AE 6030 : Advanced Potential Flow, Class Notes
Section # 2

from Georgia Tech website

Fundamentals of Fluid Dynamics

Continuum Assumption : We assume that a fluid is a continuous medium; down to the smallest dimensions of interest to us. For example: Air is a fluid and we assume its continuous for fluid dynamic purposes though it is made of discrete molecules. Some important numbers when dealing with air, M.W. 28.8 g/mol, density, 1.2 kg/m^3, Avogadro number : 6.023e23. With these numbers we get a particle density of 2.5e7 molecules per micrometer.

Streamline : A streamline is the tangent to the local flow direction at every point in the fluid.

If the tangent is defined as

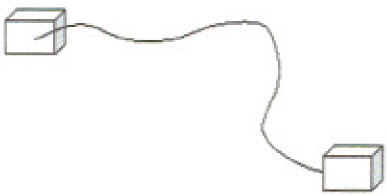
$d\vec{S} \times \vec{U} = 0$ $d\vec{S} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ Then, the condition of tangency requires for U = u i + v j + w k

Which gives

dy/dx = v/u ; dz/dy = w/v ; dz/dx = w/u.

Basic fluid motion can be described as some combination of

a. Translation, U



This is the motion of the center of mass of the fluid packet

b. Dilatation, Divergence of U, volume change

$\nabla . \vec{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

c. Rotation, or vorticity ,

$\text{Vorticity} = 2\omega = \nabla \times \vec{U}$

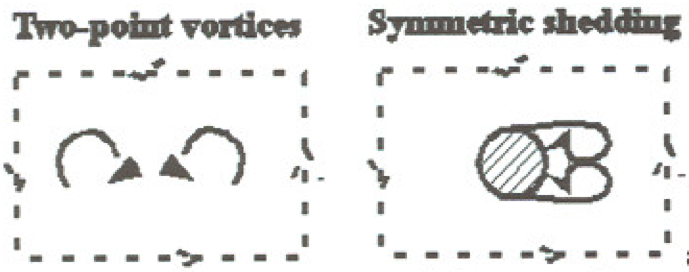
d. Shear strain,

$\varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ And so on.

Circulation :

$\Gamma = -\oint \vec{U} . d\vec{S}$ The negative sign is included such that positive circulation on a body corresponds to positive lift, and the integral is evaluated counter-clockwise.

Important Points :



a) The circulation around a closed contour with rotation and/or shear will be non-zero. It is, however, always possible to have a combination of rotation and/or shear that gives a zero circulations. See the examples below.

where is the undisturbed freestream density, is freestream speed , is the ratio of specific heats and is the freestream Mach number.

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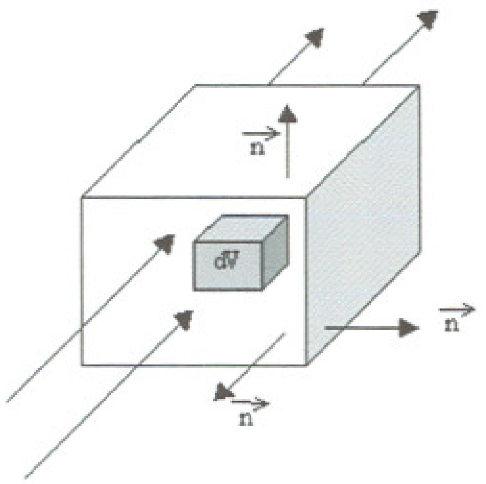
CONSERVATION EQUATIONS

These equations are expressions of the laws of physics, written in forms appropriate for flows.
(i) Mass is neither created nor destroyed : "continuity equation", or "conservation of mass".
(ii) Rate of change of momentum = Net force: "conservation of momentum"
(iii) Energy is conserved, though it may change form: "conservation of energy"

Integral forms for a control volume



$$\oiiint_{CV} \frac{\partial}{\partial t}(?)dV + \iint_{CS}(?)(\bar{u} \bullet \bar{n})dS = \oint_{CV or CV}(whatever (?) changes\ to)(dV\ \ or\ \ dS)$$

Mass:

$$\oint_{CV} \frac{\partial}{\partial t}(\rho)dV + \oint_{CS}(\rho)(\vec{u} \bullet \vec{n})d\vec{S} = 0$$

Momentum:

$$\oint_{CV} \frac{\partial}{\partial t}(\rho\vec{u})dV + \oint_{CS}(\rho\vec{u})(\vec{u} \bullet \vec{n})dS = -\oint_{CS} p\vec{n}dS + \oint_{CV} \rho\vec{f}dV + F_{shear}$$

The terms on the rhs are:
I: pressure forces acting normal to the surface, per unit area.
II: body forces per unit mass.
III: shear forces acting parallel to the surface, per unit area.

