Blasius Equation Derivation

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Developing the Incompressible Blasius Solution starts with the 2D incompressible Navier-Stokes equations (neglecting buoyancy effects):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1b}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{1c}$$

$$\rho C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] \tag{1d}$$

The two-dimensional Navier-Stokes equations are simplified by using the following assumptions that apply to boundary layers:

1. Boundary layer thickness (δ) is small, i.e. $Re \gg 1$
2. Boundary layer is laminar
3. Buoyancy effects are negligible, i.e. $Fr \gg 1$
4. Freestream velocity (U_e) is constant, or $\frac{dU_e}{dx} = 0$, $\frac{dp}{dx} = 0$ where x is the streamwise direction and y is the direction normal to the surface

where 1 and 2 are assumptions of the boundary layer equations, and 3 and 4 are assumptions of the Blasius solution. These assumptions yield the following simplifications to the Navier-Stokes equations

$$\frac{\partial}{\partial t} \approx 0 \qquad \text{(steady)} \tag{2a}$$

$$u \gg v \qquad \text{(thin boundary layer)} \tag{2b}$$

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} \qquad \text{(thin boundary layer)} \tag{2c}$$

$$\nabla p = \boldsymbol{0} \qquad \text{(const. freestream velocity)} \tag{2d}$$

Applying the assumptions (2a) – (2d) to (1a) – (1d) yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{3b}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3c}$$

with the following boundary conditions

$$u(x,0) = v(x,0) = 0, \quad u(x,\infty) = U_e \tag{3d}$$

Notice that in (3) the y-momentum equation has vanished. This is due to assumptions (2a) – (2c) turning the equation to $\frac{\partial p}{\partial y} = 0$, with assumption (2d) creating the same relation.

In order to remove the continuity equation, a stream function (Ψ) is introduced that satisfies the following relation

$$\Psi(x,y) = \int_0^y u dy \tag{4}$$

such that $$u(x,y) = \frac{\partial \Psi}{\partial y}$$

$$v(x,y) = -\frac{\partial \Psi}{\partial x}$$

Without losing generality, the stream function can take the following form

$$\Psi(x,y) = \lambda(x) f(\eta) \tag{5}$$

where $$\eta = \eta(x,y)$$

Applying (5) to (4) yields

$$u = \lambda \frac{\partial \eta}{\partial y} f' \tag{6a}$$

$$v = -\frac{d\lambda}{dx} f - \lambda \frac{\partial \eta}{\partial x} f' \tag{6b}$$

where $$f' \equiv \frac{\partial f}{\partial \eta}$$

Next, assuming that η is linear in y produces the following relations

$$\eta = yg(x) \tag{7a}$$

$$\frac{\partial \eta}{\partial x} = y\frac{dg}{dx} = \eta \frac{g'}{g} \longrightarrow y\cancel{g}\frac{g'}{\cancel{g}} = yg' \tag{7b}$$

$$\frac{\partial \eta}{\partial y} = g \tag{7c}$$

Plugging (7a) – (7c) into (6a) and (6b) to get

$$u = \lambda g f' \tag{8a}$$

$$v = -\lambda' f - \lambda \eta \frac{g'}{g} f' \tag{8b}$$

Finally, λ can be eliminated by noticing that $u(x,\infty) = U_e$ if $f'(\infty) = 1$, thus

$$\lambda = \frac{U_e}{g} \tag{9a}$$

$$\lambda' = -\frac{U_e g'}{g^2} \tag{9b}$$

which when plugged into (6b) yields the following equations

$$u = U_e f' \tag{10a}$$

$$v = \frac{U_e g'}{g^2}(f - \eta f') \tag{10b}$$

as well as the following derivatives

$$\frac{\partial u}{\partial x} = U_e \frac{\eta g'}{g} f'' \tag{11a}$$

$$\frac{\partial u}{\partial y} = U_e g f'' \tag{11b}$$

$$\frac{\partial^2 u}{\partial y^2} = U_e g^2 f''' \tag{11c}$$

Addressing the boundary conditions given in (3d) results in

$$f(0) = 0 \qquad \text{no slip} \tag{12a}$$
$$f'(0) = 0 \qquad \text{no slip} \tag{12b}$$
$$f'(\infty) = 1 \qquad \text{freestream merge} \tag{12c}$$

With (10) and (11), the simplified momentum equation (3b) turns into (with some simplifications)

$$f''' - \frac{U_e g'}{\nu g^3} f f'' = 0 \tag{13}$$

The problem now resolves down to finding a valid choice of $g(x)$ such that (13) is independent of x. To find the necessary $g(x)$, set the coefficient $-\frac{U_e g'}{\nu g^3} = 1$ and solve for g to get the following

$$g(x) = \sqrt{\frac{U_e}{2\nu x}} \tag{14}$$

Finally, combining (7a), (12), (13), (14), and into (15) results in the full definition of the Blasius solution

$$f''' + f f'' = 0 \tag{15}$$
$$\text{with the boundary conditions} \qquad f(0) = f'(0) = 0, \quad f'(\infty) = 1$$
$$\text{where} \qquad f = f(\eta), \quad \eta = y\sqrt{\frac{U_e}{2\nu x}}$$

Notice that in developing the final Blasius solution, the energy equation (3c) has not been used, thus it is completely decoupled from the continuity and momentum equations and can be solved separately from the Blasius solution.

To complete this development, values for u and v as well as the wall shear stress need to be developed. Using (14) it can be shown that

$$u(\eta) = U_e f'(\eta) \tag{16a}$$
$$\begin{aligned} v(\eta) &= \nu g(x)\left[\eta f'(\eta) - f(\eta)\right] \\ &= \sqrt{\frac{\nu U_e}{2x}}\left[\eta f'(\eta) - f(\eta)\right] \end{aligned} \tag{16b}$$

Lastly, the wall shear stress can be found to be

$$\tau_w(x) \equiv \mu \left.\frac{\partial u}{\partial y}\right|_w = \mu U_e g(x) f''(0) \tag{17}$$

and the derivation is complete.