436432 Thermofluids 4 - Boundary Layer Theory

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Contents

1 Introduction

The study of boundary layers is important for many reasons, some examples are;

- Skin friction drag (eg. on a ship or aeroplane)
- Separation (external and internal flows)
- Pressure distribution on a body
- Atmospheric boundary layers

Some References;
Most fluid mechanics text books will have chapters on boundary layers, eg.

- Gerhart and Gross "Fundamentals of Fluid Mechanics", Addison Wesley.
- Munson, Young and Okiishi "Fundamentals of Fluid Mechanics", Wiley.
- White "Fluid Mechanics", McGraw Hill.
- Smits "A physical introduction to Fluid Mechanics", Wiley.

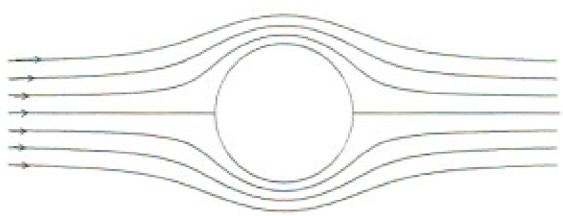
More advanced texts;

- Schlichting "Boundary Layer Theory"
- Hinze "Turbulence"
- Knudsen & Katz "Fluid Dynamics and Heat Transfer"

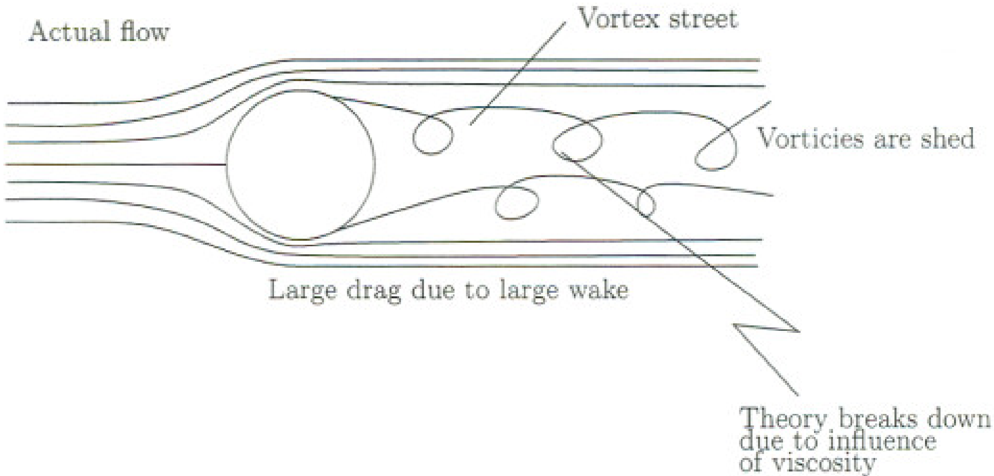
2 Background

Fluid flow is governed by the Navier-Stokes equations (developed in the early 19th century). The Navier-Stokes equations are non-linear and difficult to solve, only a few particular solutions have been found.

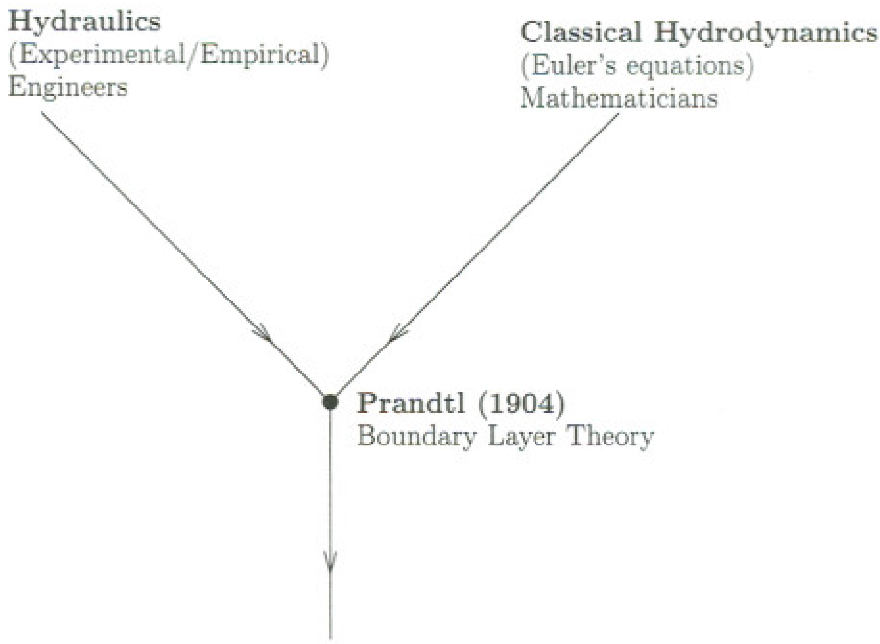
If we ignore viscosity (ie. fluid is frictionless) the N-S equations simplify to the Euler equations. Treating the fluid as frictionless is known as classical hydrodynamics [3rd year]. Euler



Drag = 0



For the two most common fluids (air and water) ν is relatively small. People could not understand why Euler failed to agree with experimental observations (d'Alembert paradox) and fluid dynamics was divided into two separate camps.



Prandtl brought together the two divergent fields of fluid dynamics. He showed that flow about a solid body can be divided into two regions. In a thin region adjacent to the body the viscous terms play an important part and this is termed the *boundary layer*. Beyond the boundary layer the flow can be considered inviscid and hence is approximated by Euler. However it should be noted the that the boundary layer has a strong influence on the boundary conditions for the inviscid (Euler) region.

3 Complete Navier-Stokes equations

Momentum equations

If we consider incompressible flow (ie. uniform density) and constant temperature fluid $\Rightarrow\ \rho = \text{const},\ \nu[T] = \text{const}$

$$x\text{-dir}: \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

$$y\text{-dir}: \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2}$$

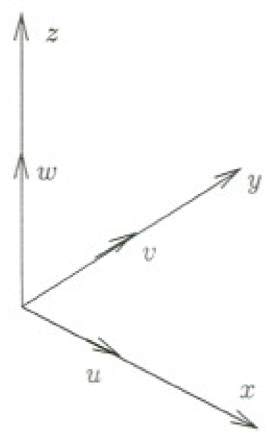
$$z\text{-dir}: \quad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

By applying continuity we can also obtain the

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

NB:



The above can be written more concisely in **vector form**

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underset{\sim}{u} \tag{5}$$

$$\nabla\cdot\underset{\sim}{u} = 0 \tag{6}$$

where

$$\underset{\sim}{u} = u\underset{\sim}{i} + v\underset{\sim}{j} + w\underset{\sim}{k} \text{ and } \nabla = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z} . \quad (7)$$

Or alternatively in **tensor form**

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \left(\frac{\partial^2 u_i}{\partial x_j x_j} \right) \quad (8)$$

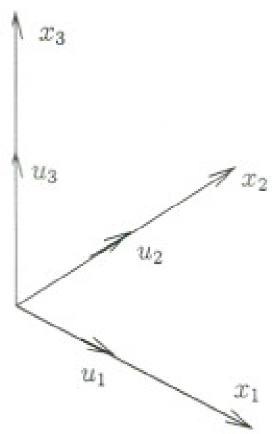
and for continuity

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (9)$$

where

$$i = 1,\ 2,\ 3$$
$$j = 1,\ 2,\ 3\ .$$

Here we use the coordinate system;

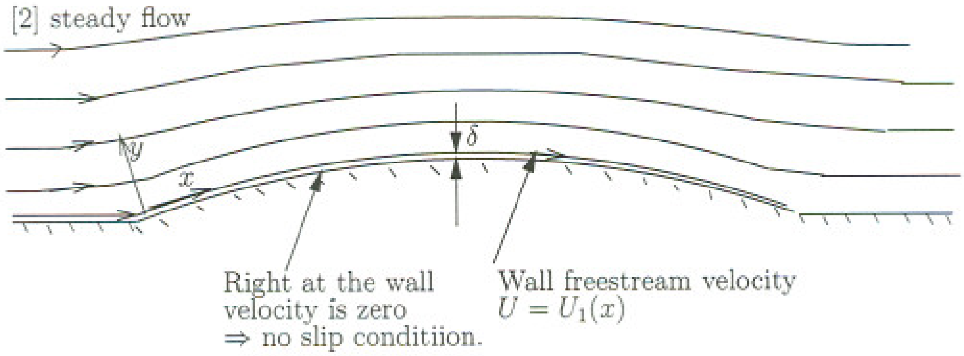


Repeated subscripts ⇒ summation (Einstein's convention), for example for $i = 1$;

$$\frac{\partial u_1}{\partial t} + u_j \frac{\partial u_1}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu \left(\frac{\partial^2 u_1}{\partial x_j x_j} \right)$$
$$\therefore \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right)$$

4 Steady [2] flow

Consider flow over a thin aerofoil;



If $\nu = 0$ N-S equations become the [2] Euler equations;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{10}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{11}$$

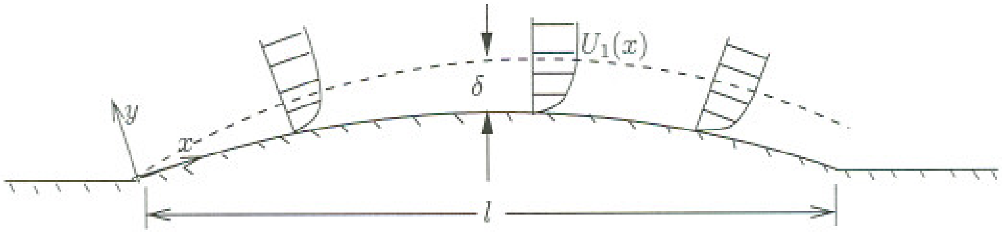
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\ . \tag{12}$$

When boundary conditions are irrotational we have potential flow, that is

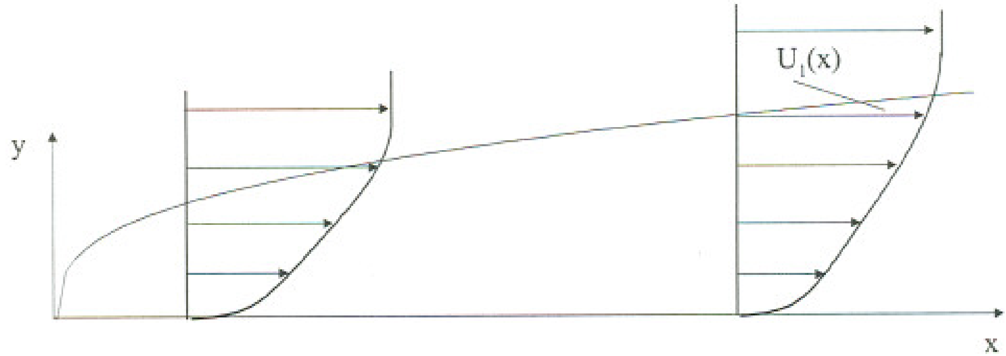
$$\nabla^2\psi = 0 \text{ and } \nabla^2\phi = 0$$

It turns out that over most of the flow Euler's equations are a valid approximation, ie. over most of the flow viscous forces are negligible compared with inertia forces and pressure gradient forces. Only in a thin layer close to the wall are the viscous forces as important as the pressure and inertial forces. In this region the velocity drops from the freestream velocity $U_1(x)$ to zero on the wall, owing to the action of viscosity. We will call this region the *boundary layer* and we will denote its thickness by δ. Prandtl's boundary layer hypothesis states as $\nu \to 0$, $\delta \to 0$.

If we magnified the boundary layer thickness then in the mean we have something like this;



Let us consider a very gentle slope $\Rightarrow \frac{\delta}{l} \ll 1$ and hence curvature ≈ 0,



Consider a laminar boundary layer which is [2] with a steady freestream flow. The equations which describe the flow are;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{13}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{14}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\ . \tag{15}$$

With boundary conditions;

$$x = 0, \quad u = U_1(0)$$

$$y = 0, \quad u = v = 0$$

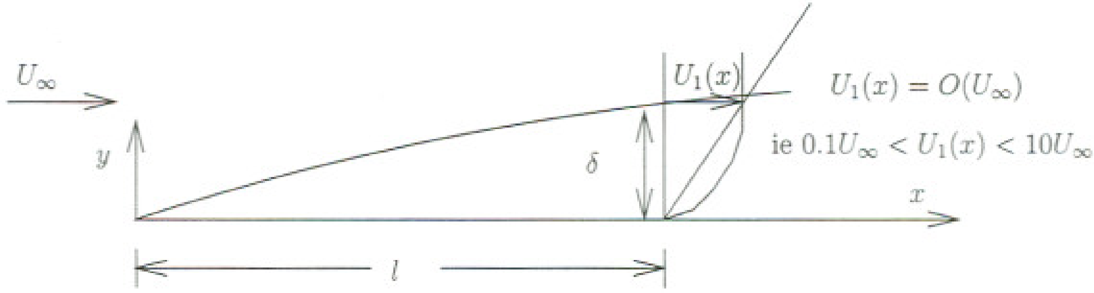
$$y/\delta = \infty, \quad u = U_1(x)$$

$$\text{and} \quad U_1\frac{\partial U_1}{\partial x} = -\frac{1}{\rho}\frac{\partial p_1}{\partial x}$$

$$\therefore \frac{1}{2}U_1^2 + \frac{p_1}{\rho} = \text{const.}$$

5 Order of magnitude argument

The above equations still present mathematical difficulties. To progress we will apply Prandtl's *order of magnitude argument*. The argument looks very rough and approximate but it gives us an idea of the relative importance of the various terms in the equations. However for $Re \to \infty$ the argument becomes asymptotically exact.



For a typical streamline a representative change in velocity $\approx O(U_\infty)$ over a length $O(l)$ hence

$$\frac{\partial u}{\partial x} = O\left(\frac{U_\infty}{l}\right) .$$

A representative value for velocity gradient

$$\frac{\partial u}{\partial y} = O\left(\frac{U_\infty}{\delta}\right) .$$

The maximum change in y for a streamline is $O(\delta)$ over a length l which gives a representative streamline slope

from the above a representative value of v is then given by

$$\frac{v}{U_\infty} = O\left(\frac{\delta}{l}\right)$$

$$\text{or } v = O\left(U_\infty \frac{\delta}{l}\right) .$$

The above result can also be obtained using continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$O\left(\frac{U_\infty}{l}\right) + O\left(\frac{v}{\delta}\right) = 0$$

$$\therefore v = O\left(U_\infty \frac{\delta}{l}\right) .$$

Note we are interested in magnitudes so we ignore the sign.

Higher derivatives

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\\ &= \frac{\Delta\left(\frac{\partial u}{\partial y}\right)}{\Delta y}\\ &= \frac{O\left(\frac{U_\infty}{\delta}\right)}{O(\delta)}\\ \therefore \frac{\partial^2 u}{\partial y^2} &= O\left(\frac{U_\infty}{\delta^2}\right)\end{aligned}$$

similarly

$$\frac{\partial^2 u}{\partial x^2} = O\left(\frac{U_\infty}{l^2}\right) .$$

Now substitute all of the above into the Navier-Stokes equations;

$$\begin{aligned}\text{in } x\text{-dir} &\rightarrow O\left(\frac{U_\infty^2}{l}\right) + O\left(\frac{U_\infty^2}{l}\right) = O\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + O\left\{\nu\left(\frac{U_\infty}{l^2} + \frac{U_\infty}{\delta^2}\right)\right\}\\ \text{in } y\text{-dir} &\rightarrow O\left(\frac{U_\infty^2\delta}{l^2}\right) + O\left(\frac{U_\infty^2\delta}{l^2}\right) = O\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) + O\left\{\nu\left(\frac{U_\infty\delta}{l^3} + \frac{U_\infty}{l\delta}\right)\right\} .\end{aligned}$$

Divide by U_∞^2/l;

$$\begin{aligned}&\text{in } x\text{-dir} \rightarrow O(1) + O(1) = O\left(\frac{-\partial p/\rho U_\infty^2}{\partial x/l}\right) + O\left(\frac{1}{\frac{U_\infty l}{\nu}}\right) + O\left(\frac{1}{\frac{U_\infty l}{\nu}}\frac{l^2}{\delta^2}\right)\\ &\text{in } y\text{-dir} \rightarrow \underbrace{O\left(\frac{\delta}{l}\right) + O\left(\frac{\delta}{l}\right)}_{\text{Inertia force terms}} = \underbrace{O\left(\frac{-\partial p/\rho U_\infty^2}{\partial y/l}\right)}_{\text{pressure gradient force terms}} \underbrace{+ O\left(\frac{1}{\frac{U_\infty l}{\nu}}\frac{\delta}{l}\right) + O\left(\frac{1}{\frac{U_\infty l}{\nu}}\frac{l}{\delta}\right)}_{\text{viscous force terms}}\end{aligned}$$

The Reynolds number is given by

$$Re = \frac{U_\infty l}{\nu}$$

and it expresses the ratio of inertial forces to viscous forces. Consider the equation in the x-dir as $Re \rightarrow \infty$,

$$O(1) + O(1) = \left(\frac{-\partial p/\rho U_\infty^2}{\partial x/l}\right) + 0 + O\left(\frac{1}{Re}\frac{l^2}{\delta^2}\right)$$

What is the order of the viscous term, ie. $O\left(\frac{1}{Re}\frac{l^2}{\delta^2}\right)$?