Energy:

$$\oint_{CV} \frac{\partial}{\partial t}\left[\rho\left(e + \frac{u^2}{2}\right)\right](\vec{u} \bullet \vec{n})dV + \oint_{CS}\left[\rho\left(e + \frac{u^2}{2}\right)\right](\vec{u} \bullet \vec{n})dS = \oint_{CV} \rho\dot{q}dV + \oint_{CV} \rho\dot{w}dV$$

Note: $u^2 = |\vec{u}|^2$

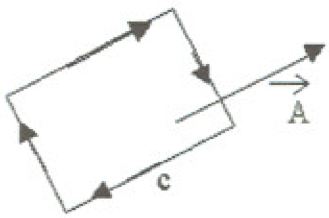
Energy per unit volume: $\rho\left(e + \frac{u^2}{2}\right)$ = r(Internal energy per unit mass + kinetic energy per unit mass).
In addition to these specific conservation laws, specific equations relating the state variables are needed to solve problems for given kinds of fluids. To solve very complicated flow problems, the boundary conditions are specified, and all of these equations are solved simultaneoulsy all over the flowfield, for each step in time. This is a task which usually requires fast computers with large memory, because we have to keep track of a large number of variables and perform a large number of calculations. Long before this became "possible" people figured out more restricted ways of solving specific problems needed to build airplanes. These "smart analytical methods" form the subject of this course.

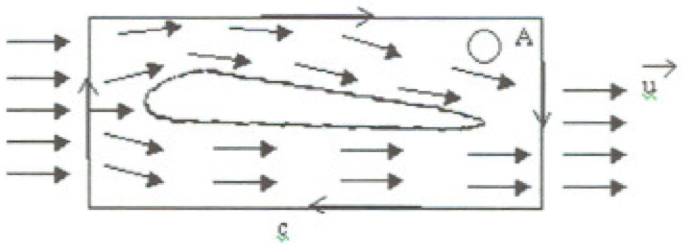
Relations Used to Reduce the Conservation Equations to Differential Form

1. Stokes' Theorem

$$\oint_C \vec{u} \bullet d\vec{l} = \oint_A (\nabla \times \vec{u}) \bullet d\vec{A}$$

where $\vec{u}$ is the vector quantity of interest, $d\vec{l}$ is the vector along the closed contour of integration C, $d\vec{A}$ is the unit vector normal to the area enclosed by C.





2. Divergence Theorem

$$\oint_A \vec{u} \bullet d\vec{A} = \oint_V (\nabla \bullet \vec{u}) \bullet d\vec{V}$$

3. Gradient Theorem

If p is a scalar field, then

$$\oint_A p\, d\vec{A} \equiv \oint_V \nabla p\, d\vec{V}$$

4. $\nabla \bullet (\rho\vec{u}) \equiv \rho\nabla \bullet \vec{u} + \vec{u} \bullet \nabla\rho$ where ρ is a scalar and $\vec{u}$ is a vector.

Substantial Derivative

The Eulerian Frame of Reference is the one fixed to the control volume. The Lagrangian frame of reference is the one fixed to a packet of fluid (a fluid element)

The rate of change of any property as seen by the fluid element is:

$$\frac{D(?)}{Dt} = \frac{\partial}{\partial t}(?) + u\frac{\partial}{\partial x}(?) + v\frac{\partial}{\partial y}(?) + w\frac{\partial}{\partial z}(?)$$

The substantial derivative is:

$$\frac{D(?)}{Dt} \equiv \frac{\partial}{\partial t}(?) + \langle \vec{u} \bullet \nabla \rangle (?)$$

. The first term on the rhs is the "local" or "unsteady" term. The second is the "convective" term.

The rate of change D()/Dt is for two reasons:

1. Things are changing at the point through which the element is moving (unsteady, local)

2. The element is moving into regions with different properties.

Using the vector identity (4) above, the conservation equations can be re-written:

Continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{u}) = 0$$

or,

$$\frac{\partial \rho}{\partial t} + \rho \nabla \bullet \vec{u} + \vec{u} \bullet \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \bullet \vec{u} = 0$$

In terms of veclocity components, this can be written as a scalar equation:

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Momentum Conservation: Differential Form

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x + \langle F_{viscous} \rangle_x$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y + \langle F_{viscous} \rangle_y$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z + \langle F_{viscous} \rangle_z$$

Knowing the properties of the particular fluid and problem being considered, the body force term and the viscous force term can be expanded. One very useful form is where the viscous stresses are related to the rate of strain of the fluid, through a linear expression.This is valid for "Newtonian Fluids". This is further simplified using the Stokes hypothesis, which permits us to delete the normal-strain terms from the strain terms, leaving only shear-strain terms. The resulting form of the momentum equation is called the Navier-Stokes equation. This is often used as the general starting point to solve problems in fluid mechanics.

