Incompressible Thermal Boundary Layer Derivation

David D. Marshall
gte552m@mail.gatech.edu

August 1, 2002

Developing the Incompressible Thermal Boundary Layer Solution starts with the energy equation from the 2D incompressible Navier-Stokes equations

$$\rho C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] \tag{1}$$

When simplified using the incompressible thermal boundary layer assumptions

1. Boundary layer thickness (δ) is small, i.e. $Re \gg 1$
2. Boundary layer is laminar
3. Buoyancy effects are negligible, i.e. $Fr \gg 1$
4. Energy changes do not significantly effect the fluid density or viscosity

yields the following for the energy equation

$$\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \tag{2}$$

Assuming dissipation effects are negligible (i.e. Eckert number is $\ll 1$), then (2) becomes

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{Pr} \frac{\partial^2 T}{\partial y^2} \tag{3}$$

Since the Blasius solution is a special case of the Falkner-Skan solution, the solution procedure proceeds with the use of the Falkner-Skan solution to the continuity and momentum equations. Utilizing the coordinate transformations from the physical space y-coordinate to the transformed coordinate, η, the right side of (3) becomes

$$\frac{\nu}{Pr} \frac{\partial^2 T}{\partial y^2} = \frac{g^2 \nu}{Pr} T'' \tag{4}$$

In order to address the left side of (3), intermediate results from the Falkner-Skan derivation for u and v are needed. These are shown here as (5a) and (5b).

$$u = U_e f' \tag{5a}$$

$$v = \frac{U_e g' (f - \eta f') - U_e' g f}{g^2} \tag{5b}$$

Utilizing the coordinate transformations used in (4) as well as (5a) and (5b), the left side of (3) becomes

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = (U_e f') \left(\frac{g' \eta}{g} T' \right) - \left(\frac{f U_e' g + U_e g' f' \eta - U_e g' f}{g^2} \right) (g T') \tag{6}$$

which when simplified becomes

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\left(\frac{U_e g'}{g}-U_e'\right)fT' \tag{7}$$

Utilizing the definition of β in the Falkner-Skan solution

$$\beta=\frac{U_e'}{\nu g^2} \tag{8}$$

and rearranging terms yields

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=g^2\nu\left(\frac{U_e g'}{g^3\nu}-\beta\right)fT' \tag{9}$$

Using another result from the Falkner-Skan solution, shown here as (10),

$$\frac{U_e g'}{g^3\nu}-\beta=-\alpha=-1 \tag{10}$$

results in (9) becoming

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=-g^2\nu fT' \tag{11}$$

Combining (4) and (9) yields the following modified equation for the energy equation:

$$-g^2\nu fT'=\frac{g^2\nu}{Pr}T''$$
$$\Rightarrow \quad T''+PrfT'=0 \tag{12}$$

The only difference between the Blasius or Falkner-Skan forms of (12) is the choice of f.

With the energy equation developed into a form independent of the x-coordinate and in the form of a homogeneous, second order ODE, an analytical solution can now be developed. First, the boundary conditions can be established as

$$\text{isothermal wall:} \quad T(0)=T_w \quad T(\infty)=T_e \tag{13}$$

$$\text{adiabatic wall:} \quad T(\infty)=T_e \quad T'(0)=0 \tag{14}$$

Notice that this assumes a constant wall temperature for the isothermal boundary conditions and a constant freestream temperature for both isothermal and adiabatic boundary conditions.

The solution of (12) is

$$T(\eta)=\int_0^{\eta} C_1 e^{Pr\int_0^{\zeta} f(s)ds}d\zeta+C_2 \tag{15}$$

Applying the adiabatic wall boundary conditions to find the constants C_1 and C_2 results in the following simple equation

$$C_1=0 \quad C_2=T_e$$
$$\Rightarrow \quad T(\eta)=T_e \tag{16}$$

Applying the isothermal wall boundary conditions to find the constants C_1 and C_2 yields

$$C_1=\frac{T_e-T_w}{\int_0^{\infty} e^{Pr\int_0^{\zeta} f(s)ds}d\zeta} \tag{17a}$$

$$C_2=T_w \tag{17b}$$

which when applied to (15) yields

$$\theta(\eta)=\frac{\int_0^{\eta} e^{PrF(\zeta)}d\zeta}{\int_0^{\infty} e^{PrF(\zeta)}d\zeta} \tag{18}$$

$$\text{where} \quad \theta(\eta)=\frac{T-T_w}{T_e-T_w}$$

$$\text{and} \quad F(\zeta)=\int_0^{\zeta} f(s)ds$$

Finally, the wall heat flux for the isothermal wall boundary conditions is

$$q_w = -k \left. \frac{\partial T}{\partial y} \right|_w = -kg(x) \frac{T_e - T_w}{\int_0^\infty e^{PrF(\zeta)} d\zeta} \tag{19}$$

and the derivation is complete.