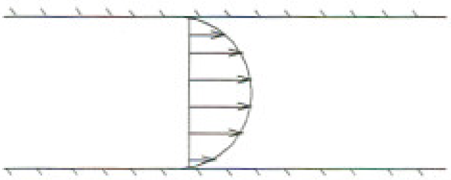
Case (a)

$$O\left(\frac{1}{Re}\frac{l^2}{\delta^2}\right) \gg O(1)$$

Here the viscous terms dominate over inertia terms. This leaves us with a balance of pressure gradient and viscous forces ie.

$$O\left(\frac{\partial p/\rho U_\infty^2}{\partial x/l}\right) = O\left(\frac{1}{Re}\frac{l^2}{\delta^2}\right)$$

Such a situation would exists for fully developed flow in straight parallel pipes and ducts.



Case (b)

$$O\left(\frac{1}{Re}\frac{l^2}{\delta^2}\right) \ll O(1)$$

This leaves a balance of inertia and pressure gradient forces ie. inviscid or Euler equation flow.

Case (c)

$$O\left(\frac{1}{Re}\frac{l^2}{\delta^2}\right) = O(1)$$

In this case inertia, pressure gradient and viscous forces are all of the same importance. This is the case Prandtl chose to represent boundary layers. It implies that

$$\begin{aligned}\frac{\delta}{l} &= O\left(\frac{1}{\sqrt{Re}}\right)\\ \therefore \delta &= O\left(\frac{l}{\sqrt{Re}}\right)\ .\end{aligned}$$

Substitute the above back into the momentum equations

$$\text{in } x\text{-dir} \rightarrow O(1) + O(1) = O\left(\frac{-\partial p/\rho U_\infty^2}{\partial x/l}\right) + O\left(\frac{1}{Re}\right) + O(1)$$

$$\text{in } y\text{-dir} \rightarrow O\left(\frac{1}{\sqrt{Re}}\right) + O\left(\frac{1}{\sqrt{Re}}\right) = O\left(\frac{-\partial p/\rho U_\infty^2}{\partial y/l}\right) + O\left(\frac{1}{Re^{3/2}}\right) + O\left(\frac{1}{\sqrt{Re}}\right)$$

Again consider $Re \to \infty$

$$\text{in } x\text{-dir} \to O(1) + O(1) = O(\text{PGT}) + 0 + O(1)$$
$$\text{in } y\text{-dir} \to 0 + 0 = O(\text{PGT}) + 0 + 0$$

We have identified the important terms in the momentum equations. it can be seen that for the y-dir the above implies

$$-\frac{\partial p}{\partial y} = 0$$
$$\to p \neq f(y)\ .$$

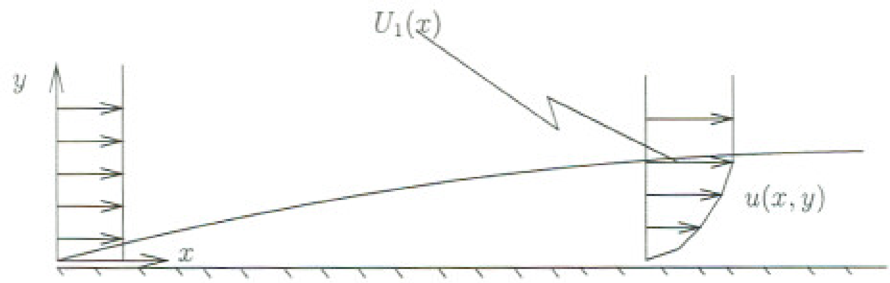
Therefore the approximate equations that govern the boundary layer flow are;

$$\boxed{\text{Momentum} \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}} \qquad (16)$$

$$\boxed{\text{Continuity} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \qquad (17)$$

the above are often referred to as Prandtl's boundary layer approximations. However for $Re \to \infty$ they are exact.

Boundary conditions



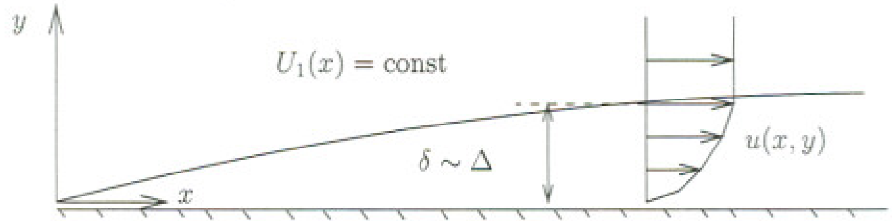
$$y = 0, \quad v = u = 0$$
$$y = \infty, \quad u = U_1(x) \quad \text{and} \quad U_1\frac{\partial U_1}{\partial x} = -\frac{1}{\rho}\frac{dp_1}{dx}$$

6 Zero pressure gradient flat plate boundary layer

Consider;

- laminar B/L
- zero pressure gradient

often called the Blasius layer.



The boundary layer equations become;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

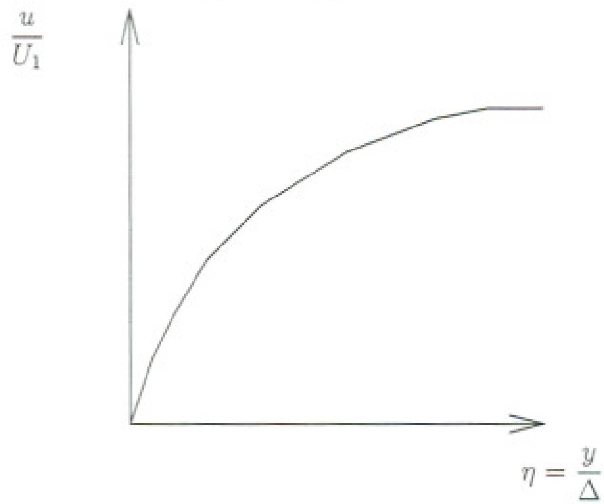
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

Δ is a length scale which is proportional to δ and will be defined later. Boundary conditions;

$$\left.\begin{aligned} u = v = 0 \quad &\text{at} \quad y = 0 \\ u = U_1 \quad &\text{at} \quad y \gg \Delta \\ u = U_1 \quad &\text{at} \quad x = 0 \end{aligned}\right\} \quad (3)$$

Prandtl suggested that a similarity solution may be valid ie.

$$\frac{u}{U_1} = F\left(\frac{y}{\Delta}\right)$$



This assumes all results can be non-dimensionalised such that they fall on one universal curve, independent of ν, ρ, U_1 and x. Put $\eta = y/\Delta$, $u = U_1 F(\eta)$

Boundary conditions require

$$
\begin{aligned}
F =& 0 \quad \text{at} \quad \eta = 0 \\
F =& 1 \quad \text{at} \quad \eta = \infty
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} =& U_1 \frac{dF}{d\eta}\frac{\partial \eta}{\partial x} \\
=& U_1 F' \frac{\partial \eta}{\partial x} \\
=& -U_1 \frac{F'}{\Delta^2} y \frac{d\Delta}{dx} \\
=& \frac{-U_1}{\Delta}\frac{d\Delta}{dx}\eta F'
\end{aligned}
$$

Using (2)

$$
\begin{aligned}
v =& -\int_0^y \left(\frac{\partial u}{\partial x}\right) dy \\
=& \int_0^y \frac{d\Delta}{dx}\frac{U_1}{\Delta}\eta F' dy \\
=& U_1 \frac{d\Delta}{dx}\int_0^\eta \eta F' d\eta \\
=& U_1 \frac{d\Delta}{dx}([\eta F]_0^\eta - \int_0^\eta F d\eta) \\
=& U_1 \frac{d\Delta}{dx}\left(\eta F - \int_0^\eta F d\eta\right)
\end{aligned}
$$

Change variables to get rid of integrals

$$
\text{Put} \quad \int_0^\eta F d\eta = \frac{f}{2} \quad \text{ie,} \quad F = \frac{f'}{2}
$$

therefore

$$
\begin{aligned}
u =& U_1 \frac{f'}{2} \\
\frac{\partial u}{\partial x} =& \frac{-U_1}{\Delta}\frac{d\Delta}{dx}\eta\frac{f''}{2} \\
v =& U_1 \frac{d\Delta}{dx}\left(\eta\frac{f'}{2} - \frac{f}{2}\right) \\
\frac{\partial u}{\partial y} =& \frac{U_1}{2}\frac{f''}{\Delta} \\
\frac{\partial^2 u}{\partial y^2} =& \frac{U_1}{2}\frac{f'''}{\Delta^2}
\end{aligned}
$$

Substitute above into (1)

$$\frac{-U_1^2}{\Delta}\frac{d\Delta}{dx}\frac{\eta f' f''}{4} + \frac{U_1^2}{\Delta}\frac{d\Delta}{dx}\frac{\eta f' f''}{4} - \frac{U_1^2}{\Delta}\frac{d\Delta}{dx}\frac{f f''}{4} = \frac{\nu U_1}{\Delta^2}\frac{f'''}{2}$$

$$\therefore \quad \underbrace{\frac{-U_1}{\nu}\Delta\frac{d\Delta}{dx}}_{\text{function of } x \text{ alone}} = \underbrace{\frac{2f'''}{f f''}}_{\text{function of } \eta \text{ alone}} \tag{4}$$

The above is only possible if both sides are equal to a constant (ie. separation of variables). From inspection the constant must be negative, so put

$$\frac{-U_1}{\nu}\Delta\frac{d\Delta}{dx} = -\kappa \tag{5}$$

$$\frac{2f'''}{f f''} = -\kappa \tag{6}$$

Where κ is a positive constant, further it is a *universal* constant since $\frac{2f(\eta)'''}{f(\eta)f''(\eta)}$ is a universal function. However the value of κ is arbitrary and the choice for κ simply fixes the definition of Δ eg. $y/\Delta = 1$ when $u/U_1 = 0.5$. For historical reasons we put $\kappa = 2$. Note we have gone from a p.d.e to an o.d.e with the simple assumption of $u/U_1 = F(\eta)$.

<u>Solution</u>

$$\frac{-U_1}{\nu}\Delta\frac{d\Delta}{dx} = -2$$

$$\therefore \int_0^{\Delta} \Delta d\Delta = \frac{2\nu}{U_1}\int_0^x dx$$

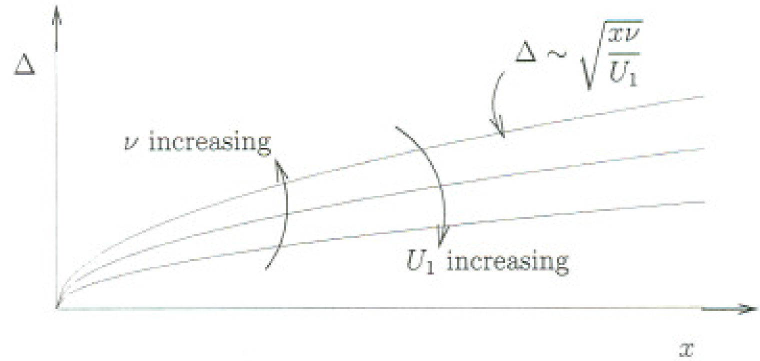
$$\frac{\Delta^2}{2} = \frac{2\nu}{U_1}x$$

$$\Delta = 2\sqrt{\frac{\nu x}{U_1}}$$

or

$$\eta = \frac{y}{\Delta} = \frac{y}{2}\sqrt{\frac{U_1}{\nu x}}$$

This tells us the growth of the boundary layer.



From (6)

$$f''' + ff'' = 0 \quad \text{3rd order non linear ode} \tag{7}$$

Zero pressure gradient flat plate B/L (c'ont)

We will attempt to find an analytical solution (alternative is to solve numerically). Have

$$\eta = \frac{y}{2}\sqrt{\frac{U_1}{\nu x}}, \quad u = U_1\frac{f'}{2}, \quad f(0) = 0$$

Boundary conditions

$$\begin{aligned} u =& 0 \quad \text{at} \quad y = 0 \rightarrow f'(0) = 0 \\ u =& U_1 \quad \text{at} \quad y \gg \Delta \rightarrow f'(\infty) = 2 \\ u =& U_1 \quad \text{at} \quad x = 0 \end{aligned}$$

We wish to determine the skin friction coefficient where

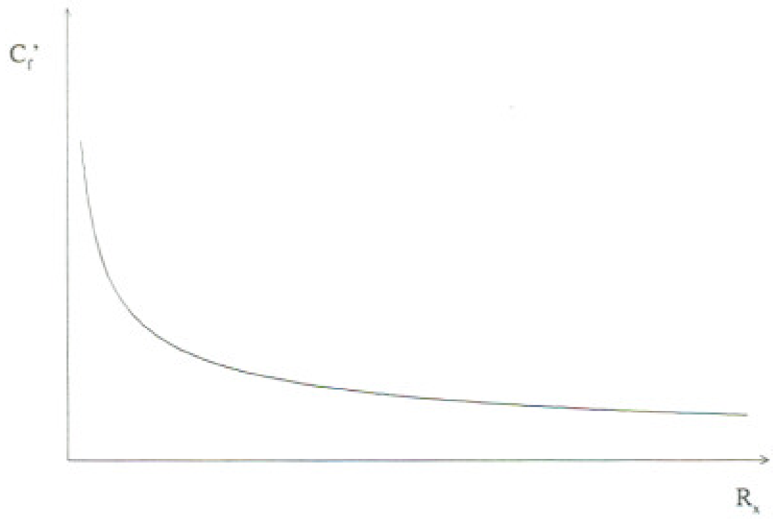
$$C_f' = \frac{\tau_0}{\frac{1}{2}\rho U_1^2}, \quad \tau = \mu\frac{\partial u}{\partial y}, \quad \tau_0 = \mu\frac{\partial u}{\partial y}\bigg|_{y=0}$$

$$\begin{aligned} \frac{\tau}{\rho} &= \frac{\nu U_1 f''}{2\Delta} \\ \therefore \frac{\tau_0}{\rho} &= \frac{\nu U_1 f''(0)}{2\Delta} \\ &= \frac{\nu U_1}{2\cdot 2}\sqrt{\frac{U_1}{\nu x}} f''(0) \\ &= \frac{1}{4}\frac{\nu^{1/2}U_1^{3/2}}{x^{1/2}} f''(0) \\ \therefore C_f' &= \frac{1}{\frac{1}{2}U_1^2}\frac{1}{4}\frac{\nu^{1/2}U_1^{3/2}}{x^{1/2}} f''(0)\ . \end{aligned}$$

The Reynolds number is given by $R_x = \frac{xU_1}{\nu}$ hence

$$\boxed{C_f' = \frac{f''(0)}{2}\frac{1}{(R_x)^{\frac{1}{2}}}} \tag{8}$$

Because f is a universal function $f''(0)/2$ is a universal constant and if we could solve (7) we can find $f''(0)$ and hence obtain the local skin friction formula, for a z.p.g laminar boundary layer ie.