Energy Conservation

$$\rho\frac{D\left(e + \frac{u^2}{2}\right)}{Dt} = \rho\dot{q} - \nabla \bullet (p\hat{u}) + \rho(\vec{f} \bullet \hat{u}) + Q'_{viscous} + W'_{viscous}$$

The Euler equation

If the Reynolds number $Re \equiv \frac{\rho U D}{\mu}$ = (Inertial Force divided by Viscous Force) >>1 in our flow problem, we can neglect the viscous stress terms. Thus the differential form of the momentum equation reduces to:

$$\frac{\partial}{\partial t}(\rho u) + \nabla \bullet (\rho \vec{u})u = -\frac{\partial p}{\partial x} + \rho f_x$$

$$\frac{\partial}{\partial t}(\rho v) + \nabla \bullet (\rho \vec{u})v = -\frac{\partial p}{\partial y} + \rho f_y$$

$$\frac{\partial}{\partial t}(\rho w) + \nabla \bullet (\rho \vec{u})w = -\frac{\partial p}{\partial z} + \rho f_z$$

Here u,v,w are the Cartesian components along x,y,z of the vector $\vec{u}$, and fx, fy and fz are components of the body force vector. From the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{u}) = 0$$

, and the substantial derivative, we can reduce th emomentum equation to:

$$\rho \frac{Du}{Dt} = -\nabla p + \rho \vec{f}$$ Euler equation.

Kelvin's Theorem

From Euler's equation

$$\rho\frac{D\bar{u}}{Dt} = -\nabla p$$

$$\frac{D}{Dt}\oint\hat{u}\bullet d\hat{s} = \oint\frac{\nabla p}{\rho}\bullet d\hat{s}$$

$$\frac{D}{Dt}\Gamma = -\oint\frac{dp}{\rho}$$

This is called Kelvin's Theorem. It relates the rate of change of the total circulation to the line integral of the (pressure change divided by density) around a closed contour. The integral on the r.h.s. vanishes if either of the following holds:

a) density is constant: r = constant. Note that the condition is that density is constant. If the flow Mach number is below 0.3, we can neglect density changes due to velocity changes: this is the assumption of "incompressible flow". However, if there are temperature changes due to other reasons (e.g., hot air rising), there can be density differences within the closed contour of integration.

b) p = f(r) This is called "barotropic". It means that pressure is a function only of density, i.e, there is a unique relation between pressure and density everywhere inside the closed contour. One counter-examples to this is a case where there is a shock. The isentropic cases is a special case of barotropic, where $p = r^{(g/(g-1))}$.

Using Stokes' Theorem,

$$\oint\vec{u}\bullet d\hat{s} = \iint_S(\nabla\times\vec{u})(d\hat{s})$$

Thus,

$$\frac{D}{Dt}\Gamma = 0$$

implies:

$$(\nabla\times\vec{u} = 0)$$

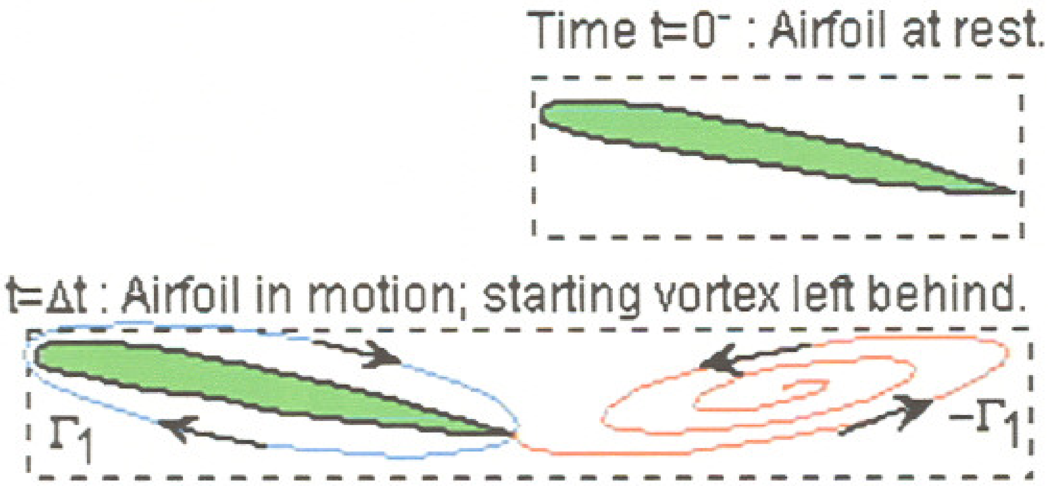
This means that if a packet of fluid starts out in an incompressible or barotropic flow, with a set amount of circulation, and does not encounter shocks or heat transfer, then its circulation will not change.

Kelvin's theorem for incompressible, barotropic flow:

The time rate of change of circulation around a closed contour consisting of the same fluid elements is zero. i.e.,

$$\frac{D}{Dt}\Gamma = 0$$

Example: Airfoil starts moving at time t = 0.



