:** If a system has ***k*** physical quantities of relevance that depend on depend on ***r*** independent dimensions, then there are a total of ***k-r*** independent dimensionless products $\pi_1, \pi_2, \ldots, \pi_{k-r}$. The behavior of the system is describable by a dimensionless equation

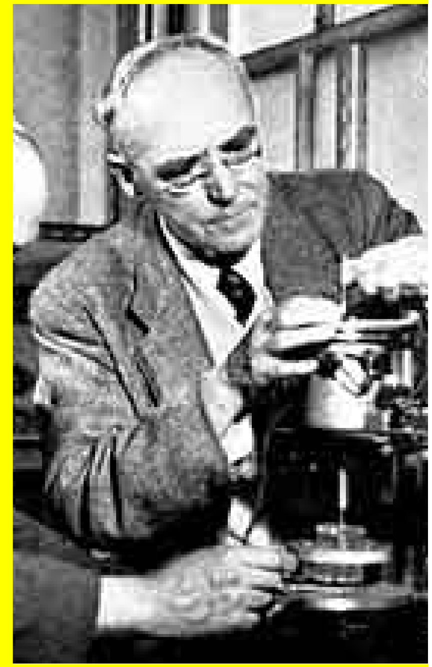
$$F(\pi_1, \pi_2, \ldots, \pi_{k-r})=0$$

Exponent Method

1. List all k variables involved in the problem
2. Express each variables in terms of [M] [L] [T] dimensions (r)
3. Determine the required number of dimensionless parameters ($k - r$)
4. Select a number of repeating variables = r
   (All dimensions must be included in this set and each repeating variable must be independent of the others.)
5. Form a dimensionless parameter π by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an unknown exponent.
6. Solved for the unknown exponents.
7. Repeat this process for each non-repeating variable
8. Express result as a relationship among the dimensionless parameters – $F(\pi_1, \pi_2, \pi_3, \ldots) = 0$.

G. I. Taylor's 1947 Analysis





Published U.S. Atomic Bomb was 18 kiloton device

Nuclear Explosion Shock Wave

The propagation of a nuclear explosion shock wave depends on: E, r, ρ, and t.

$$r = f(E, \rho, t)$$

$n = 4$ No. of variables
$r = 3$ No. of dimensions
$n - r = 1$ No. of dimensionless parameters

E	ρ	r	t
ML^2T^{-2}	ML^{-3}	L	T

Select "repeating" variables:
E, t, and ρ
Combine these with the rest of the variables: r

$$\pi_1 = r \times (E^a t^b \rho^c)$$

$$M^0 L^0 T^0 = (L)(ML^2T^{-2})^a (T)^b (ML^{-3})^c$$

$$M: \quad 0 = a + c \quad \Rightarrow \quad c = -a$$

$$L: \quad 0 = 1 + 2a - 3c \Rightarrow \quad a = -\frac{1}{5}$$

$$T: \quad 0 = -2a + b \quad \Rightarrow \quad b = -\frac{2}{5}$$

$$\pi_1 = RE^{-1/5} t^{-2/5} \rho^{1/5} = \frac{R}{E^{1/5} t^{2/5} \rho^{-1/5}}$$