The problem is to find $f''(0)$. Expand F into a Taylor series

$$f = c_0 + c_1\eta + \frac{c_2\eta^2}{2!} + \frac{c_3\eta^3}{3!} + \cdots \tag{9}$$

$$f' = c_1 + \frac{2c_2\eta}{2!} + \frac{3c_3\eta^2}{3!} + \frac{4c_4\eta^3}{4!} + \cdots \tag{10}$$

From BC's $f(0) = f'(0) = 0$ hence $c_0 = c_1 = 0$. Substitute (9) and (10) into (7) and collect like powers,

$$c_3 + c_4\eta + \left(\frac{c_2^2}{2} + \frac{c_5}{2}\right)\eta^2 + \cdots = 0$$

This must converge (ie. $= 0$) for all values of η,

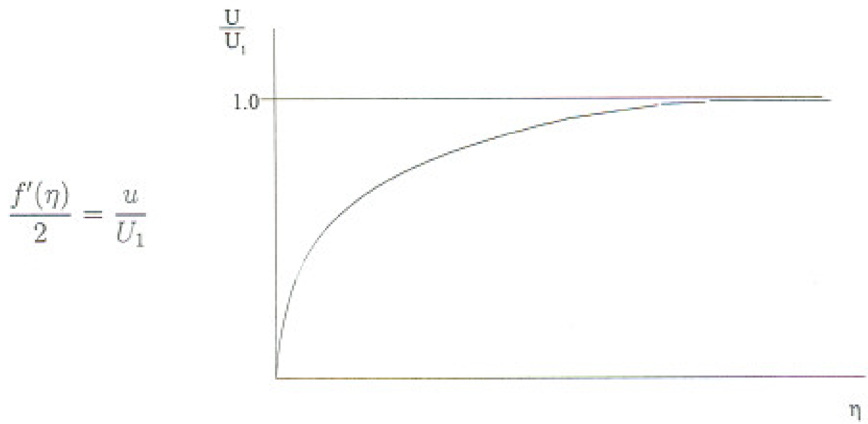
$$\therefore c_3 = c_4 = \left(\frac{c_2^2}{2} + \frac{c_5}{2}\right) = 0\ .$$

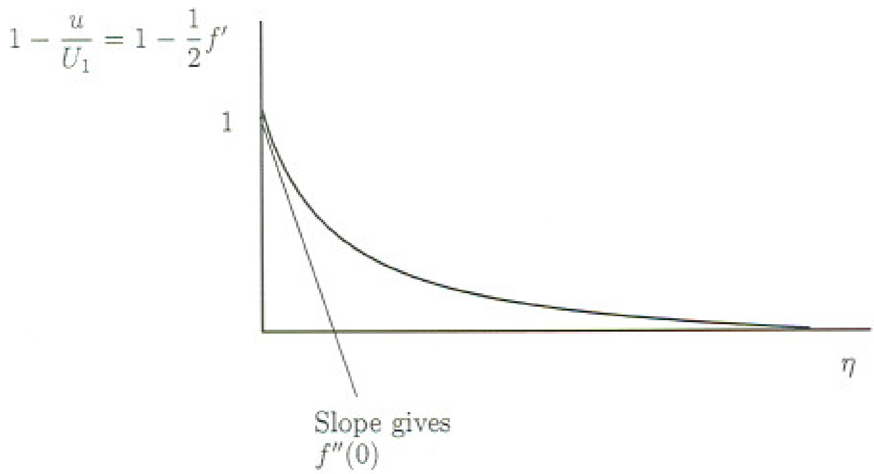
It turns out that all non-zero coefficients in the series can be expressed in terms of c_2,

$$f = \frac{c_2\eta^2}{2!} - \frac{c_2^2\eta^5}{5!} + \frac{11c_2^3\eta^8}{8!} - \frac{375c_2^4\eta^{11}}{11!} + \cdots$$

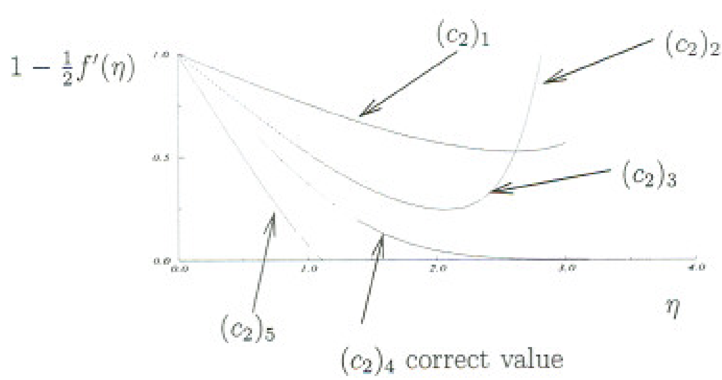
Boundary conditions

The constant c_2 is unknown but we can evaluate it from the freestream boundary conditions ie. $f'(\infty) = 2$.





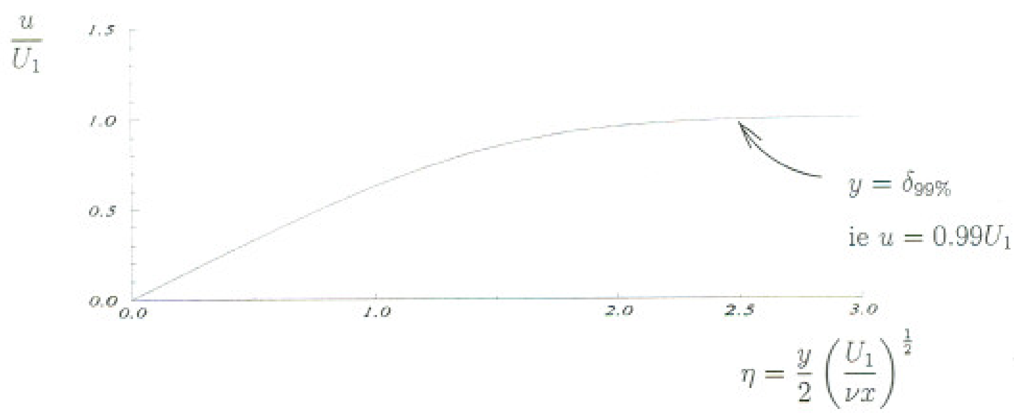
Using trial and error method chose c_2 until we get a solution that satisfies the boundary condition ie.



Further details of this method can be found in Schlicting 'B/L theory' p. 126, Duncan, Thom & Young 'Mechanics of Fluids' p. 260 and Knudsen & Katz 'Fluid dynamics and heat transfer' p. 253. Blasius got $c_2 = 1.32824$, $\Rightarrow f''(0)/2 = 0.664$, substitute into (8)

$$\boxed{C_f' = 0.664\,(R_x)^{-\frac{1}{2}}} \qquad \text{first successful B/L equation (1930's)}$$

We have now also obtained the universal velocity profile in the form of a Taylor series expansion



The relation between the length scale Δ and the boundary layer thickness δ (99% thickness), for the case when we chose $\kappa = 2$ is then

$$\Delta \approx \frac{\delta}{2.5}\,.$$

We can also evaluate the average skin friction (C_f) for a plate of length l

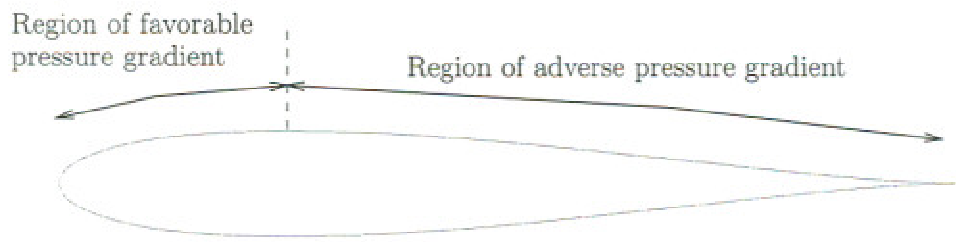
$$\begin{aligned} C_f &= \frac{1}{l}\int_0^l C_f' dx \\ &= \frac{1}{\frac{U_1 l}{\nu}}\int_0^{\frac{U_1 l}{\nu}} \frac{0.664}{R_x^{\frac{1}{2}}} dR_x \\ &= f\left(\frac{U_1 l}{\nu}\right) \qquad \boxed{\text{Exercise; find}} \end{aligned}$$

Note the difference between;

- Local skin friction coefficient, $C_f' = \dfrac{\tau_0}{\frac{1}{2}\rho U_1^2}$
- Average skin friction coefficient, $C_f = \dfrac{D}{\frac{1}{2}\rho U_1^2 l}$ where D =drag/unit span

7 Effect of pressure gradients

In general all bodies have streamwise pressure gradients (except a flat plate aligned with the flow, Blasius soln.).



The momentum equation says

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \, .$$

To get a feeling for what goes on at the wall consider $y \to 0$ (ie. $v = u = 0$) therefore

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y\to 0} \, .$$

Now shear stress $\tau \approx \mu\frac{\partial u}{\partial y}$ actually $\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ but from the order of magnitude argument it was found $\frac{\partial v}{\partial x} \ll \frac{\partial u}{\partial y}$.

Close to the wall
$$\boxed{\frac{dp}{dx} = \mu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \left(\frac{\partial \tau}{\partial y}\right)_{y=0}} \qquad (11)$$

We also know

In the freestream
$$\boxed{\frac{\partial u}{\partial y} = 0, \quad \tau = 0} \qquad (12)$$

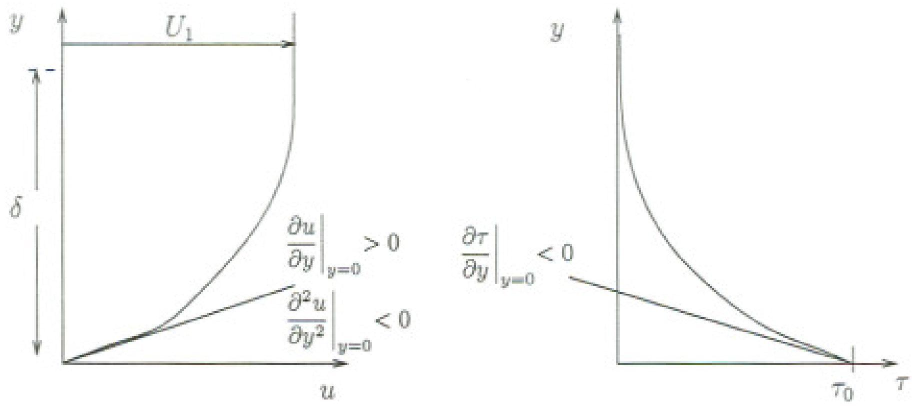
Using (11) and (12) lets deduce the shape of the velocity and shear stress profiles for different cases.

Case (1)

$\tau_0 > 0$ ie. $\left(\frac{\partial u}{\partial y}\right)_{y=0} > 0$, there are two possibilities.

1a) A favourable pressure gradient

$$\Rightarrow \frac{dp}{dx} < 0$$
$$\Rightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} < 0 \quad \text{ie;} \quad \left.\frac{\partial \tau}{\partial y}\right|_{y=0} < 0$$

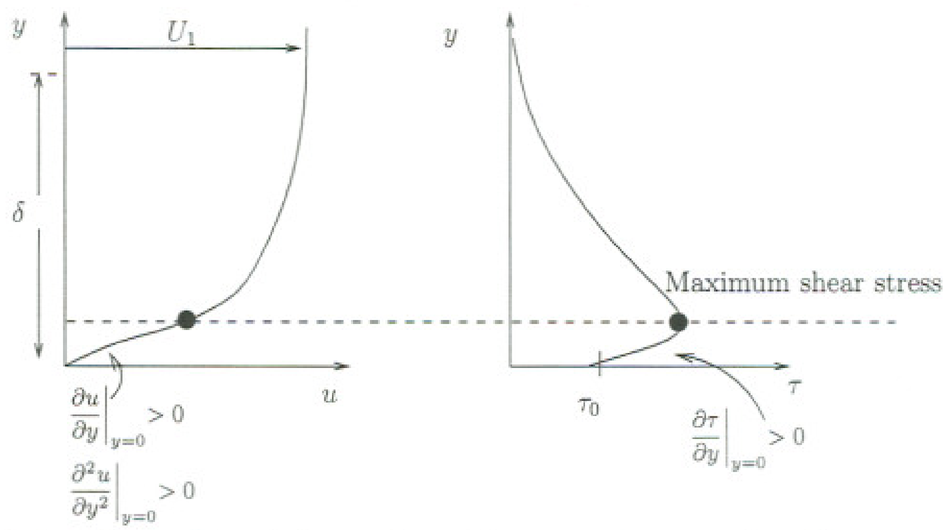


This is termed a healthy profile and maximum shear stress is at the wall.

1b) An adverse pressure gradient

$$\Rightarrow \frac{dp}{dx} > 0$$

$$\Rightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} > 0 \quad \text{ie;} \quad \left.\frac{\partial \tau}{\partial y}\right|_{y=0} > 0$$



This is a sick profile and the maximum shear stress is not at the wall.

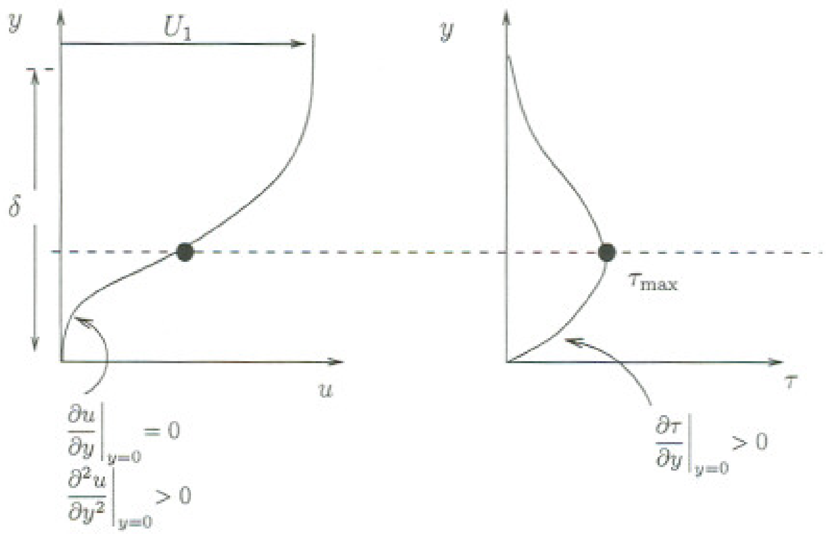
Case (2)

$\tau_0 = 0$

$$\text{ie.} \quad \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$$

this only occurs in the case of an adverse pressure gradient

$$\Rightarrow \frac{dp}{dx} > 0, \text{ie.} \quad \left.\frac{\partial \tau}{\partial y}\right|_{y=0} > 0\,.$$



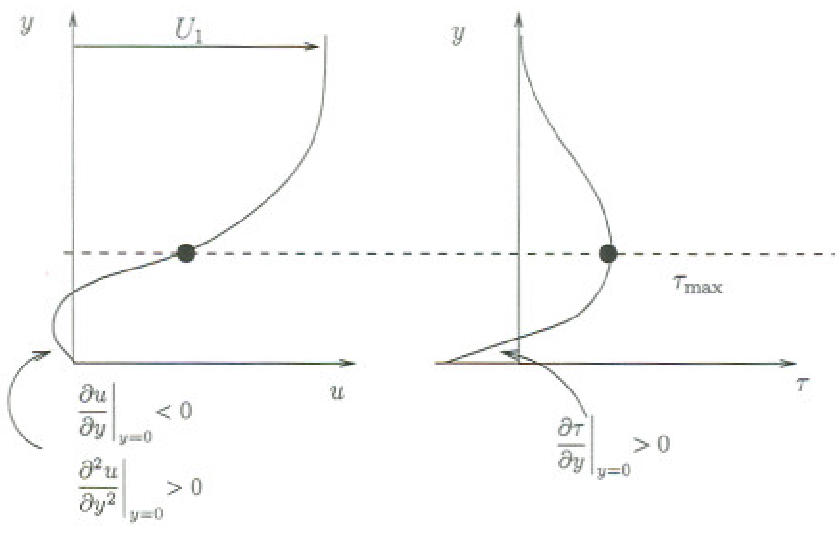
Separation profile

Case (3) $\tau_0 < 0$

$$\text{ie.} \quad \left(\frac{\partial u}{\partial y}\right)_{y=0} < 0$$

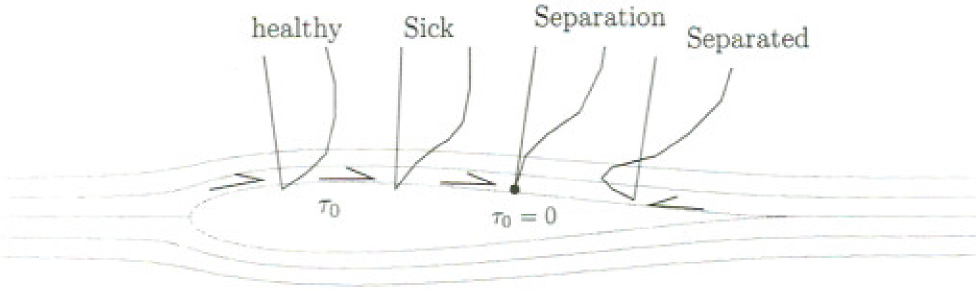
again this is for an adverse pressure gradient

$$\Rightarrow \frac{dp}{dx} > 0, \quad \text{ie.} \quad \left.\frac{\partial \tau}{\partial y}\right|_{y=0} > 0 \, .$$

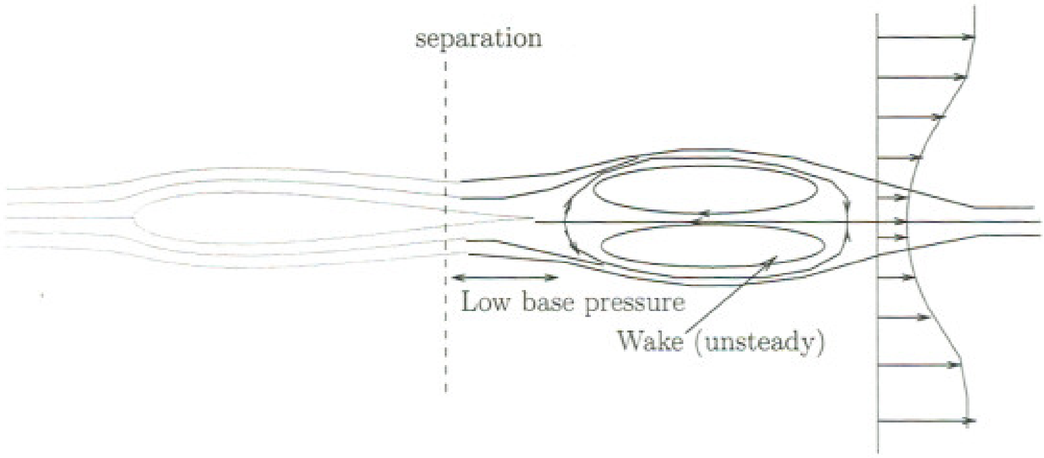


This is a separated profile, note a region of flow reversal near the wall.