Derivation of the Steady Bernoulli Equation from the Euler equation, integrated along a streamline:

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}$$

Steady: $\frac{\partial}{\partial t} = 0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Multiply the u-component equation by dx:

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

A Streamline is defined as a locus of points tangent to the flow direction:
Along a streamline,

$$udz = wdx$$

$vdx = udy$ Using these relations, the u-component equation above becomes:

$$u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + u\frac{\partial u}{\partial z}dz = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

$udu = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$, so that $\frac{1}{2}d(u^2) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$

$$\frac{1}{2}d(u^2+v^2+w^2) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right)$$

$$dp+\frac{1}{2}d(u^2) = 0$$

$$\left(\int_{p1}^{p2} dp\right)+\frac{\rho}{2}\int_{u1}^{u2} d(u^2) = const$$

$$(p2-p1)+\frac{1}{2}\rho(u2^2-u1^2) = const$$

$$p1+\frac{1}{2}\rho u1^2 = p2+\frac{1}{2}\rho u2^2 = const$$

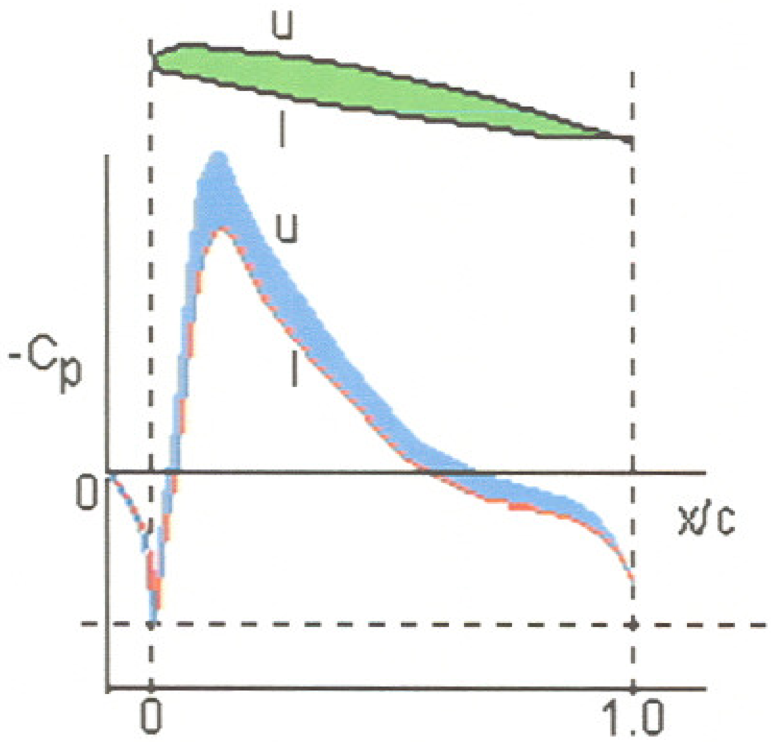
Along a streamline in steady incompressible flow.

Pressure Coefficient

$C_p \equiv \frac{p-p_\infty}{q_\infty}$ where $q_\infty = \frac{1}{2}\rho u_\infty^2$

Using the Bernoulli equation, this can also be written as

$$C_p = 1-\left(\frac{u}{u_\infty}\right)^2$$



Characteristics of the figure above:

1. Leading edge stagnation point: pressure coefficient is +1.0

2. Near the leading edge, and downstream of the stagnation point, the flow acccelerates beyond the freestream velocity, so that the pressure coefficient becomes negative on both the upper and lower surfaces.
3. The point of minimum pressure is upstream of 1/4-chord for most airfoils.
4. The pressure coefficient increases again, and becomes positive (flow slower than freestream) over the aft portion of the airfoil, both surfaces.
5. At the trailing edge, there is again stagnation. If we neglect all viscous losses, the pressure coefficient at the trailing edge should this reach +1.0 again. However, due to viscous losses, some of the stagnation pressure is lost, so that the pressure coefficient at the trailing edge stagnation is less than +1.0
6. Note that the pressure distributions on the upper and lower surfaces are quite similar. There is a small difference between them, either because of the angle of attack or the camber, or due to both. This difference is what gives the lift.

