Contents of today's lecture

- Blasius solution for laminar flat-plate boundary layer where external velocity is constant ($\partial U/\partial x = 0$)
- About solution methods for laminar boundary layers
- Thwaites' method as an example of integral methods

Blasius solution for laminar flat-plate boundary layer

In case of a flat-plate boundary layer with constant external velocity $U = U_\infty$ also pressure is constant and the boundary-layer equations further reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u(x,0) = v(x,0) = 0 \quad \text{and} \quad u(x,\delta) = U = U_\infty$$

Blasius found a similarity solution for this problem. This means that the solution is independent of x when scaled properly. It is quite obvious that U and δ are appropriate scales.

Blasius solution for laminar flat-plate boundary layer

We already know from our order-of-magnitude analysis that $\delta \sim x/\sqrt{Re_x}$.
Thus, the following similarity variable may be a good choice

$$\eta = \frac{y}{x}\sqrt{Re_x} = y\sqrt{\frac{U}{\nu x}}$$

then

$$\frac{u}{U} = \frac{u}{U}(\eta)$$

We can inherently satisfy the continuity equation by working with the stream function

$$\psi(x, y) = \sqrt{\nu x U} f(\eta)$$

Blasius solution for laminar flat-plate boundary layer

Then, the velocity components are

$$u = \frac{\partial \psi}{\partial y} = U f'(\eta) \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U}{x}}(\eta f' - f)$$

Evaluating all three terms of the momentum equation and simplifying, we obtain

$$f f'' + 2f''' = 0$$

with the boundary conditions $f(0) = f'(0) = 0$ and $f'(\infty) = 1$. This is an ordinary but nonlinear differential equation. It can only be solved numerically owing to its nonlinearity — or by using complex series expansions as Blasius did in 1908. As soon as f is solved, then $u(\eta) = U f'(\eta)$.

Blasius solution for laminar flat-plate boundary layer

Some results from the Blasius solution:

$$\delta = 5.26x/\sqrt{Re_x}$$

$$\delta_1 = 1.72x/\sqrt{Re_x}$$

$$\delta_2 = 0.664x/\sqrt{Re_x}$$

$$H_{12} = 2.59$$

$$c_f = 0.664/\sqrt{Re_x}$$

where is the skin-friction coefficient defined as

$$c_f \equiv \frac{\tau_0}{\frac{1}{2}\rho U^2}$$

Solution approaches for laminar boundary layers

The Blasius solution is restricted to the case of 2-dimenional flat-plate boundary layer with $\mathrm{d}U/\mathrm{d}x = 0$. Other similarity solutions have been found for a number of special cases such as for flow past a wedge by Falkner and Skan $U(x) \sim x^m$. However, more general 2-dimensional cases with arbitrary pressure gradient (or $\mathrm{d}U/\mathrm{d}x$) must be solved by means of:

1) solving the boundary-layer equations numerically (numerical solution should approach exact solution when $\Delta x, \Delta y \to 0$)

2) some approximate method based on the integral momentum equation or on the integral kinetic-energy equation or both.

Solution approaches for laminar boundary layers

The first approach is more general and involves no other inaccuracies than the numerical error which can easily be kept negligibly small today. The numerical methods can also quite easily be extended for 3-dimensional boundary-layer problems. However, numerical solutions of the boundary-layer equations are beyond the scope of this course.

Methods representing the second approach can be more restricted considering practical use in aerodynamic design. However, some of those methods may be very useful as they often allow solution in closed form. Such results may be very useful for checking purposes and for producing quick initial estimates. We shall next introduce one of these methods.

Thwaites' integral method

The problem is to close the integral momentum equation

$$\frac{\mathrm{d}\delta_2}{\mathrm{d}x} + \frac{\delta_2}{U}\frac{\mathrm{d}U}{\mathrm{d}x}(H_{12}+2) = \frac{c_f}{2}$$

and then to integrate it up to x starting from some initial point x_0. The key question is: "how to express H_{12} and c_f as functions of δ_2 and U?"

Thwaites' method is based on the assumption that the velocity profiles are uni-parametric, *i.e.* defined by only one parameter λ that is defined as

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = -\lambda\frac{U}{\delta_2^2}$$

where the subscript "0" stands for a value on the wall surface.

Thwaites' integral method

λ is related to the external pressure (or velocity) gradient as on the wall surface the boundary-layer equations reduce to

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu \left(\frac{\partial^2 u}{\partial y^2}\right)_0$$

and Bernoulli tells us that

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{2}\frac{\partial U^2}{\partial x} = -U\frac{\partial U}{\partial x}$$

thus

$$\lambda = -\frac{\delta_2^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = \frac{\delta_2^2}{\nu}\frac{\partial U}{\partial x}$$

Thwaites' integral method

Multiply the integral momentum equation by $U\delta_2$ and rearrange to obtain

$$\frac{U}{2}\frac{\mathrm{d}\delta_2^2}{\mathrm{d}x} - \nu\lambda(H+2) = \nu l$$

where l is defined by

$$\left(\frac{\partial U}{\partial y}\right)_0 = l\frac{U}{\delta_2}$$

and it is related with the skin-friction coefficient as $\tau_0 = \mu l U/\delta_2$ and $c_f = 2l\nu/(\delta_2 U)$. Thwaites assumed that $l = l(\lambda)$ only and that $H_{12} = H_{12}(\lambda)$ only (uni-parametricity). Thence, he could write

$$\frac{U}{\nu}\frac{\mathrm{d}\delta_2^2}{\mathrm{d}x} = 2\{-\lambda[H_{12}(\lambda)+2] + l(\lambda)\} \equiv F(\lambda)$$

Thwaites' integral method

Now, $F(\lambda)$ is the function that must be closed using empirical data and/or exact solutions for the special cases such as the Blasius and Falkner-Skan solutions, etc. Thwaites found the following linear approximation

$$F(\lambda) = 0.45 - 6\lambda$$

After inserting this, multiplying by U^5, and rearranging we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(U^6\delta_2^2) = 0.45\nu U^5$$

This can be readily integrated as

$$\delta_2^2(x) = \frac{1}{U^6(x)}\left[0.45\nu\int_0^x U^5(x)\mathrm{d}x + \left(\delta_2^2 U^6\right)_{\text{at } x=0}\right]$$

Thwaites' integral method

When δ_2 is solved, then λ is also known. H_{12} and l, and thus δ_1 and c_f can be computed using Thwaites' formulas. These are given in the lecture notes (pages 19 and 20). Separation takes place when $\lambda < -0.09$.

Note, however, that in reality H_{12} and l may depend also on some other parameters than λ, say, on $\mathrm{d}\lambda/\mathrm{d}x$ for instance. The uni-parametricity is especially questionable in adverse pressure-gradient cases ($\mathrm{d}U/\mathrm{d}x < 0$). It is thus important to remember that **this is not an exact method** albeit it has turned out to be quite accurate in many cases.

Two-parameter methods, such as Head's method, are more accurate and reliable, but less practical and thus more or less outdated nowadays.

What did we learn today?

- We studied the exact similarity solution by Blasius for the flat-plate boundary layer with constant external velocity
- We took a brief look at the general solution methods for laminar 2-dimensional boundary layers
- We went through the derivation of Thwaites' integral method