Therefore overall picture is something like this



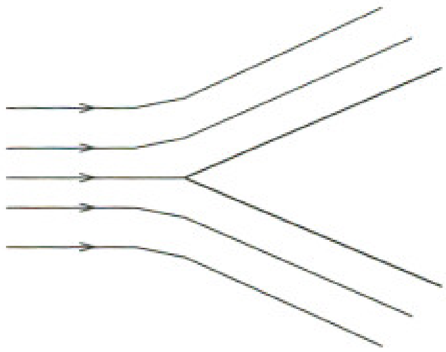
On leading edge of body pressure gradient is favourable and on trailing edge pressure gradient is adverse. The resulting streamline pattern is more like



Because the base pressure is low, drag is large. To reduce drag we want separation points to move to the rear of the body → gives a higher pressure recovery and higher base pressure. This can be achieved by a long tail piece (ie. streamlining) which reduces the strength of the adverse pressure gradient thus delaying separation.

8 Falkner and Skan similarity solutions

Finding solutions for the case with pressure gradients (ie $U_1 = U_1(x)$) is a more difficult problem. A class of similarity solutions has been found for flow past a wedge.



this is consistent with a freestream velocity distribution given by

$$U_1 = ax^m$$

Governing equations;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_1\frac{dU_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \leftarrow \quad \text{momentum}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \leftarrow \quad \text{continuity}$$

Note

$$-\frac{1}{\rho}\frac{dp}{dx} = U_1\frac{dU_1}{dx} \quad \text{where} \quad U_1 = U_1(x)$$

Try to find a similarity solution ie. put

$$\frac{u}{U_1} = F(\eta) \quad \text{where} \quad \eta = \frac{y}{\Delta}$$

also note that

$$\frac{\partial \eta}{\partial x} = -\frac{yd\Delta}{\Delta^2 dx} = -\frac{\eta d\Delta}{\Delta dx} \qquad \text{and}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\Delta}$$

and Δ is a length scale yet to be defined.

Find expressions for each term in the momentum equation;

$$u = U_1 F(\eta) \quad \leftarrow \quad \text{we are assuming a similarity solution}$$

$$\frac{\partial u}{\partial x} = \frac{dU_1}{dx}F - \underbrace{\eta F' \frac{U_1 d\Delta}{\Delta dx}}$$

Note the extra term because we now must include derivatives of U_1 .

Substitute above into continuity,

$$\Rightarrow v = -\int_0^\eta F d\eta \cdot \Delta \frac{dU_1}{dx} + U_1 \frac{dU_1}{dx} \int_0^\eta \eta F' d\eta$$

To simplify notation put $\int_0^\eta F d\eta = f$ or $F = f'$ hence $u = U_1 f'$

$$\frac{\partial u}{\partial x} = \frac{dU_1}{dx} f' - \eta f'' \frac{U_1 d\Delta}{\Delta dx}$$

$$v = -\Delta \frac{dU_1}{dx} f + U_1 \frac{d\Delta}{dx} \underbrace{(\eta f' - f)}_{\text{after integrating by parts}}$$

$$\frac{\partial u}{\partial y} = \frac{U_1 f''}{\Delta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_1 f'''}{\Delta^2}$$

Now put it all into the momentum equation,

$$U_1 f' \left(\frac{dU_1}{dx} f' - \eta f'' \frac{U_1 d\Delta}{\Delta dx} \right) + \left(-\Delta \frac{dU_1}{dx} f + U_1 \frac{d\Delta}{dx} [\eta f' - f] \right) U_1 \frac{f''}{\Delta} = U_1 \frac{dU_1}{dx} + \frac{\nu U_1}{\Delta^2} f'''$$

$$\frac{\Delta^2}{\nu} \frac{dU_1}{dx} (f')^2 - \frac{\Delta^2}{\nu} \frac{dU_1}{dx} f f'' - \frac{\Delta U_1}{\nu} \frac{d\Delta}{dx} f f'' = \frac{\Delta^2}{\nu} \frac{dU_1}{dx} + f'''$$

Universal functions of η alone

$\Rightarrow$ coefficients must all be constants for a solution

Put $\frac{\Delta^2}{\nu} \frac{dU_1}{dx} = \kappa_1$ where κ_1 ia a universal constant. But what about $\frac{\Delta U_1}{\nu} \frac{d\Delta}{dx}$? For a similarity solution to exist (work) this must also be a constant. Question; is it possible for $\frac{\Delta U_1}{\nu} \frac{d\Delta}{dx}$ to be a constant when $\frac{\Delta^2}{\nu} \frac{dU_1}{dx} = \kappa_1 = \text{const}$?

Answer;

$$\begin{aligned}
\text{Have} \quad U_1 &= ax^m \\
\therefore \frac{dU_1}{dx} &= amx^{m-1} \\
\therefore \frac{\Delta^2}{\nu} &= \frac{\kappa_1}{amx^{m-1}} \\
\therefore \Delta^2 &= \frac{\kappa_1 \nu}{amx^{m-1}} \\
\therefore \frac{d\Delta}{dx} &= \sqrt{\frac{\kappa_1 \nu}{am}} \frac{d\left(x^{\frac{1-m}{2}}\right)}{dx} \\
\therefore \frac{d\Delta}{dx} &= \sqrt{\frac{\kappa_1 \nu}{am}} \frac{1-m}{2} x^{\left(\frac{1-m}{2}-1\right)} \\
\therefore \frac{\Delta U_1}{\nu}\frac{d\Delta}{dx} &= \sqrt{\frac{\kappa_1 \nu}{am}} x^{\frac{1-m}{2}} \frac{ax^m}{\nu} \sqrt{\frac{\kappa_1 \nu}{am}} \frac{1-m}{2} x^{\left(\frac{1-m}{2}-1\right)} \\
\therefore \frac{\Delta U_1}{\nu}\frac{d\Delta}{dx} &= \frac{\kappa_1 \nu}{am} \frac{ax^{\left(\frac{1}{2}-\frac{m}{2}+\frac{1}{2}-\frac{m}{2}-1+m\right)}}{\nu} \left(\frac{1-m}{2}\right) \\
\therefore \boxed{\frac{\Delta U_1}{\nu}\frac{d\Delta}{dx} = \kappa_1 \left(\frac{1-m}{2m}\right)}
\end{aligned}$$

So the answer is yes $\frac{\Delta U_1}{\nu}\frac{d\Delta}{dx}$ is a constant provided the velocity distribution is $U_1 = ax^m$. In other words a similarity solution exists (or is permitted) for this velocity distribution. So the equation to solve can be written as,

$$\begin{aligned}
\kappa_1 (f')^2 - \kappa_1 f f'' - \kappa_1 \left(\frac{1-m}{2m}\right) f f'' &= \kappa_1 + f''' \\
\therefore \kappa_1 (f')^2 - \kappa_1 \left(\frac{2m+1-m}{2m}\right) f f'' &= \kappa_1 + f''' \\
\therefore \kappa_1 (f')^2 - \kappa_1 \left(\frac{m+1}{2m}\right) f f'' &= \kappa_1 + f''' \\
f''' + \kappa_1 \left(\frac{m+1}{2m}\right) f f'' + \kappa_1 (1 - (f')^2) &= 0
\end{aligned}$$

Let $\kappa_1 \left(\frac{m+1}{2m}\right) = 1$ for historical reasons. Note κ_1 is an arbitrary universal constant and depends on how we define η (ie. how we define Δ). To be consistent with text books we put $\kappa_1 = \beta$.

$$\boxed{f''' + \beta \left(\frac{m+1}{2m}\right) f f'' + \beta (1 - (f')^2) = 0} \qquad \text{Non-linear 3rd order ODE} \qquad (1)$$

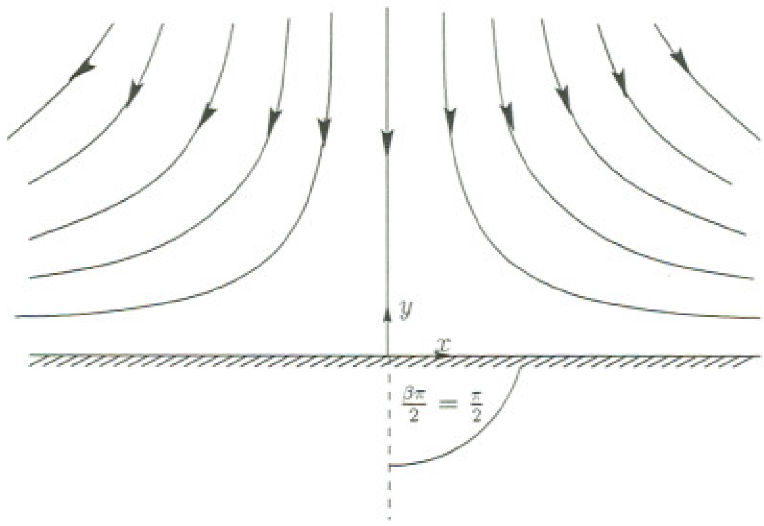
with boundary conditions

$$\eta = 0 \rightarrow f = f' = 0$$
$$\eta = \infty \rightarrow f' = 1$$

Here we have converted a non-linear partial differential equation set to an ordinary differential equation by assuming a similarity solution, as we did for the flat plate Blasius solution. The ODE can be solved numerically or by a series solution.

It can be shown that $U_1 = ax^m$ corresponds to flow over a wedge of semi angle= $\frac{\pi m}{m+1} = \frac{\pi\beta}{2}$. Consider the following cases;

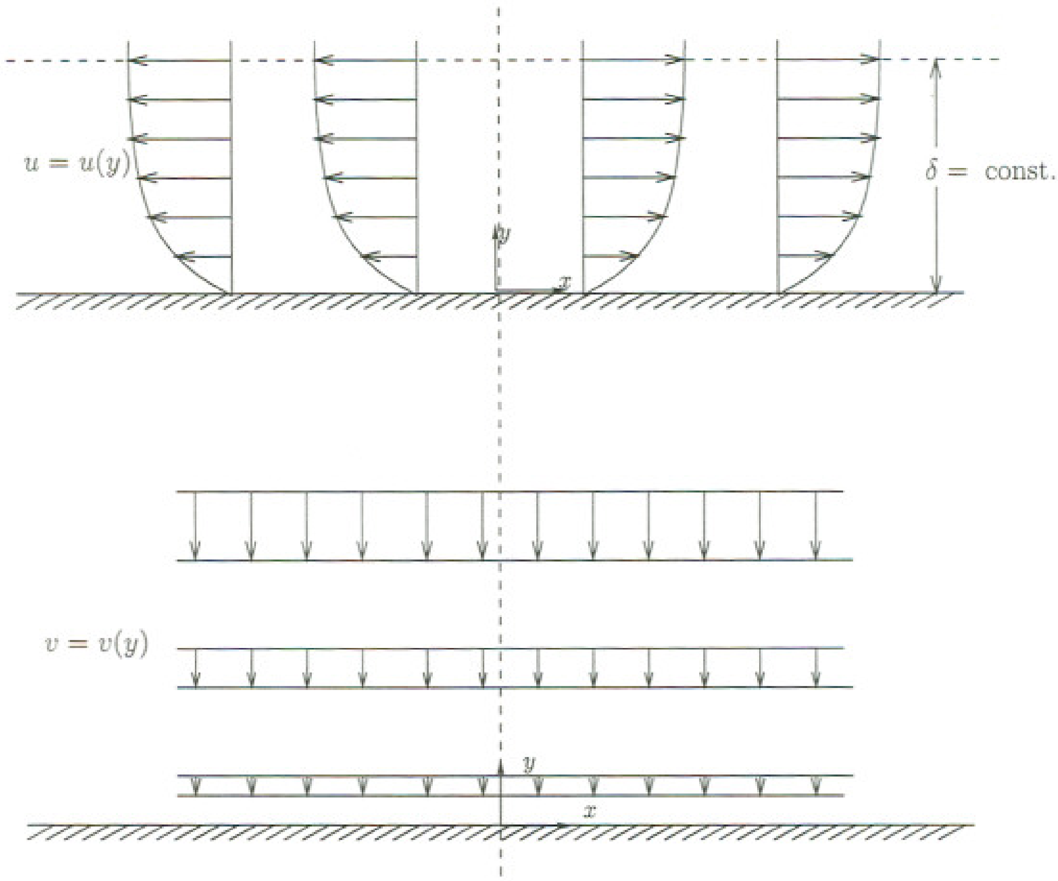
- For a flat plate $m = \beta = 0$ and we get the Blasius equation as obtained in previous lectures.
- If we put $m = 1$ (ie $\beta = \pi/2$) we get plane (ie. [2]) stagnation point flow and this happens to be one of the few exact solutions to the N-S equation.



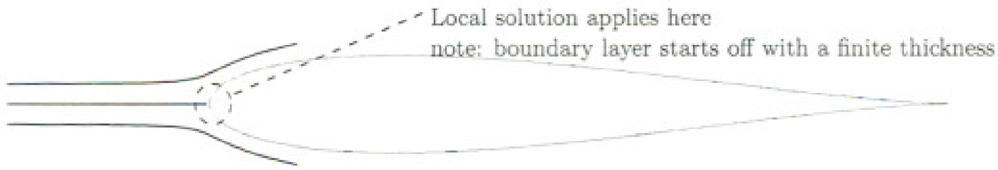
Also with $m = 1$,

$$\frac{d\Delta}{dx} = 0$$
$$\therefore \Delta = \text{ const } \quad \text{hence} \quad \delta = \text{ const}$$

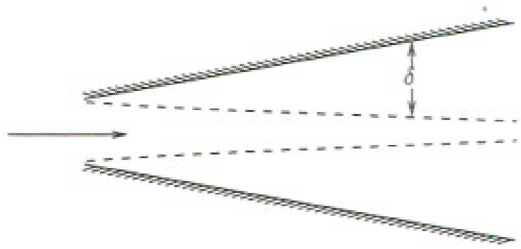
The velocity profiles within the boundary layer would look like this



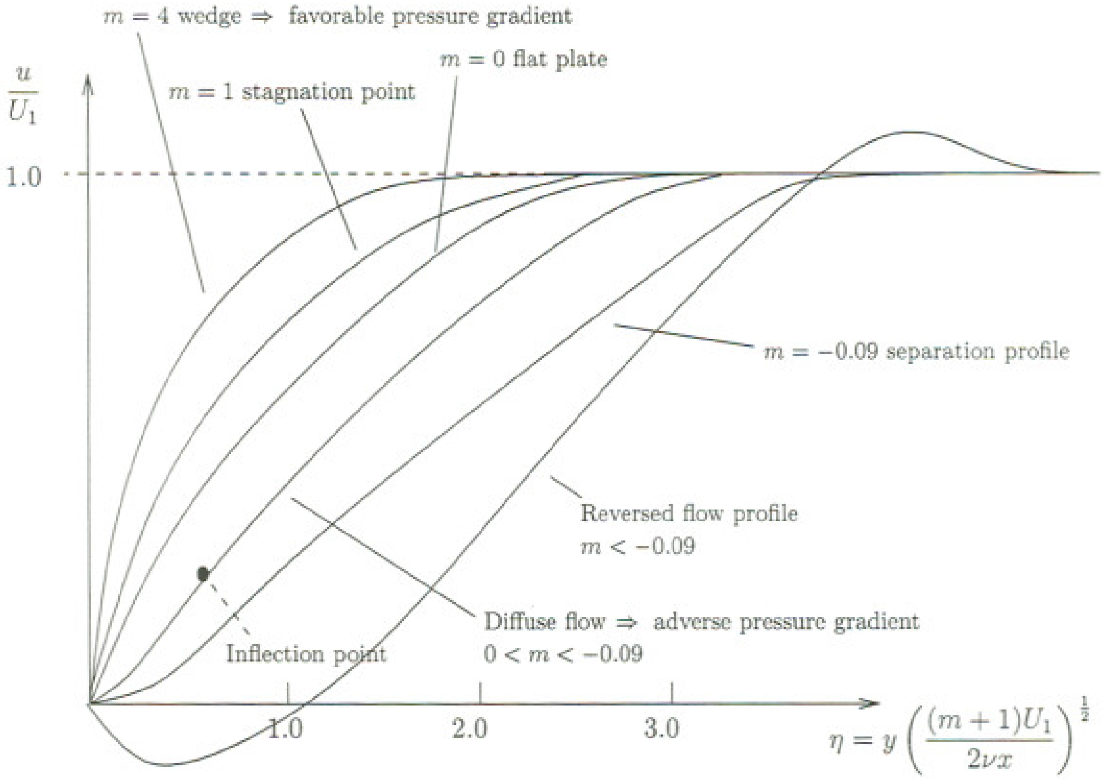
Such a solution would apply at the stagnation point on an aerofoil eg.



- If we put m negative we have diffuse flow, eg.