Kutta-Jowkowski Theorem for steady flow

Reference: *Eskinazi, S., "Vector Mechanics of Fluids and Magnetofluids". Academic Press, NY 1967, p.284.*

Apply Newton's 2nd law of motion to steady flow across a control surface:

$$\vec{F} = -\oint_S [p\hat{n} - (\rho u)[\hat{u} \bullet \hat{n}]dS = 0$$

From the Bernoulli equation for steady flow,

$$p = p_0 - \frac{1}{2}\rho u^2$$

Since p_0 is constant, $\oint_S p_0 \hat{n} dS = 0$

Use the divergence theorem:

$$\oint_S \hat{u} \bullet \overrightarrow{ndS} = \oint_V (\nabla \times \hat{u}) dV$$

$$\vec{F} = -\oint_V \rho \left[\frac{1}{2}\nabla u^2 - \hat{u}(\nabla \bullet \hat{u}) - (\hat{u} \bullet \nabla)\hat{u} \right] dV$$

Note that

$(\hat{u} \bullet \nabla)\hat{u} = \left[\frac{1}{2}(\nabla u^2) + (\nabla \times \hat{u}) \times \hat{u} \right]$. Substituting,

$$\vec{F} = \oint_V \rho [u \times (\nabla \times \hat{u}) - \hat{u}(\nabla \bullet \hat{u})] dV$$

Note that: $\nabla \bullet \hat{u}$ is the Dilatation.
$\nabla \times \hat{u}$ is the Vorticity.
If the flow is constant-density, the dilatation is zero. Note: even if it is incompressible flow (i.e., Mach number is so low that density changes due to velocity changes are negligible), there can be density changes due to heating or species gradients.
Thus the Kutta-Jowkwski theorem in steady constant-density flow is:

$$\vec{F} = \oint_V \rho [u \times (\nabla \times \hat{u})] dV$$

or,

$$\overrightarrow{dF} = \rho [u \times (\nabla \times \hat{u}) - \hat{u}(\nabla \bullet \hat{u})]$$

This acts perpendicular to both the velocity vector and to $\nabla \times \hat{u}$, the vorticity vector.

Notes:

1. The Lift, which is the force perpendicular to the freestream velocity vector, results from the combination of vorticity and a freestream.
2. Force can come from vorticity and dilatation. The resultant force in compressible flow may not be perpendicular to the freestream velocity vector.
3. If we can somehow get the correct value of vorticity, we can calculate the lift without explicity considering viscosity.

POTENTIAL FLOW

Objective:
Get a method to describe flow velocity fields and relate them to surface shapes consistently. Once the velocity field is known, the pressures and hence the loads can be calculated.

Strategy:
Describe the flowfield as the effect of variations in one scalar quantity.

Consider the vector identity:
$\nabla\times(\nabla\phi)\equiv 0$, i.e., The curl of the gradient of a scalar function is zero.
Thus, if we have a vector relation of the form
$\nabla\times\vec{u}\equiv 0$
it should be possible to express the vector u as the gradient of the scalar function f.
We defined "vorticity" as
$\xi\equiv\nabla\times\vec{u}$. This is a measure of rotation in the flow; in fact the vorticity is half the angular velocity.
Thus, $\nabla\times\vec{u}\equiv 0$ implies "irrotational flow".
Thus if we have irrotational flow, the vector u can describe the gradient of some scalar function f. This function is called the "velocity potential. Thus,

$\vec{u}\equiv\nabla\phi$ or, $u=\dfrac{\partial\phi}{\partial x}$; $v=\dfrac{\partial\phi}{\partial y}$; $w=\dfrac{\partial\phi}{\partial z}$

Unsteady Bernoulli Equation
From Euler's equation,
$$\frac{\partial}{\partial t}\vec{u}+(\vec{u}\bullet\nabla)\vec{u}+\frac{1}{\rho}\nabla p=0$$
Use the vector identity:
$$(\vec{u}\bullet\nabla)(\vec{u})\equiv\frac{1}{2}\nabla(\vec{u}\bullet\vec{u})-(\vec{u}\times\nabla\times\vec{u})$$
If the flow is irrotational,
$\nabla\times\vec{u}=0$. With this, the Euler equation above becomes
$$\frac{\partial}{\partial t}\nabla\phi+\nabla\left(\frac{u^2}{2}\right)+\frac{1}{\rho}\nabla p=0$$
From the definition of a barotropic fluid,
$\rho=f1(p)$, so that $\dfrac{\nabla p}{\rho}=f2(p)\nabla p$
$$\frac{\nabla p}{\rho}=\nabla\int f2(p)dp \quad\text{or}\quad \frac{\nabla p}{\rho}=\nabla\int\frac{dp}{\rho}$$

Substituting in the Euler equation,

$$\nabla\left|\frac{\partial}{\partial t}\phi+\left(\frac{u^2}{2}\right)+\left(\frac{\int dp}{\rho}\right)\right| = 0$$

$$\left|\frac{\partial}{\partial t}\phi+\left(\frac{u^2}{2}\right)+\left(\frac{\int dp}{\rho}\right)\right| = H(t)$$

This is the unsteady Bernoulli equation.

If we replace f by $\phi-\int H(t))dt$, the velocity field is not modified. Also, if

$$\left.\frac{\partial\phi}{\partial t}\right|_\infty = \left.\int\frac{dp}{\rho}\right|_\infty = 0$$

then $H(t) = \frac{u_\infty^2}{2}$

$$\left| \frac{\partial}{\partial t}\phi + \left(\frac{u^2}{2}\right) + \left(\frac{\int dp}{\rho}\right) \right| = H(t)$$ This is the unsteady Bernoulli equation.