$$F(\pi_1) = 0 \quad \Rightarrow \quad \pi_1 = C$$

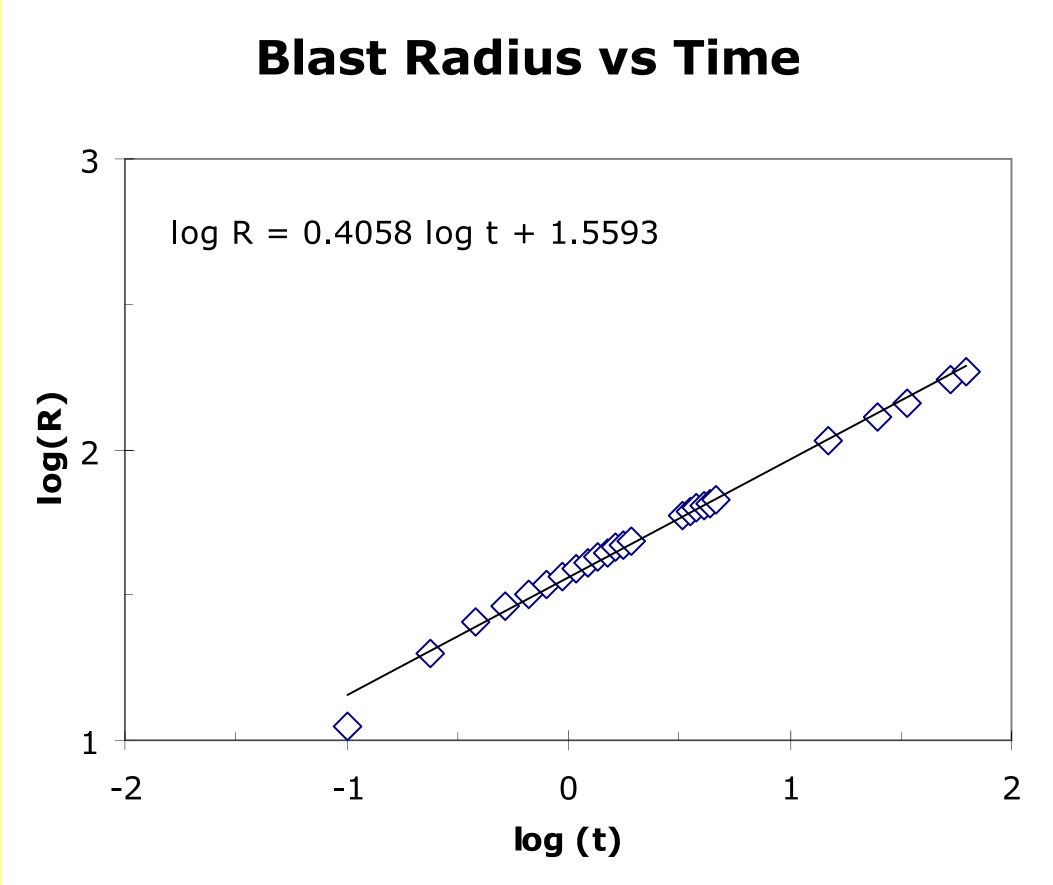
$$\frac{R}{E^{1/5} t^{2/5} \rho^{1/5}} = C \quad \Rightarrow \quad R = CE^{1/5} t^{2/5} \rho^{-1/5}$$

$$\Rightarrow$$

$$R = C\left(\frac{Et^2}{\rho}\right)^{1/5}$$

$$R = (E/\rho)^{1/5}\, t^{2/5}$$

$$\log R = 0.4 \log t + 0.2 \log(E/\rho)$$



$$0.2 \log(E/\rho) = 1.56$$

$$\rho = 1 \text{ kg/m}^3$$

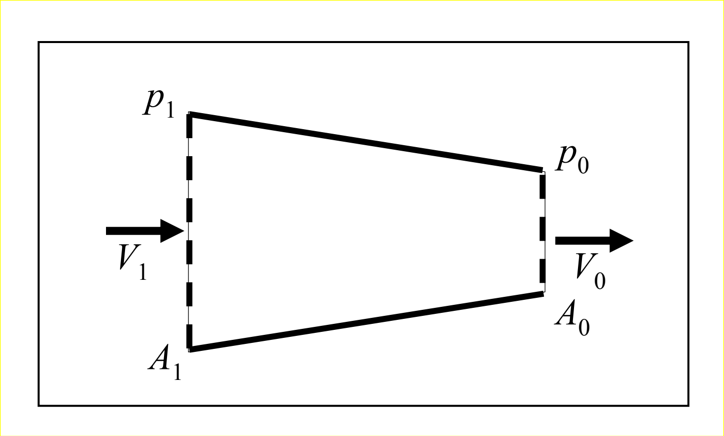
$$\Downarrow$$

$$E = 7.9\text{x}10^{13} \text{ J}$$

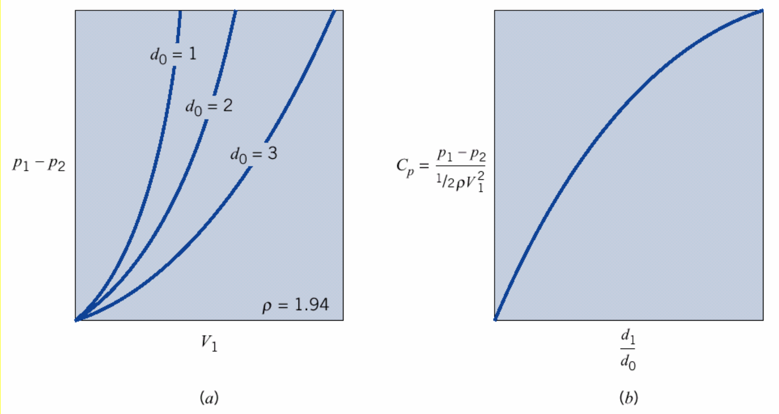
$$= 19.8 \text{ kilotons TNT}$$

Dimensional Analysis in the Lab

- Want to study pressure drop as function of velocity (V_1) and diameter (d_o)
- Carry out numerous experiments with different values of V_1 and d_o and plot the data



ΔP	ρ	V_1	d_1	d_2
$ML^{-1}T^{-2}$	ML^{-3}	LT^{-1}	L	L



5 parameters:
$\Delta p, \rho, V_1, d_1, d_o$

2 dimensionless parameter groups:
$\Delta P/(\rho V^2/2), (d_1/d_o)$

Much easier to establish functional relations with 2 parameters, than 5

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