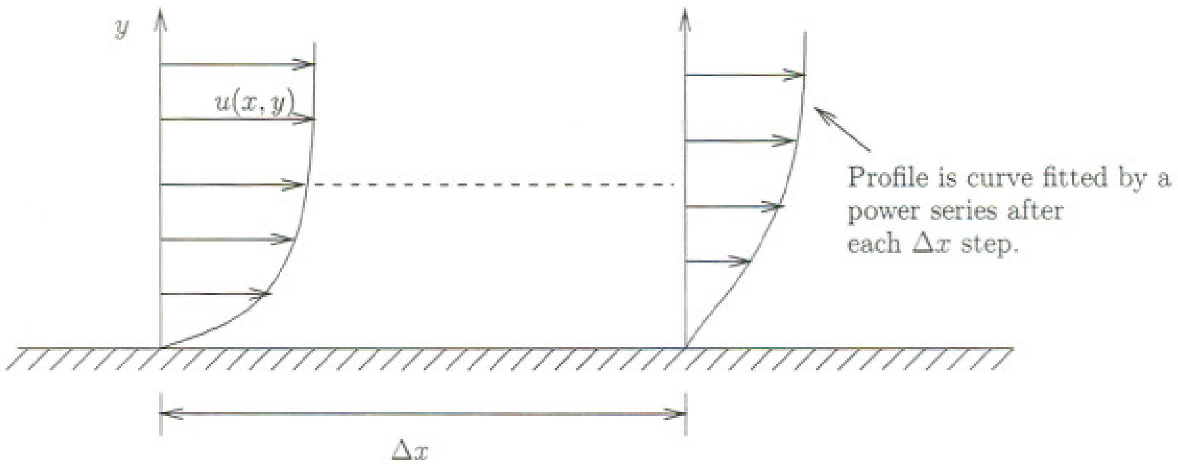


Family of Falkner and Skan solutions



9 Viscid-Inviscid interactions

So far we have considered analytical solutions to the boundary layer equations, the alternative is to solve numerically.



This is called the step by step method. Given;

- a specified pressure gradient $\rightarrow \frac{1}{\rho}\frac{dp}{dx} = f(x)$
- an initial velocity profile

$$\rightarrow u(x_0, y) = a_1 y + \frac{a_2}{2!}y^2 + \frac{a_3}{3!}y^3 + \cdots$$

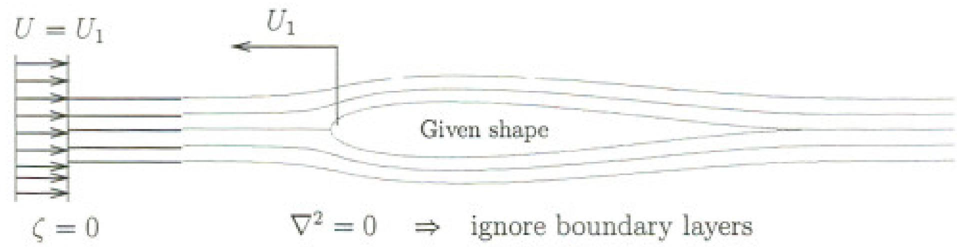
use the momentum and continuity equations to find $\frac{\partial u}{\partial x}$. Hence,

$$u(x_0 + \Delta x, y) = u(x_0, y) + \frac{\partial u}{\partial x}\Delta x$$

and the profile shape evolves with streamwise distance.

However the presence of a boundary layer acts to modify the pressure gradient, hence we must iterate until we get convergence. Procedure for a given body shape:

Step 1

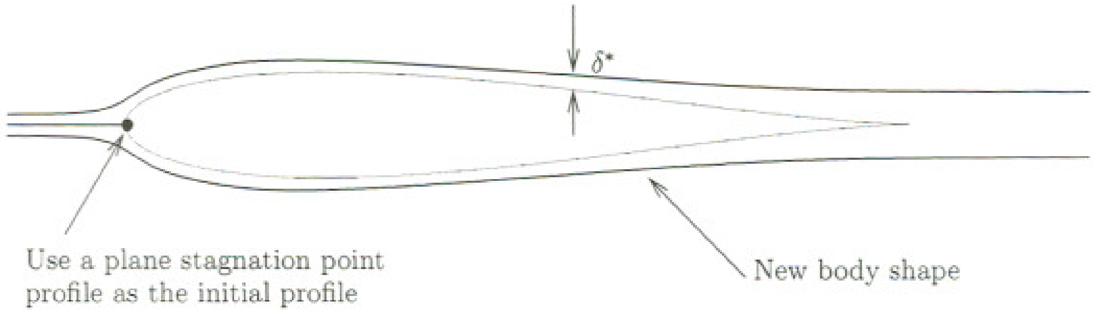


Step 2

Use above to work out surface velocity (ie. inviscid, flow has slip at the wall). This gives $U_1 = U_1(x)$, use this freestream velocity in the boundary layer calculation.

Step 3

Perform B/L calculation using the numerical method



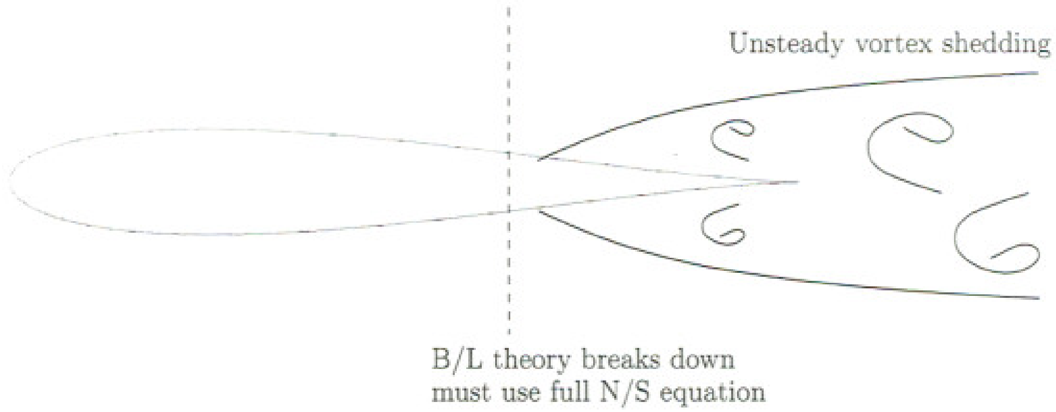
The outer potential flow 'feels' that with the presence of the B/L the body shape is different. This difference is the displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy$$

Go back to step 1 and repeat until we get convergence

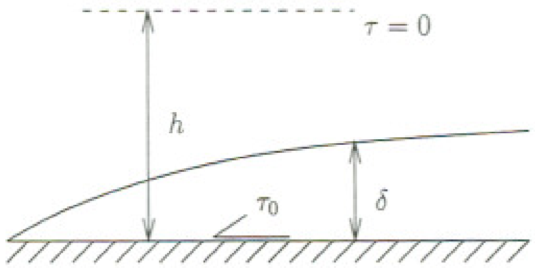
$$\Rightarrow \boxed{\text{Viscous} \rightarrow \text{inviscous interaction}}.$$

This works fine provided B/L are attached and thin.



10 Momentum integral equation

An alternative approach to solving the boundary layer problem is to consider the integral form of the momentum equation (Theodore von Karman).



Have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$
$$= U_1\frac{dU_1}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U_1\frac{dU_1}{dx}\right) dy = \frac{-\tau_0}{\rho}$$

$$\text{using continuity} \quad \rightarrow \quad v = -\int_0^y \frac{\partial u}{\partial x} dy$$

$$\Rightarrow \int_0^h \left(u\frac{\partial u}{\partial x} - \underbrace{\frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}dy}_{\text{integrate by parts}} - U_1\frac{dU_1}{dx}\right) dy = \frac{-\tau_0}{\rho}$$

$$\int_0^h \left(2u\frac{\partial u}{\partial x} - U_1\frac{\partial u}{\partial x} - U_1\frac{dU_1}{dx}\right) dy = \frac{-\tau_0}{\rho}$$

$$\text{or} \qquad \int_0^h \frac{\partial}{\partial x}\left[u\left(U_1 - u\right)\right] dy + \frac{dU_1}{dx}\int_0^h \left(U_1 - u\right) dy = \frac{\tau_0}{\rho}$$

Put $h = \infty$, have;

$$\text{Displacement thickness} \quad \rightarrow \delta^* = \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy$$
$$\therefore \delta^* U_1 = \int_0^\infty (U_1 - u) dy$$

$$\text{Momentum thickness} \quad \rightarrow \theta = \int_0^\infty \frac{u}{U_1}\left(1 - \frac{u}{U_1}\right) dy$$
$$\therefore \theta U_1^2 = \int_0^\infty u(U_1 - u) dy$$

Hence,

$$\frac{\tau_0}{\rho} = \frac{d}{dx}(U_1^2\theta) + \delta^* U_1\frac{dU_1}{dx}$$

divide by U_1^2 and manipulate

$$\rightarrow \boxed{\frac{d\theta}{dx} + \frac{(H+2)\theta}{U_1}\frac{dU_1}{dx} = \frac{C_f'}{2}} \quad \text{von Karman integral momentum equation}$$

where

$$H = \frac{\delta^*}{\theta}$$

$$C_f' = \frac{\tau_0}{\frac{1}{2}\rho U_1^2}$$

$$\frac{d\theta}{dx} \rightarrow \quad \text{Inertia term}$$

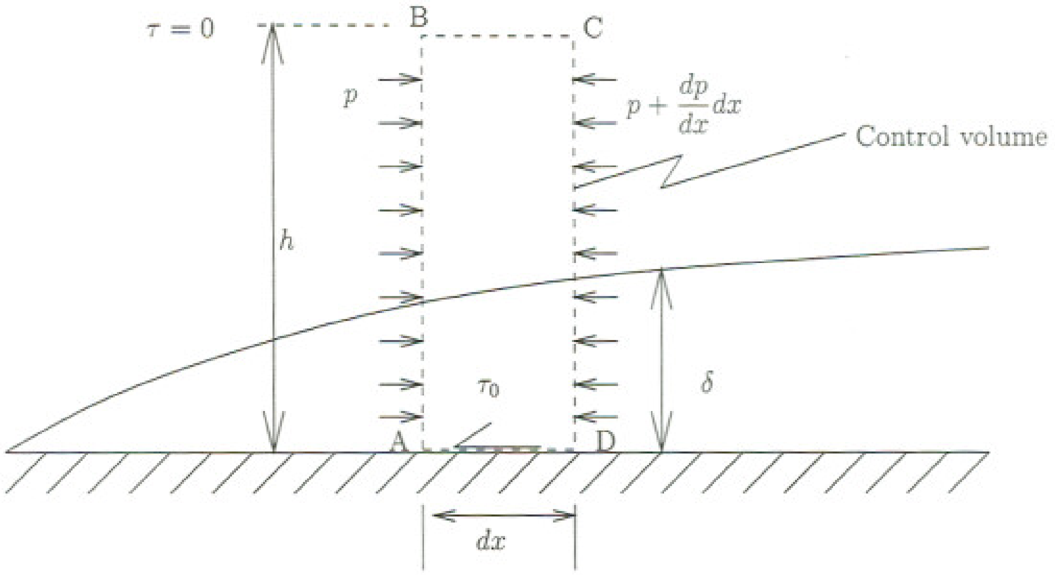
$$\frac{(H+2)\theta}{U_1}\frac{dU_1}{dx} \rightarrow \quad \text{pressure gradient term}$$

$$\frac{C_f'}{2} \rightarrow \quad \text{wall skin friction term}$$

Hence for the zero pressure gradient layer

$$\boxed{\frac{d\theta}{dx} = \frac{C_f'}{2}}$$

The above result can also be obtained using control volume analysis.



Balance of mass

$$\begin{aligned}
\text{Mass in through } AB &= \int_0^h \rho u dy \\
\text{Mass out through } CD &= \int_0^h \rho u dy + \frac{\partial}{\partial x}\int_0^h \rho u dy dx \\
\Rightarrow \text{Mass out through } BC &= \int_0^h \rho u dy - \left(\int_0^h \rho u dy + \frac{\partial}{\partial x}\int_0^h \rho u dy dx\right)
\end{aligned}$$

Momentum balance (x–dir.)

$$\begin{aligned}
\text{Momentum in through } AB &= \int_0^h \rho u^2 dy \\
\text{Momentum out through } CD &= \int_0^h \rho u^2 dy + \frac{\partial}{\partial x}\int_0^h \rho u^2 dy dx \\
\text{Momentum out through } BC &= -U_1 \frac{\partial}{\partial x}\int_0^h \rho u dy dx
\end{aligned}$$

Net efflux of x momentum

$$\begin{aligned}
&= - U_1 \frac{\partial}{\partial x}\int_0^h \rho u dy dx + \int_0^h \rho u^2 dy + \frac{\partial}{\partial x}\int_0^h \rho u^2 dy dx - \int_0^h \rho u^2 dy \\
&= - U_1 \frac{\partial}{\partial x}\int_0^h \rho u dy dx + \frac{\partial}{\partial x}\int_0^h \rho u^2 dy dx
\end{aligned}$$

$$\text{Resultant external } x \text{ component of force} = -h\frac{dp}{dx}dx - \tau_0 dx$$

Equating above expressions gives

$$h\rho U_1 \frac{dU_1}{dx} - \tau_0 = -U_1 \frac{\partial}{\partial x}\int_0^h \rho u dy + \frac{\partial}{\partial x}\int_0^h \rho u^2 dy$$

Introduce

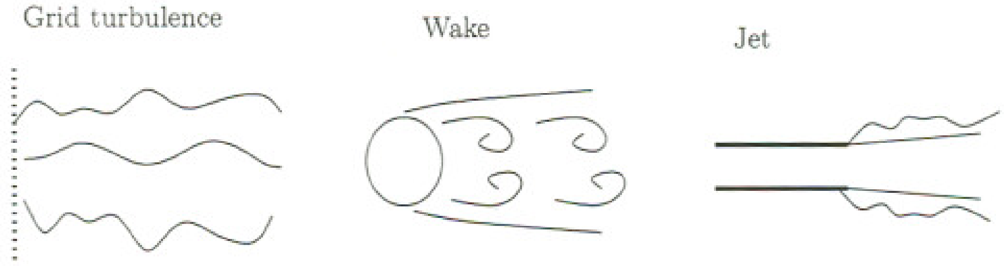
$$\begin{aligned}
\delta^* &= \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy \\
\theta &= \int_0^\infty \frac{u}{U_1}\left(1 - \frac{u}{U_1}\right) dy
\end{aligned}$$

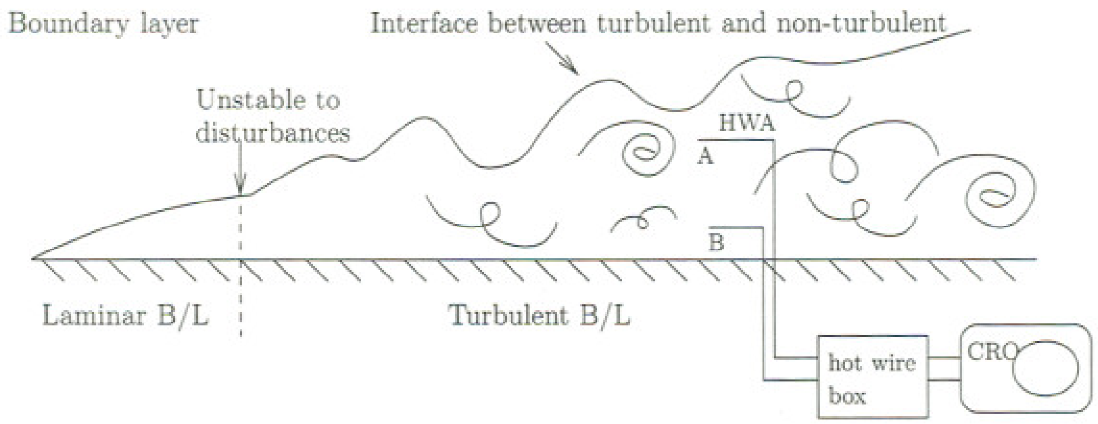
Hence show

$$\Rightarrow \boxed{\frac{d\theta}{dx} + \frac{(H+2)\theta}{U_1}\frac{dU_1}{dx} = \frac{C_f'}{2}}$$

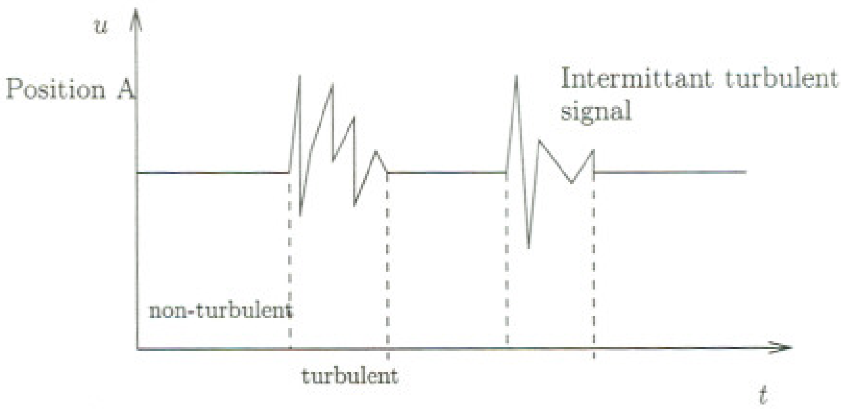
11 Turbulence

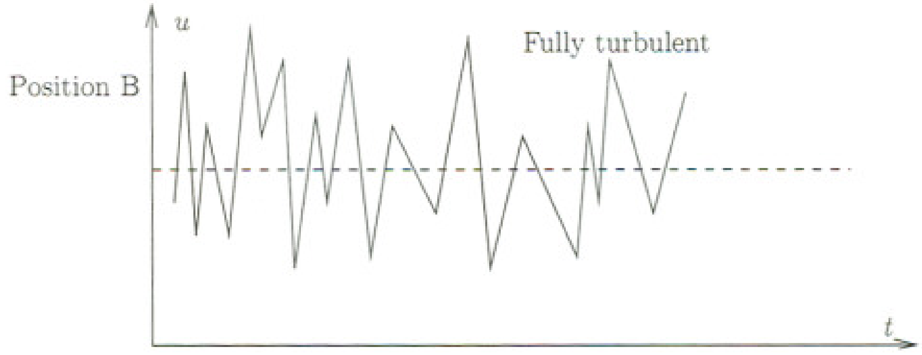
So far we have considered the laminar boundary layer. However in many practical engineering examples the boundary layer is turbulent. Wall turbulence is one example of turbulent flow, other examples are grid turbulence, wakes and jets.