If we replace f by $\phi - \int H(t))dt$, the velocity field is not modified. Also, if
$\left.\frac{\partial\phi}{\partial t}\right|_{\infty} = \left.\int\frac{dp}{\rho}\right|_{\infty} = 0$ then $H(t) = \frac{u_{\infty}^2}{2}$

POTENTIAL EQUATION

Continuity equation in differential form is:

$$\frac{D\rho}{Dt} + \rho(\nabla \bullet \hat{u}) = 0$$

, or,

$$\frac{\partial\rho}{\partial t} + (\hat{u} \bullet \nabla)\rho + \rho\nabla \bullet \hat{u} = 0$$

Use the definition of the velocity potential: $u = \nabla\Phi$, and the definition

$$\nabla^2 \equiv \nabla \bullet \nabla = \frac{\partial^2}{\partial x^2}(?) + \frac{\partial^2}{\partial y^2}(?) + \frac{\partial^2}{\partial z^2}$$

The continuity equation reduces to:

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{\hat{u}}{\rho} \bullet \nabla\rho + \nabla^2\Phi = 0$$

Special Case of Constant-Density Flow:

If the density r is constant, this reduces to: $\nabla^2\Phi = 0$. This is the Laplace equation.
Note:

1. The Laplace equation describes unsteady potential flow is **r is constant**. In most low-speed aerodynamics problems, assuming incompressible flow also ensures constant density.
2. This does NOT mean that the flowfield solution is the same for an airfoil at a degrees steady angle of attack in steady flow, and an airfoil passing through a degrees during an unsteady maneuver or during a change in flow conditions.

General case of compressible flow

Differentiate the unsteady Bernoulli equation with respect to time:

$$-\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\phi + \left(\frac{|u|^2}{2}\right)\right) = \frac{\partial}{\partial t}\left(\frac{\int dp}{\rho}\right)$$

For barotropic flow,

$$\rho = f(p)$$

$$\frac{\partial}{\partial t}\int\frac{1}{\rho}dp = \frac{\partial}{\partial t}\int f(p)dp$$

This can be written as $\frac{\partial}{\partial t}\int\frac{1}{\rho}dp = \frac{\partial}{\partial t}F(p)$

$$\frac{\partial}{\partial t}\int\frac{1}{\rho}dp = \frac{\partial}{\partial t}F(p) = \frac{dF}{dp}\frac{\partial p}{\partial t} = f(p)\frac{\partial p}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial\rho}{\partial t}$$

For the case of isentropic flow,

$$\frac{\partial}{\partial t}\int\frac{1}{\rho}dp = \frac{a^2}{\rho}\frac{\partial\rho}{\partial t}$$

$$a^2 = \left.\frac{dp}{d\rho}\right|_{isentropic} = \gamma RT$$

Substituting,

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} = -\frac{1}{a^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\phi + \left(\frac{|u|^2}{2}\right)\right)$$

Momentum equation for barotropic flow is:

$$-\nabla\left(\frac{\partial}{\partial t}\phi + \left(\frac{|u|^2}{2}\right)\right) = \frac{\nabla p}{\rho} = \frac{1}{\rho}\frac{dp}{d\rho}\nabla\rho = \frac{a^2}{\rho}\nabla\rho$$

Multiply by $\frac{\vec{u}}{a^2}$

$$\frac{\vec{u}\bullet\nabla\rho}{\rho} = -\frac{1}{a^2}\left[\frac{\partial}{\partial t}\left(\frac{|u|^2}{2}\right) + u\bullet\nabla\left(\frac{|u|^2}{2}\right)\right] = 0$$

Substituting, we get the full potential equation:

$$\nabla^2\Phi - \frac{1}{a^2}\left[\frac{\partial^2}{\partial t^2}\Phi + \frac{\partial}{\partial t}\left(\frac{|u|^2}{2}\right) + u\bullet\nabla\left(\frac{|u|^2}{2}\right)\right] = 0$$

Notes:

1. Incorporates the continuity and momentum equations for barotropic flow.

2. One scalar equation for f replaces 4 scalar equations for u,v,w and r.

Boundary Conditions:

1. Disturbances due to the body cannot grow as distance from the body increases.

Wake goes to infinity in potential flow, and waves go to infinity in supersonic flow, since there is no viscosity to dissipate them.