The boundary layer starts off laminar but at some critical Reynolds number it becomes unstable to disturbances (eg. noise, vibration, freestream turbulence) and it becomes turbulent. The interface between the turbulent fluid and non-turbulent fluid is non-stationary.





11.1 Boundary layer equations and Reynolds averaging

Here we use N/S equations

$$\left.\begin{aligned} \frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \\ \frac{Dv}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \\ \frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \end{aligned}\right\} \tag{1}$$

and continuity

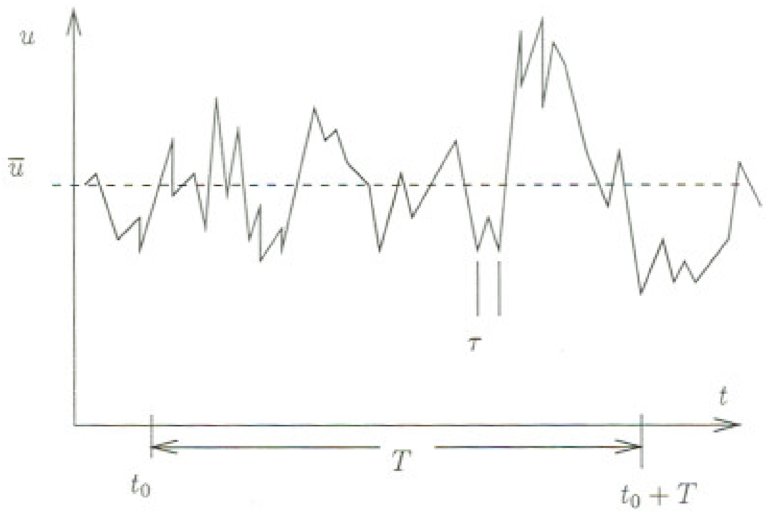
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad \text{(total derivative)}$$
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Decompose the variables into a mean and fluctuating part;

$$\left.\begin{aligned} u &= \overline{u} + u' \\ v &= \overline{v} + v' \\ w &= \overline{w} + w' \\ \rho &= \overline{\rho} + \rho' \\ p &= \overline{p} + p' \end{aligned}\right\} \tag{3}$$



$$\overline{u} = \frac{1}{T}\int_{t_0}^{t_0+T} u dt$$

Provided the sampling period T is sufficiently large, the result will converge ie. $T \to \infty$. Actually only require $T \gg \tau$ where τ is a characteristic time scale of the turbulence. (see Hinze 'Turbulence' for details). Note we are considering steady flow in the mean sense.

At moderately low Mach numbers $M < 0.2$

$$\frac{\rho'}{\rho} \approx M^2 \approx 0.04 \quad \text{for} \quad M = 0.2$$

hence for these conditions ignore density fluctuations.

Substitute (3) into (1) and (2) and take time average of the equations, ie. consider turbulence as a statistical mechanics problem. When taking averages note the following

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \overline{u}}{\partial x}$$

$$\overline{\int u ds} = \int \overline{u} ds$$

$$\overline{u + v} = \overline{u} + \overline{v}$$

$$\overline{u} = \overline{\overline{u} + u'} = \overline{\overline{u}} + \overline{u'} = \overline{u} + \overline{u'}$$

$$\Rightarrow \overline{u'} = 0 \text{ ie. } \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} u' dt = 0$$

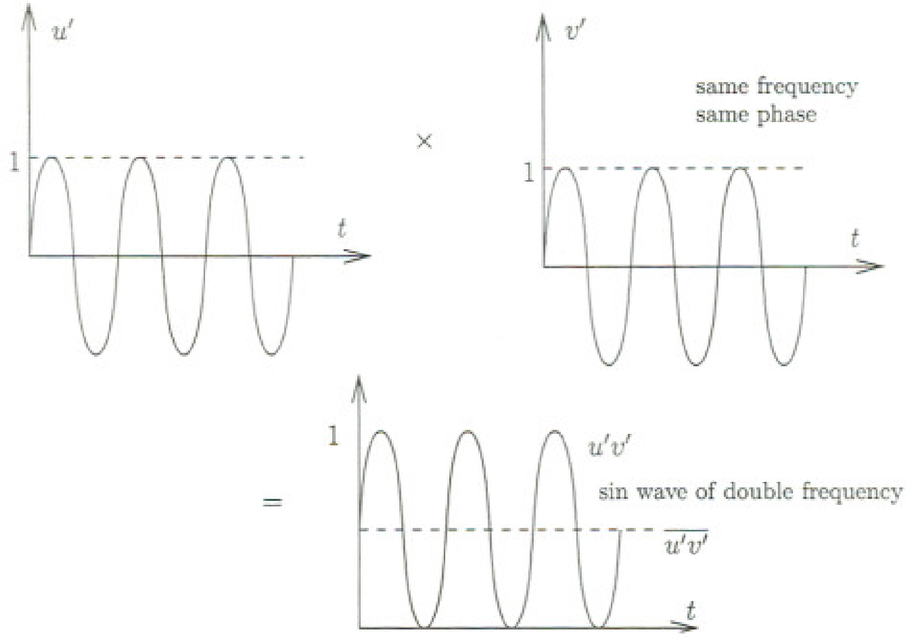
Hence area of fluctuating quantities must be zero as implied by the decomposition given in (3). Also,

$$\overline{\overline{u}\overline{v}} = \overline{\overline{u}} \cdot \overline{\overline{v}} = \overline{u}\overline{v}$$

$$\overline{\overline{u}u'} = \overline{\overline{u}} \cdot \overline{u'} = \overline{u}\overline{u'} = 0 \quad (\text{since } \overline{u'} = 0)$$

$$\overline{uv} = \overline{(\overline{u} + u')(\overline{v} + v')} = \overline{\overline{u}\overline{v} + \overline{u}v' + \overline{v}u' + \overline{v'}v'} = \overline{u}\overline{v} + \overline{u'v'}$$

Although $\overline{u'} = 0$, $\overline{v'} = 0$ in general $\overline{u'v'} \neq 0$



Special case: phase shift is $\pi/2$ eg. $u' = \sin(\omega t)$, $v' = \cos(\omega t)$ then $\overline{u'v'} = 0$.

Writing the result in tensor form;

$$\left.\begin{aligned} \rho\left\{\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}\right\} = -\frac{\partial \overline{p}}{\partial x_i} + \mu\left\{\frac{\partial^2 \overline{u_i}}{\partial x_j x_j}\right\} - \rho\left\{\frac{\partial \overline{u_i' u_j'}}{\partial x_j}\right\} \\ \frac{\partial \overline{u_i}}{\partial x_i} = 0 \end{aligned}\right\} \text{Osbourne Reynolds equations} \tag{4}$$

The corresponding laminar flow equations are

$$\left.\begin{aligned} \rho\left\{u_j\frac{\partial u_i}{\partial x_j}\right\} = -\frac{\partial p}{\partial x_i} + \mu\left\{\frac{\partial^2 u_i}{\partial x_j x_j}\right\} \\ \frac{\partial u_i}{\partial x_i} = 0 \end{aligned}\right\} \tag{5}$$

Laminar equations and turbulent equations are identical if we put $u_i = \overline{u_i}$ except for the extra term

$$-\rho\left\{\frac{\partial \overline{u_i' u_j'}}{\partial x_j}\right\}$$

The terms

$$\overline{u_1'^2},\ \overline{u_2'^2},\ \overline{u_3'^2}$$

are called Reynolds normal stresses. While the terms

$$\overline{u_1'u_2'},\ \overline{u_1'u_3'},\ \overline{u_2'u_3'}$$

are called the Reynolds shear stresses. We can construct a symmetric stress tensor

$$\frac{\tau_{ij}}{\rho}$$

Reynolds stresses for most of the flow dominate over viscous stress terms.

Let us consider a [2] (in the mean sense) turbulent boundary layer. Applying an order of magnitude argument (details given in Hinze, Schlichting) the boundary layer equations are found to be;

$$\left.\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \end{aligned}\right\} \text{Prandtl's B/L eq}^{n\cdot} \text{ for turbulent}$$

Compare this to the laminar case, recall

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{aligned}\right\} \text{Prandtl's B/L eq}^{n\cdot} \text{ for laminar.}$$

In the turbulent equation there is an extra term

$$-\frac{\partial \overline{u'v'}}{\partial y}$$

this is a gradient of a Reynolds shear stress and only this component of the Reynolds stresses is appreciable. This was discovered by an order of magnitude argument. Hence we can write the approximate B/L equation as

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial}{\partial y}\left\{\frac{\tau}{\rho}\right\}$$

$$\text{where} \quad \frac{\tau}{\rho} = \nu\frac{\partial \overline{u}}{\partial y} - \overline{u'v'} \quad \text{for turbulent flow}$$

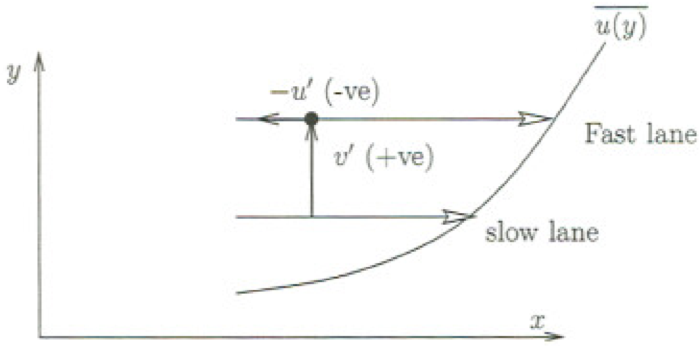
$$\frac{\tau}{\rho} = \nu\frac{\partial \overline{u}}{\partial y} \quad \text{for laminar flow}$$

$\frac{\tau}{\rho}$ → Is the total shear stress

$\nu\frac{\partial \overline{u}}{\partial y}$ → is the viscous shear stress and represents the contribution from molecular transport of momentum

$-\overline{u'v'}$ → is the Reynolds shear stress and represents the contribution from large scale turbulent transport of momentum

It turns out that $-\overline{u'v'}$ is a positive quantity, consider two y levels in the B/L;



A +ve v' causes a lump of fluid to move from the slow lane to the fast lane. This in turn causes a $-u'$ perturbation to the fast lane. Hence more often than not the product $u'v'$ is negative and hence $\overline{u'v'}$ is negative, ie. $-\overline{u'v'}$ is positive.

Summary of equations

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\tau = \nu\frac{\partial \overline{u}}{\partial y} - \overline{u'v'}$$

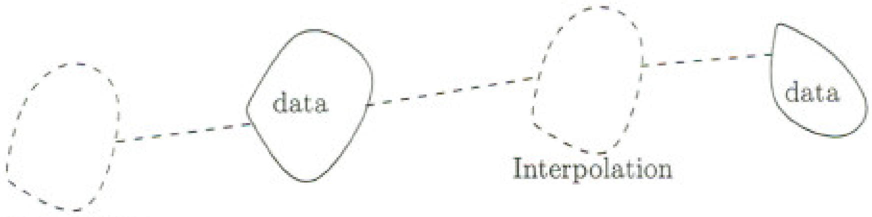
Three equations, four unknowns ($\overline{u}$, $\overline{v}$, τ, $-\overline{u'v'}$), compare to laminar flow where we do not have $\overline{u'v'}$ hence 3 equations 3 unknowns.

It can be shown that for turbulent flow the laminar von Karman momentum integral equation is still valid ie.

$$\frac{d\theta}{dx} + \frac{(H+2)\theta}{U_1}\frac{dU_1}{dx} = \frac{C_f'}{2}\,.$$

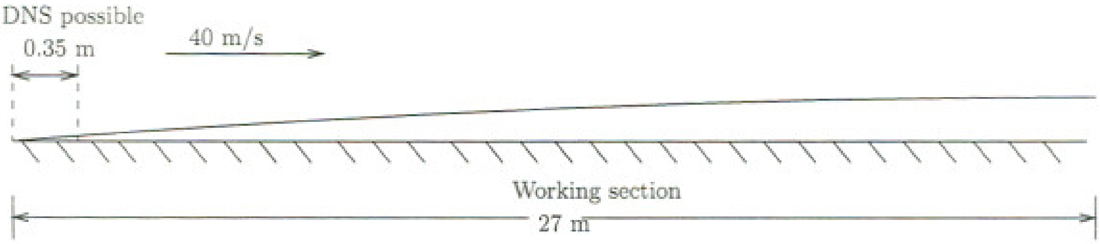
12 Closure problem

When we take an average (ie. when we use a statistical approach) we loose information and always end up with more unknowns than equations. The problem of having more unknowns than equations is called the closure problem. To get the missing equation requires a closure hypothesis. For example use experimental data with interpolation and extrapolation or use some turbulence model (eg. eddy viscosity).



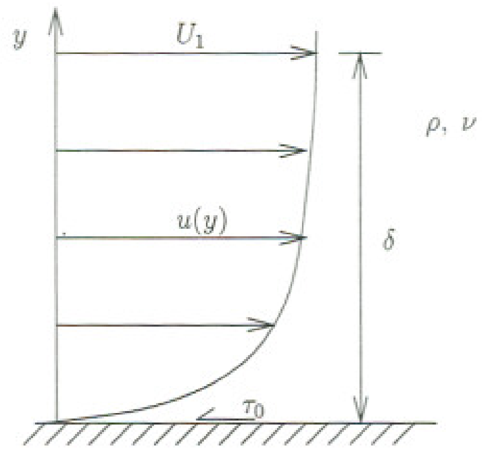
The alternative is to perform a full direct numerical simulation (DNS), (ie. use full Navier Stokes equations no B/L approx. no averaging). This requires a supercomputer and the calculation may take several months. Even with modern computers this approach can only be applied to low Reynolds numbers.

High Reynolds number boundary layer tunnel



The singular unsolved problem in classical physics is turbulence.

13 Prandtl's law of the wall



Change of notation all u, v are averages (eg $\overline{u}$, $\overline{v}$)

Prandtl's law of the wall says $u = f(y, \tau_0, \nu, \rho)$, we have 5 variables and 3 fundamental quantities

$$\Rightarrow 5 - 3 = 2\Pi \text{ products}$$

Introduce a new velocity 'scale'

$$\left[\sqrt{\frac{\tau_0}{\rho}}\right] = LT^{-1}$$

since the above has dimensions consistent with a velocity call it the friction velocity ie.

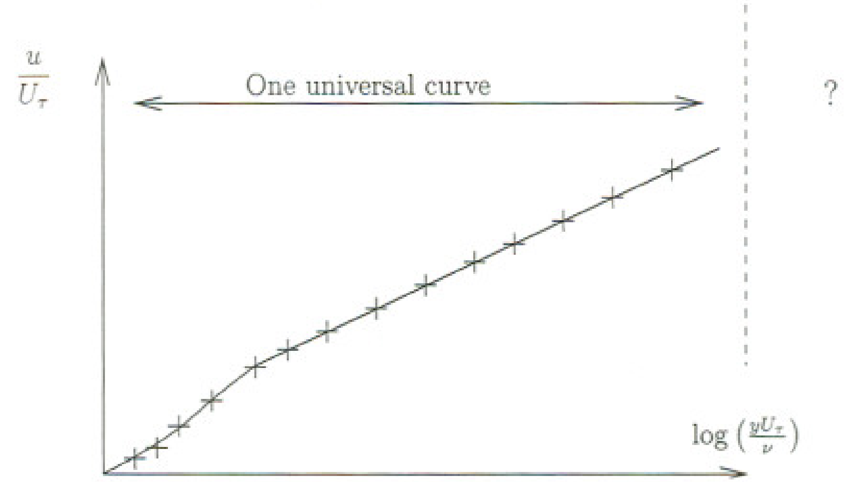
$$\sqrt{\frac{\tau_0}{\rho}} = U_\tau \leftarrow \text{ the friction velocity .}$$

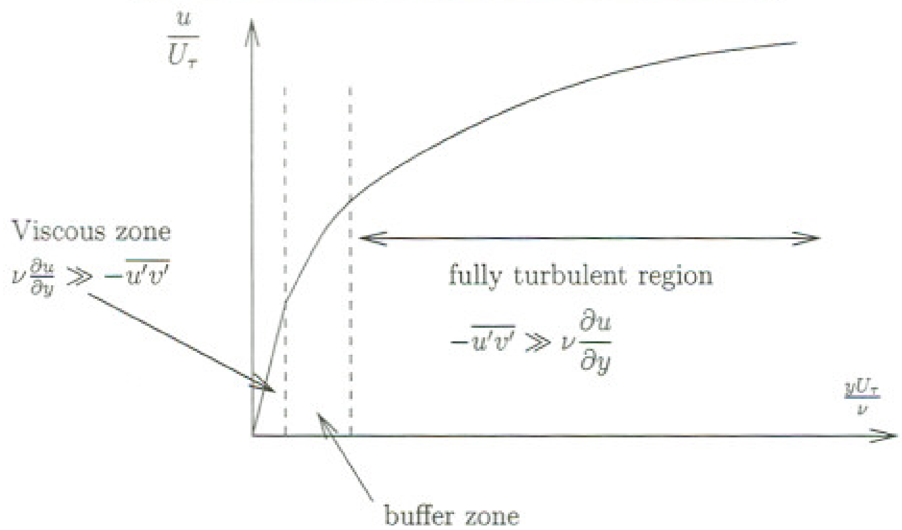
Hence Prandtl's law of the wall can be written as $u = f(y, U_\tau, \nu)$, we have 4 variables and 2 fundamental quantities

$$\Rightarrow 4 - 2 = 2\Pi \text{ products}$$

forming the Π products gives

$$\frac{u}{U_\tau} = f\left(\frac{yU_\tau}{\nu}\right)$$





In analogy with laminar flow

$$\frac{\tau}{\rho} = \epsilon\frac{\partial u}{\partial y} \leftarrow \text{ for fully turbulent region}$$

ϵ is the 'viscosity' due to the large eddies $\Rightarrow$ Eddy viscosity. Unlike ν this is a function of the fluid motion.

Let us assume $\epsilon = f(\tau/\rho, y)$ not a function of ν this assumption is called Reynolds similarity hypothesis ie. turbulent motions are independent of ν.

$$\left.\begin{aligned}\text{Number of variables} &= 3\\ \text{Number of fundemental quantities} &= 2\end{aligned}\right\} 3-2 = 1\Pi$$

Since there is only one Π product this means;

$$f(\Pi_1) = 0$$
$$\Rightarrow f\left(\frac{\epsilon}{y\sqrt{\frac{\tau}{\rho}}}\right) = 0$$
$$\text{ie. } \frac{\epsilon}{y\sqrt{\frac{\tau}{\rho}}} = \kappa \quad \text{where } \kappa \text{ is a universal constant}$$

But $\tau = \tau_0$ at $y = 0$ therefore $\tau \approx \tau_0$ for small y. Hence

$$\left.\begin{aligned}\frac{\epsilon}{y\sqrt{\frac{\tau_0}{\rho}}} &= \kappa\\ \therefore \frac{\epsilon}{yU_\tau} &= \kappa\end{aligned}\right\} \text{for small } y$$

Now the closure equation is

$$\frac{\tau}{\rho} = \epsilon\frac{\partial u}{\partial y}$$

but for small y

$$\begin{aligned}\frac{\tau_0}{\rho} &= \kappa y U_\tau \frac{\partial u}{\partial y}\\ \therefore U_\tau^2 &= \kappa y U_\tau \frac{\partial u}{\partial y}\\ \therefore \frac{\partial u}{\partial y} &= \frac{U_\tau}{\kappa y}\\ \therefore u &= \frac{U_\tau}{\kappa}\ln y + c'\end{aligned}$$

where c' is a constant of integration

$$\frac{u}{U_\tau} = \frac{1}{\kappa} \ln y + c$$

$$\text{but } \frac{u}{U_\tau} = f\left(\frac{yU_\tau}{\nu}\right)$$

$$\text{hence } c = \frac{1}{\kappa} \ln \frac{U_\tau}{\nu} + \underbrace{A}_{\text{universal}}$$

The eddy viscosity closure hypothesis therefore implies the existence of a logarithmic law of the wall

$$\Rightarrow \boxed{\frac{u}{U_\tau} = \frac{1}{\kappa} \ln \frac{yU_\tau}{\nu} + A} \quad \text{Prandtl's logarithmic law of the wall}$$

14 Alternative derivation of the log-law

The logarithmic law of the wall can also be obtained using dimensional reasoning. This is a more rigorous approach since it does not rely on the eddy viscosity assumption. Again assume a law of the wall ;

$$\begin{aligned} u =& f(y, U_\tau, \nu) \\ \frac{u}{U_\tau} =& f\left(\frac{yU_\tau}{\nu}\right) \quad \text{Law of the wall - verified by experiment} \end{aligned} \tag{1}$$

Far from the wall the mean relative motion;

$$\begin{aligned} U_1 - u =& g(y, U_\tau, \delta) \\ \frac{U_1 - u}{U_\tau} =& g\left(\frac{y}{\delta}\right) \quad \text{Velocity defect law - verified by experiment} \end{aligned} \tag{2}$$

If there is a region of overlap then

$$\frac{\partial u}{\partial y}_{\text{from (1)}} = \frac{\partial u}{\partial y}_{\text{from (2)}} .$$

From (1)

$$u = U_\tau f\left(\frac{yU_\tau}{\nu}\right)$$

$$\frac{\partial u}{\partial y} = U_\tau f'\left(\frac{yU_\tau}{\nu}\right)\frac{U_\tau}{\nu} \qquad \text{(where } ' \text{ means the derivative w.r.t the argument)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{U_\tau^2}{\nu} f'\left(\frac{yU_\tau}{\nu}\right) . \tag{3}$$

From (2)

$$\begin{aligned} \frac{u}{U_\tau} &= \frac{U_1}{U_\tau} - g\left(\frac{y}{\delta}\right) \\ u &= U_\tau\left(\frac{U_1}{U_\tau} - g\left(\frac{y}{\delta}\right)\right) \\ \therefore \frac{\partial u}{\partial y} &= \frac{-U_\tau}{\delta} g'\left(\frac{y}{\delta}\right) . \end{aligned} \tag{4}$$

Equating (3) with (4)

$$\frac{U_\tau^2}{\nu} f'\left(\frac{yU_\tau}{\nu}\right) = \frac{-U_\tau}{\delta} g'\left(\frac{y}{\delta}\right)$$

$$\underbrace{\frac{yU_\tau}{\nu} f'\left(\frac{yU_\tau}{\nu}\right)}_{\text{function of } yU_\tau/\nu \text{ alone}} = \underbrace{\frac{-y}{\delta} g'\left(\frac{y}{\delta}\right)}_{\text{function of } y/\delta \text{ alone}} = \frac{1}{\kappa}$$

This is possible only if both sides = a universal constant = $1/\kappa$.

Take the L.H.S;

$$\begin{aligned}
\frac{df\left(\frac{yU_\tau}{\nu}\right)}{d\left(\frac{yU_\tau}{\nu}\right)} &= \frac{1}{\kappa\left(\frac{yU_\tau}{\nu}\right)} \\
f\left(\frac{yU_\tau}{\nu}\right) &= \frac{1}{\kappa}\ln\left(\frac{yU_\tau}{\nu}\right) + A \\
\therefore \frac{u}{U_\tau} &= \frac{1}{\kappa}\ln\left(\frac{yU_\tau}{\nu}\right) + A \qquad \text{logarithmic law of the wall} \qquad (5)
\end{aligned}$$

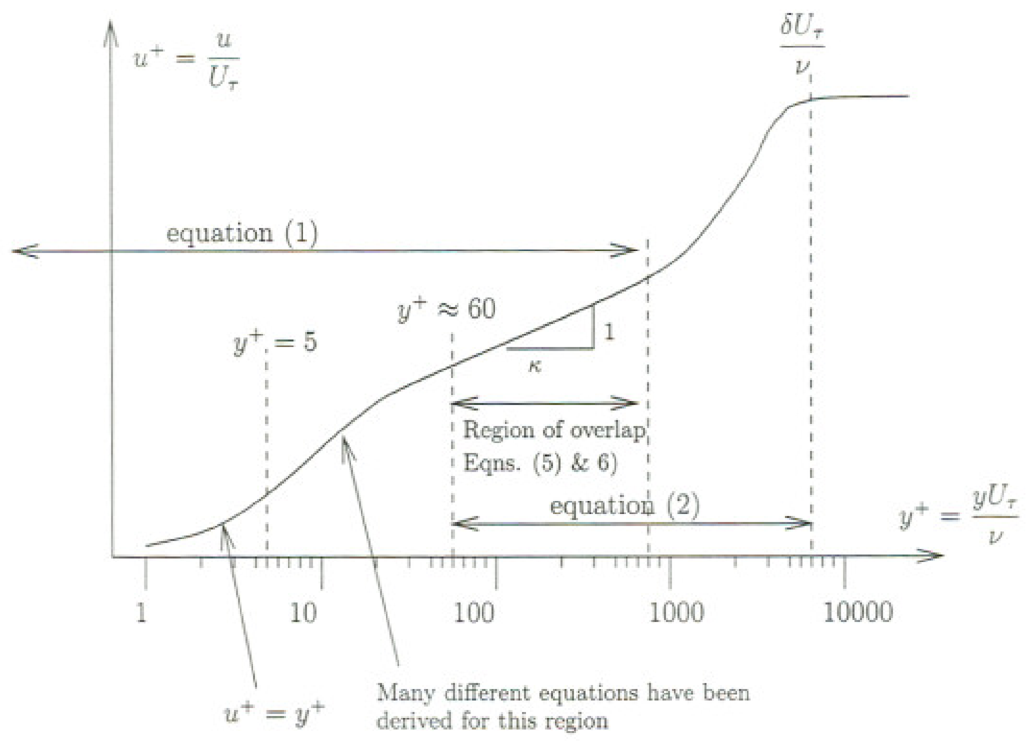
Take the R.H.S;

$$\begin{aligned}
\frac{-dg\left(\frac{y}{\delta}\right)}{d\left(\frac{y}{\delta}\right)} &= \frac{1}{\kappa\left(\frac{y}{\delta}\right)} \\
g\left(\frac{y}{\delta}\right) &= -\frac{1}{\kappa}\ln\frac{y}{\delta} + c \\
\frac{U_1 - u}{U_\tau} &= -\frac{1}{\kappa}\ln\frac{y}{\delta} + c \qquad \text{logarithmic velocity defect law} \qquad (6)
\end{aligned}$$

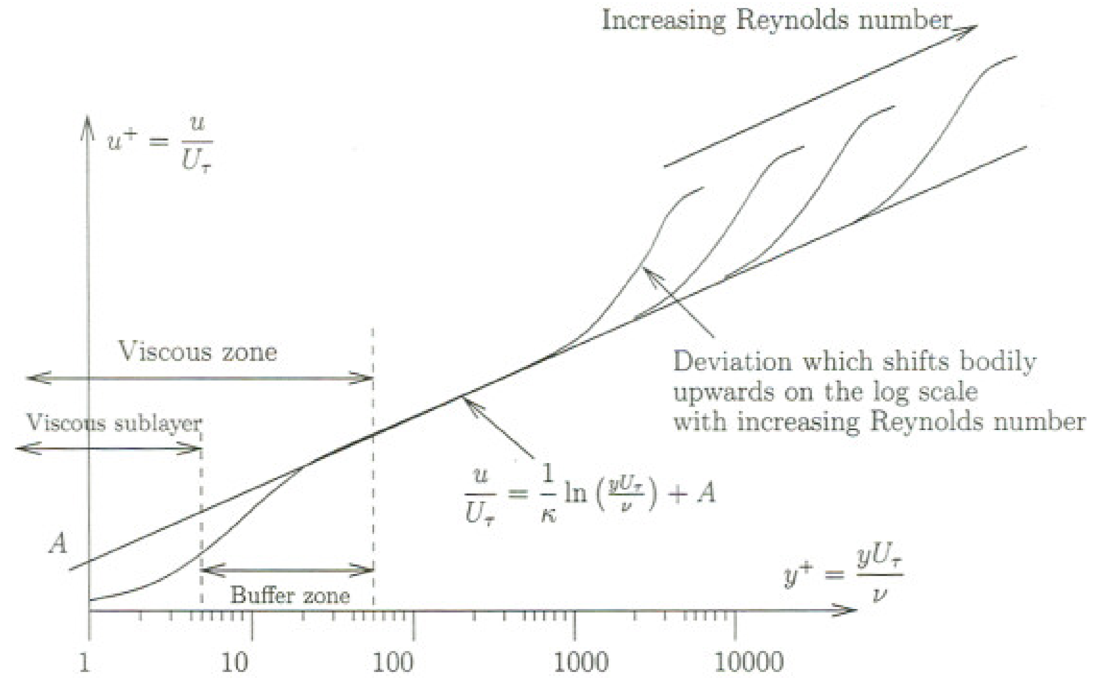
- κ is the Karman constant, it is a universal constant and from experiments $\kappa = 0.41$.
- A is the universal smooth wall constant and experiments have found $A = 5.0$.
- c is a characteristic constant which depends on the flow geometry, ie. for zero pressure gradient $c = 2.3$.

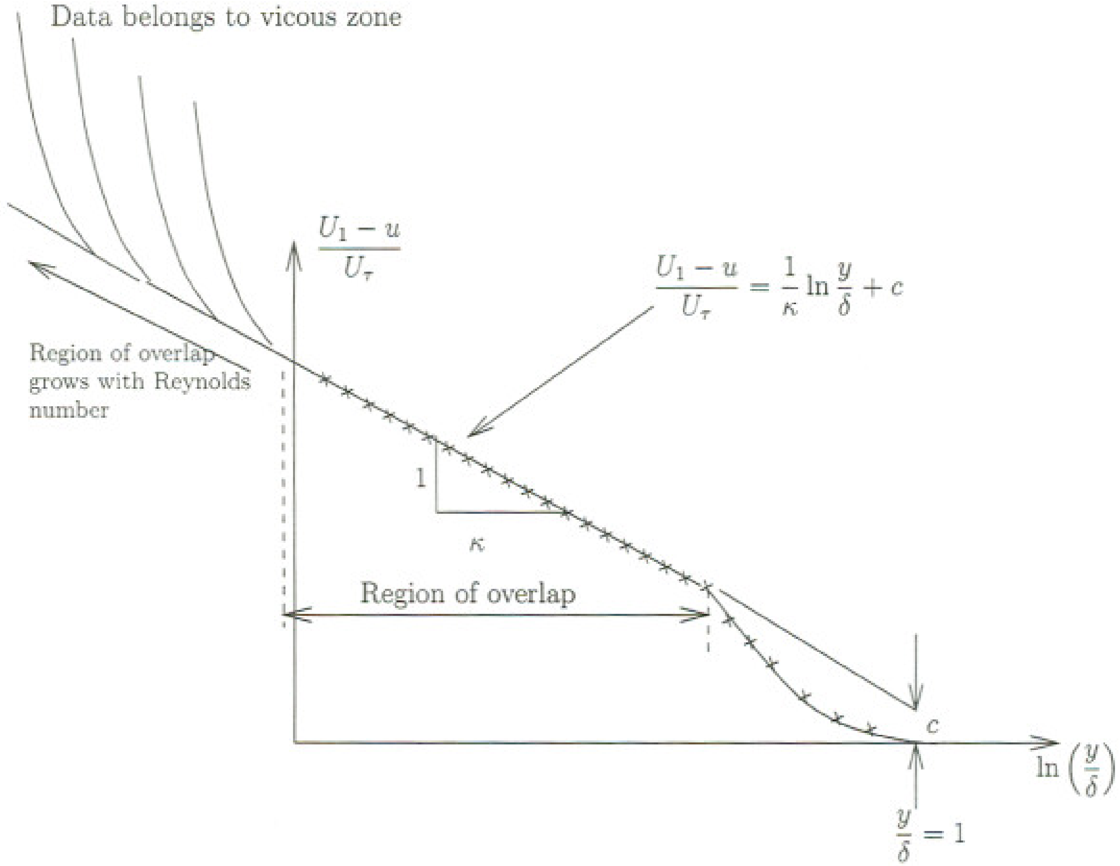
Note there is much debate about the exact values, more recent experiments suggest $\kappa = 0.44$, $A = 6.3$.

$$\left.\begin{aligned}
\frac{yU_\tau}{\nu} &= y^+ \\
\frac{u}{U_\tau} &= u^+ \\
\frac{y}{\delta} &= \eta
\end{aligned}\right\} \qquad \text{Often use the notation}$$



If we plot several profiles with different Reynolds numbers we get a family of profiles that all collapse onto the law of the wall and the region of collapse increases with increasing Reynolds number.