2. The flow at the body surface must follow the body motion.

A 3-dimensional, deforming body shape can be described by

B(x,y,z,t) = 0

The the surface boundary condition is

$$\frac{DB}{Dt} = 0 \quad \text{or,} \quad \frac{\partial B}{\partial t} + u\frac{\partial B}{\partial x} + v\frac{\partial B}{\partial y} + w\frac{\partial B}{\partial z} = 0$$

Note: For steady flow, this reduces to $\vec{u}\bullet\nabla B = 0$

Describing the surface by B(x,y,z,t) = z-za(x,y,z,t)=0, we see that

$$\frac{\partial B}{\partial x} = 1 \quad \text{, and} \quad \frac{Dz}{Dt} = w = \frac{\partial z_a}{\partial t} + u\frac{\partial}{\partial x}(z_a) + v\frac{\partial z_a}{\partial y}$$

LINEARIZATION OF THE POTENTIAL EQUATION

The assumptions made in linearization are:
1. The time derivatives of the velocity components are all of the same order of magnitude.
2. The spatial derivatives are all of the same order of magnitude.
The full potential equation can be written as

$$\nabla^2\phi - \frac{1}{a^2}\left(\Phi_{tt} + u_t^2 + \vec{u}\bullet\nabla\left(\frac{u^2}{2}\right)\right) = 0$$

Define perturbation quantities:

$$u' = u - u_\infty = \frac{\partial\phi}{\partial x}\ ;\ v' = \frac{\partial\phi}{\partial y}\ ;\ w' = \frac{\partial\phi}{\partial z}$$

Small-perturbation assumption:

Assume that $u', v', w' \ll u_\infty$ and that the disturbance motions of the body are not too rapid; that is, the time rates of change are not very large.

Surface Boundary Condition

$$w = \frac{\partial}{\partial t}z_a + u_\infty\frac{\partial z_a}{\partial x} + u'\frac{\partial z_a}{\partial x} + v'\frac{\partial z_a}{\partial y}$$

Assume that body slopes are small.

$$\frac{\partial z_a}{\partial x} \ll 1\ ;\ \frac{\partial z_a}{\partial y} \ll 1 \quad \text{so that} \quad \begin{pmatrix} u'\frac{\partial z_a}{\partial x} \\ v'\frac{\partial z_a}{\partial y} \end{pmatrix} \ll u_\infty\frac{\partial z_a}{\partial x}$$

This leads to the linearized body surface boundary condition: $w \approx \frac{\partial z_a}{\partial t} + u\frac{\partial z_a}{\partial x}$
For lifting surfaces, a more explicit form is:
Bu(x,y,z,t) = z -zu(x,y,z,t)=0
Bl(x,y,z,t) = z -zl(x,y,z,t) = 0

Planar Wing Approximation

Apply the boundary condition at z = 0, instead of at z = za.

$w \approx \frac{\partial z_u}{\partial t} + u\frac{\partial z_u}{\partial x}$ at z =0+, and $w \approx \frac{\partial z_l}{\partial t} + u\frac{\partial z_l}{\partial x}$ at z = 0-

Assuming that the time and space derivatives are of the same order of magnitude,

$$\frac{\partial u}{\partial t} + \vec{u}\bullet\nabla\left(\frac{u^2}{2}\right) \approx 2u_\infty\frac{\partial}{\partial t}u' + u_\infty^2\frac{\partial}{\partial x}u' = 2u_\infty\phi_{tx} + u_\infty^2\phi_{xx}$$

Substituting in the potential equation,

$$\nabla^2\phi - \frac{1}{a^2}(\phi_{tt} + 2u_\infty\phi_{tx} + u_\infty\phi_{xx}) = 0$$

$$\nabla^2\phi - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + u_\infty\frac{\partial}{\partial x}\right)^2\phi = 0$$

We can replace a by its value in the undisturbed freestream without incurring 1st order errors.

$$\nabla^2\phi - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + u_\infty\frac{\partial}{\partial x}\right)^2\phi = 0$$

Acceleration Potential

The Euler equation is

$$\frac{D\bar{u}}{Dt} = -\nabla\int\frac{dp}{\rho}$$

Thus the acceleration vector is the gradient of a scalar quantity which will be defined as the Acceleration Potential Y, such that

$$\nabla\Psi \equiv \frac{D\bar{u}}{Dt}$$ with components $\frac{Du}{Dt} = \frac{\partial\Psi}{\partial x}$, $\frac{Dv}{Dt} = \frac{\partial\Psi}{\partial y}$, $\frac{Dw}{Dt} = \frac{\partial\Psi}{\partial z}$

Thus $\Psi = \frac{Du}{Dt} = \frac{D}{Dt}(\nabla\phi)$.

Thus, Y also satisfies the linearized potential equation

$$\nabla^2\psi - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + u_\infty\frac{\partial}{\partial x}\right)^2\psi = 0$$

Linearized Pressure Coefficient

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho u_\infty^2} = \frac{p - p_\infty}{\rho_\infty}\left(\frac{2}{u_\infty^2}\right) = -\left(\frac{2}{u_\infty^2}\right)\psi$$ **assuming that** $\frac{\rho - \rho_\infty}{\rho_\infty} \ll 1$

Under this condition, of course, the compressible case reduces to the incompressible case. The density change due to velocity change is negligibly small.