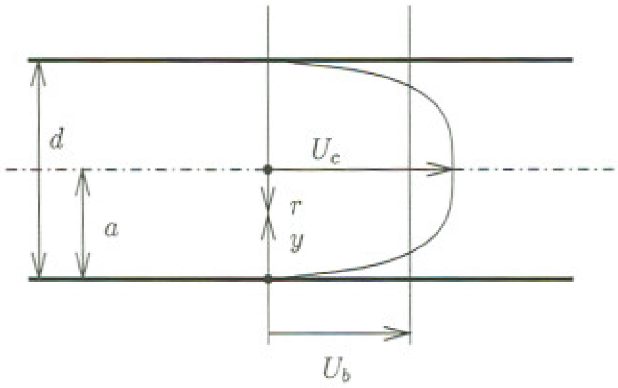


15 Fully developed turbulent pipe flow

Flow in a pipe is a form of wall turbulence. Hence the law of the wall and defect law are equally valid except we replace δ with the pipe radius a and the constant c will be different since this is a characteristic constant dependent on the flow geometry. But κ and A are universal so they are the same for fully developed turbulent pipe flow as for turbulent boundary layers, in fact their numerical values are most easily determined from pipe flow experiments.

$$\frac{u}{U_\tau} = \frac{1}{\kappa}\ln\left(\frac{yU_\tau}{\nu}\right) + A \tag{5}$$

$$\frac{U_1 - u}{U_\tau} = -\frac{1}{\kappa}\ln\frac{y}{\delta} + c \tag{6}$$



U_b =bulk velocity $= \frac{1}{A}\int_A U dA$

U_c =center line velocity

Note fully developed means there is no freestream, the 'boundary layers' extend all the way to the center line.

For high Reynolds numbers it is valid to assume (5) goes all the way to the wall. However in the region of the center line the velocity profile will deviate from (5), so introduce a deviation function ie.

$$\frac{u}{U_\tau} = \frac{1}{\kappa}\ln\left(\frac{yU_\tau}{\nu}\right) + A + H\left(\frac{y}{a}\right) \tag{7}$$

Deviation function, goes to zero for small y

Put $y = a$

$$\frac{U_c}{U_\tau} = \frac{1}{\kappa}\ln\left(\frac{aU_\tau}{\nu}\right) + A + H(1) \tag{8}$$

Subtract (7) from (8) to give

$$\frac{U_c - u}{U_\tau} = -\frac{1}{\kappa}\ln\left(\frac{y}{a}\right) + H(1) - H\left(\frac{y}{a}\right) \tag{9}$$

The above is only a function of y/a

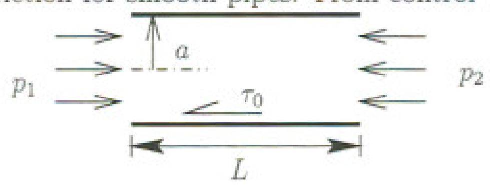
$$\Rightarrow \frac{U_c - u}{U_\tau} = g\left(\frac{y}{a}\right)$$

which is consistent with (2).

15.1 Friction formula

We wish to determine the skin friction for smooth pipes. From control volume analysis,

$$L \cdot 2\pi a \tau_0 = (p_1 - p_2)\pi a^2$$
$$\therefore \tau_0 = \frac{p_1 - p_2}{L}\frac{a}{2}$$



Define the friction factor f;

$$\frac{f}{8} = \frac{U_\tau^2}{U_b^2}$$
$$\Rightarrow \tau_0 = \frac{f}{8}\rho U_b^2\,.$$

Note f is analogous to C_f' for a boundary layer, recall

$$\frac{C_f'}{2} = \frac{U_\tau^2}{U_1^2}\,.$$

Hence

$$f\frac{1}{2}\rho U_b^2 = d\frac{p_1 - p_2}{L}$$

Using (2) a expression for f as a function of Reynolds number can be found.

$$\begin{aligned}\frac{U_c - U_b}{U_\tau} =& \frac{1}{A}\int_A \left(\frac{U_c - u}{U_\tau}\right) dA \\ =& \frac{1}{A}\int_A \left(-\frac{1}{\kappa}\ln\left(\frac{y}{a}\right) + \underbrace{H(1) - H\left(\frac{y}{a}\right)}_{\text{neglect contributions from these terms}}\right) dA\end{aligned}$$

$$\therefore \frac{U_c - U_b}{U_\tau} = \frac{3}{2\kappa} \tag{10}$$

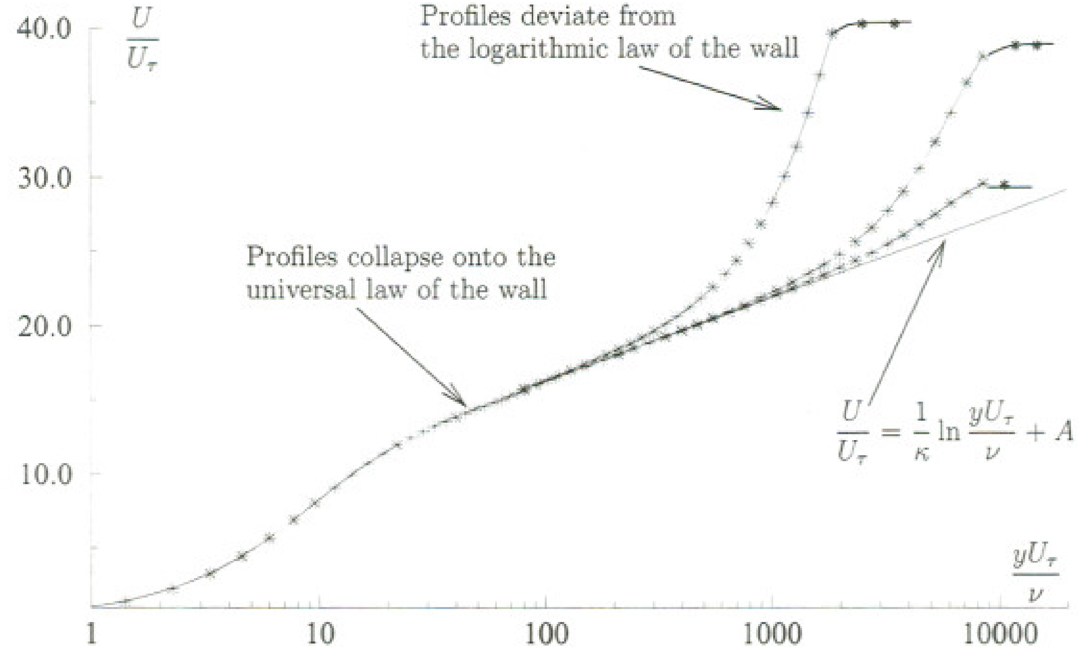
Substitute (8) into (10)

$$\frac{U_b}{U_\tau} - \frac{1}{\kappa}\ln\frac{aU_\tau}{\nu} - A - H(1) = \frac{-3}{2\kappa}$$
$$\therefore \frac{U_b}{U_\tau} = \frac{1}{\kappa}\ln\frac{aU_\tau}{\nu} + A + H(1) - \frac{3}{2\kappa}$$
$$\therefore \sqrt{\frac{8}{f}} = \frac{1}{\kappa}\ln\left(Re\sqrt{\frac{f}{8}}\right) + D \qquad \leftarrow \text{Prandtl's smooth pipe friction formula}$$

where

$$Re = \frac{dU_b}{\nu} \qquad \text{Pipe Reynolds number}$$
$$D = \text{ A universal constant}$$

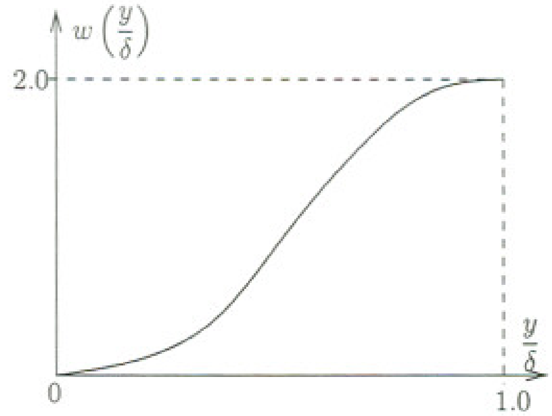
16 Turbulent boundary layers



The amount of deviation from the logarithmic law depends on the large scale flow geometry (ie. the pressure gradient) as well as the Reynolds number. We can characterise the deviation from the logarithmic law by the wake function $w(y/\delta)$ ie

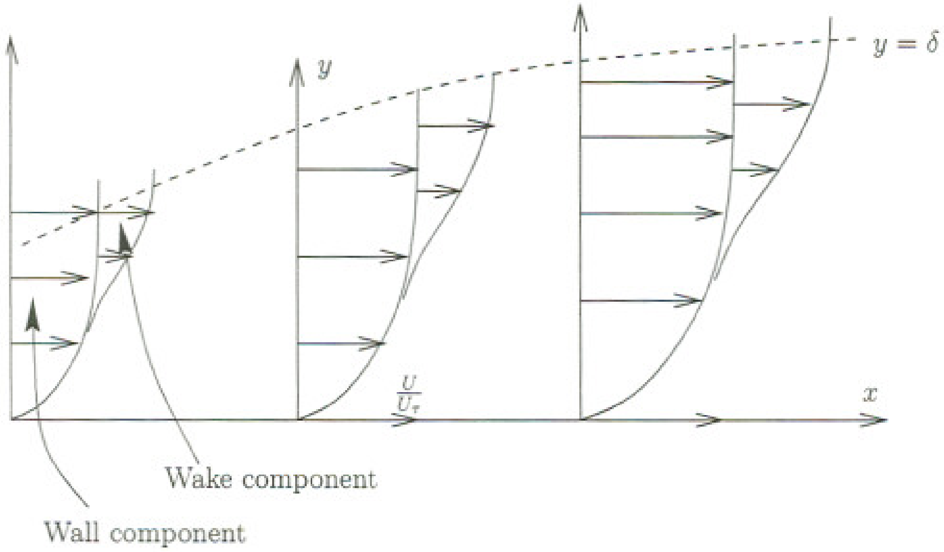
$$\text{Deviation} = \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right)$$

Where Π is called the wake factor and is a function of x. $w(y/\delta)$ is a universal function, that is, it is common to all [2D] turbulent boundary layers.



Hence we can express the velocity profile in terms of a law of the wall/ law of the wake

$$\frac{U}{U_\tau} = \underbrace{\frac{1}{\kappa}\ln\frac{yU_\tau}{\nu} + A}_{\text{Law of the wall}} + \underbrace{\frac{\Pi}{\kappa}w\left(\frac{y}{\delta}\right)}_{\text{Law of the wake}} \tag{a}$$



16.1 Zero pressure gradient layer

There are layers where Π = constant and such layers are called equilibrium layers. An example is the zero pressure gradient layer, where

$$\Pi = 0.55 \leftarrow \text{this gives us closure.}$$

For this case it is possible to derive a law of skin friction, using the integral momentum equation in conjunction with the law of the wall, law of the wake. The von Karman skin friction law is one example.

Now for a zero pressure gradient U_1 =constant, so the integral momentum equation becomes

$$\begin{aligned}\frac{C_f'}{2} =& \frac{d\theta}{dx}\\ =& \frac{d\left(\frac{\theta U_1}{\nu}\right)}{d\left(\frac{xU_1}{\nu}\right)}\\ \therefore \frac{C_f'}{2} =& \frac{dR_\theta}{dR_x}\end{aligned} \tag{1}$$

At $y = \delta$, $U = U_1$ hence (a) implies

$$\frac{U_1}{U_\tau} = \frac{1}{\kappa}\ln\frac{\delta U_\tau}{\nu} + A + \frac{\Pi}{\kappa}w(1) \tag{2}$$

Subtract (a) from (2)

$$\frac{U_1 - U}{U_\tau} = \frac{-1}{\kappa} \ln\left(\frac{y}{\delta}\right) + \frac{\Pi}{\kappa} w(1) - \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right) \tag{b}$$

This represents a similarity solution, ie. it is of the form

$$\frac{U_1 - U}{U_\tau} = f(\eta) \tag{3}$$

Put

$$S = \sqrt{\frac{2}{C_f'}} = \frac{U_1}{U_\tau} \tag{4}$$

Use (b) to find expressions for δ^* and θ;

$$\begin{aligned}
\frac{\delta^*}{\delta} &= \int_0^1 \left(1 - \frac{U}{U_\tau}\right) d\eta \\
&= \frac{U_\tau}{U_1} \int_0^1 \left(\frac{U_1 - U}{U_\tau}\right) d\eta \\
&= \frac{1}{S} \underbrace{\int_0^1 f(\eta) d\eta}_{\text{definite integral of a universal function} \Rightarrow \text{universal constant}}
\end{aligned}$$

$$\therefore \frac{\delta^*}{\delta} = \frac{C_1}{S} \qquad \text{actually we don't use this result, its just included for completeness}$$

$$\begin{aligned}
\frac{\theta}{\delta} &= \int_0^1 \frac{U}{U_\tau} \left(1 - \frac{U}{U_\tau}\right) d\eta \\
&= \int_0^1 \left(\frac{U_1 - U}{U_1}\right) d\eta - \int_0^1 \left(\frac{U_1 - U}{U_1}\right)^2 d\eta \\
&= \frac{1}{S} \int_0^1 \left(\frac{U_1 - U}{U_\tau}\right) d\eta - \frac{1}{S^2} \int_0^1 \left(\frac{U_1 - U}{U_\tau}\right)^2 d\eta \\
\therefore \frac{\theta}{\delta} &= \frac{C_1}{S} - \frac{C_2}{S^2}
\end{aligned} \tag{5}$$

where

$$C_1 = \int_0^1 f(\eta) d\eta, \qquad C_2 = \int_0^1 f^2(\eta) d\eta \, .$$

From (2)

$$\begin{aligned}
\frac{U_1}{U_\tau} &= \frac{1}{\kappa} \ln \frac{\delta U_\tau}{\nu} + A + \frac{\Pi}{\kappa} w(1) \\
\therefore \ln\left(\frac{\delta U_\tau}{\nu}\right) &= \kappa \left(S - A - \frac{\Pi}{\kappa} w(1)\right) \\
\therefore \frac{\delta U_\tau}{\nu} &= \exp\left\{\kappa\left(S - A - \frac{\Pi}{\kappa} w(1)\right)\right\}
\end{aligned} \tag{6}$$

From (5)

$$\frac{\theta U_1}{\nu} = \delta \frac{U_1}{\nu}\left(\frac{C_1}{S} - \frac{C_2}{S^2}\right)$$
$$\therefore R_\theta = \frac{\delta U_\tau}{\nu} S \left(\frac{C_1}{S} - \frac{C_2}{S^2}\right)$$

Hence from (6)

$$R_\theta = \left(C_1 - \frac{C_2}{S}\right) \exp\left\{\kappa \left(S - A - \frac{\Pi}{\kappa} w(1)\right)\right\}$$

Put $A + \frac{\Pi}{\kappa} w(1) = \phi(1)$

$$\therefore \frac{dR_\theta}{dS} = \left(\frac{C_2}{S^2} + \kappa C_1 - \frac{\kappa C_2}{S}\right) \exp\left\{\kappa \left(S - \phi(1)\right)\right\} \tag{7}$$

Now (1) implies

$$\frac{dR_\theta}{dR_x} = \frac{1}{S^2}\ . \tag{8}$$

Hence using (7) and (8) we obtain the differential equation;

$$\frac{dR_x}{dS} = \frac{dR_x}{dR_\theta} \cdot \frac{dR_\theta}{dS}$$
$$= S^2 \left(\frac{C_2}{S^2} + \kappa C_1 - \frac{\kappa C_2}{S}\right) \exp\left\{\kappa \left(S - \phi(1)\right)\right\}$$
$$\therefore \frac{dR_x}{dS} = (C_2 + \kappa C_1 S^2 - \kappa C_2 S) \exp\left\{\kappa \left(S - \phi(1)\right)\right\}\ .$$

Initial conditions are $R_x = 0$, $S = 0$. Solving by integration gives

$$\boxed{R_x = \left\{C_1 S^2 - \left(\frac{2C_1}{\kappa} + C_2\right) S + \frac{2}{\kappa}\left(\frac{C_1}{\kappa} + C_2\right)\right\} \exp\left\{\kappa \left(S - \phi(1)\right)\right\}}$$

Since in practice S is large (eg. $20 < S < 30$) we retain only the S^2 terms giving

$$R_x = C_1 S^2 \exp\left\{\kappa \left(S - \phi(1)\right)\right\}$$
$$\Rightarrow S = \phi(1) + \frac{1}{\kappa} \ln\left(\frac{R_x}{C_1 S^2}\right)$$

hence

$$\boxed{\sqrt{\frac{1}{C_f'}} = A' + B' \log_{10}(R_x C_f')} \quad \rightarrow \text{Karman law of skin friction}$$

With $\kappa = 0.51$, $A = 5.1$ and $\Pi = 0.55$ get

$$A' \approx 4.15, \quad B' \approx 1.7\ .$$

$$\text{Or} \quad \sqrt{\frac{2}{C'_f}} = A'' + B'' \log_{10}\left(\frac{R_x C'_f}{2}\right)$$