Summary of restrictions in the linearized potential equation

1. No body forces
2. Viscous forces are negligible (regions of shear and rotation will be modeled using vortices and vortex sheets as needed).
3. Barotropic (includes isentropic flow)
4. Small body slopes.
5. Small perturbations in all flow parameters.
6. Changes with time are not too rapid.

Linearization is not valid for transonic or hypersonic flows.

Summary of Linearized Potential Equation Forms

Reference: Pierce, G.A., "Potential Flow Solutions". Course Notes, given in AE6030/6031.

The general form of th elinearized potential flow equation in stationary coordinates with uniform freestream velocity along the x-axis is:

$$\nabla^2\phi - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + u_\infty\frac{\partial}{\partial x}\right)^2\phi = 0$$

a. Steady Incompressible Flow : we get the Laplace equation

$\nabla^2\phi = 0$

The expression for a steady source is:

$$\phi(x,y,z) = \frac{\phi_0}{\sqrt{x^2+y^2+z^2}}$$

b. Unsteady incompressible flow: We still get the Laplace equation. $\nabla^2\phi = 0$

Note that the f here is f(x,y,z,t).

Assume that the potential can be factored into a space-dependent part and a time-dependent part. i.e., f (x,y,z,t) = g(x,y,z)h(t).

The Laplace equation reduces to:

$h(t)\nabla^2 g = 0$

The solution for a source is:

$$\phi(x,y,z,t) = \frac{\phi_0(t)}{\sqrt{x^2+y^2+z^2}}$$

c. Steady subsonic flow:

$$\nabla^2\phi = M_\infty^2\frac{\partial^2\phi}{\partial x^2}$$

Use the transformation: $\bar{x} = x$; $\bar{y} = \beta y$; $\bar{z} = \beta z$; $\beta = \sqrt{1-M_\infty^2}$. This gives:

$\bar{\nabla}^2\phi(\bar{x},\bar{y},\bar{z}) = 0$. The expression for a steady source is:

$$\phi(x,y,z) = \frac{\phi_0}{\sqrt{x^2+\beta^2(y^2+z^2)}}$$

d. Unsteady Subsonic Flow:

$$\nabla^2\phi - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + u_\infty\frac{\partial}{\partial x}\right)^2\phi = 0$$

Use a Lorentz transformation: $\bar{x} = x$; $\bar{y} = \beta y$; $\bar{z} = \beta z$; $\beta = \sqrt{1-M_\infty^2}$;

$\bar{t} = t + \dfrac{M_\infty x}{a_\infty\beta^2}$;

$\omega = a_\infty\beta^2\nu$ The linearized potential equation is:

$$\bar{\nabla}^2\phi = \frac{1}{a_\infty^2\beta^4}\frac{\partial^2\phi}{\partial \bar{t}^2}$$

Using separation of variables as before, assume that the potential can be factored into a function of space and a function of time. $\phi(\bar{x},\bar{y},\bar{z},\bar{t}) = g(\bar{x},\bar{y},\bar{z})h(\bar{t})$

this reduces to two ordinary differential equations:

$h'' + a^2\beta^4\vartheta^2 h = 0$

$\bar{\nabla}^2 g + \vartheta^2 g = 0$

The solution for a source is: $\phi(x,y,z,t) = \frac{\phi_0}{R}e^{i\omega\tau}$

where $\tau = t + \frac{1}{a_\infty\beta^2}(M_\infty x - R)$, and $R = \sqrt{x^2 + \beta^2(y^2 + z^2)}$

e. Unsteady Supersonic Flow

$$\nabla^2\phi - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + u_\infty\frac{\partial}{\partial x}\right)^2\phi = 0$$

Use the modified Lorentz transformation:

$\bar{x} = x$; $\bar{y} = i\beta y$; $\bar{z} = i\beta z$; $\beta = \sqrt{M_\infty^2 - 1}$; $\bar{t} = t - \frac{M_\infty x}{a_\infty\beta^2}$

The linearized potential equation becomes:

$$\bar{\nabla}^2\phi = \frac{1}{a_\infty^2\beta^4}\frac{\partial^2\phi}{\partial\bar{t}^2}$$

Separating variables, $\phi(\bar{x},\bar{y},\bar{z},\bar{t}) = \frac{\phi_0}{\bar{R}}e^{i\omega\left(\bar{t} \pm \frac{\bar{R}}{a_\infty\beta^2}\right)}$ where $R = \sqrt{x^2 - \beta^2(y^2 + z^2)}$

Transforming back to the original coordinates,

$\phi(x,y,z,t) = \frac{\phi_0}{\bar{R}}e^{i\omega\left(t - \frac{(M_\infty x \pm R)}{a_\infty\beta^2}\right)}$ where $R = \sqrt{x^2 - \beta^2(y^2 + z^2)}$