The Dynamics of Fluids

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Fall 1999

Lecture 11

The doublet

The potential doublet is the derivative of the potential source, i.e.

$$F = \frac{\mu}{z} \tag{1}$$

The doublet can be thought of as a superposition of a sink and a source, both of equal and infinite strength, placed infinitely close together, how odd. See figure 1.

Let

$$\begin{aligned} F &= \lim_{\epsilon \to 0} \frac{\mu}{2\epsilon} \left[\ln(z+\epsilon) - \ln(z-\epsilon) \right], \\ &= \lim_{\epsilon \to 0} \frac{\mu}{2\epsilon} \ln \frac{z+\epsilon}{z-\epsilon}. \end{aligned} \tag{2}$$

Note that in the above, we have chosen the source strength to be infinite and the distance between sink and source to be zero by taking the limit as $\epsilon \to 0$. Note that for small ϵ we can expand

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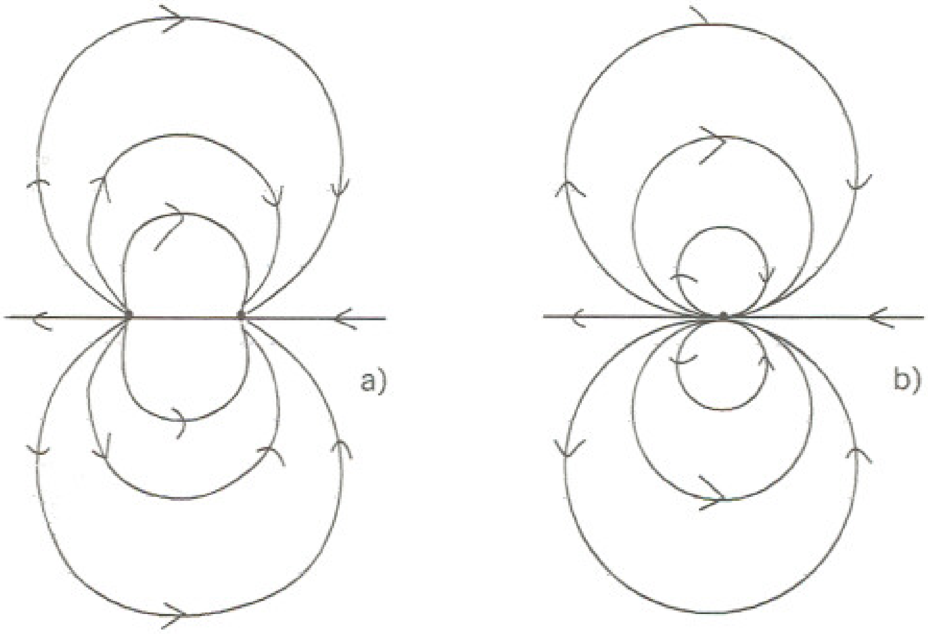

Figure 1: (a) The streamlines for a sink and source of equal strength placed apart. (b) The streamlines for the doublet.

$$\begin{aligned}\ln\frac{z+\epsilon}{z-\epsilon} &= \ln\frac{1+\epsilon/z}{1-\epsilon/z}\\ &= \ln\left(1+2\epsilon/z+\ldots\right)=\frac{2\epsilon}{z}+\ldots,\end{aligned} \tag{3}$$

so that

$$F=\lim_{\epsilon\to 0}\frac{\mu}{2\epsilon}\left(\frac{2\epsilon}{z}+\ldots\right)=\frac{\mu}{z}. \tag{4}$$

Let's now find the equations for the streamlines of a potential doublet. With

$$\begin{aligned}F &= \frac{\mu}{z}=\frac{\mu}{x+i\,y}=\mu\frac{x-i\,y}{x+i\,y},\\ &= \mu\frac{x}{x^2+y^2}-i\,\mu\frac{y}{x^2+y^2}.\end{aligned} \tag{5}$$

Therefore the streamfunction for the potential doublet is identified as

$$\psi=-\mu\frac{y}{x^2+y^2},\ \text{or}\ x^2+y^2+\frac{\mu}{\psi}y=0. \tag{6}$$

In the above equation we *complete the square* and rewrite the equation for the streamline as

$$x^2+y^2+\frac{\mu}{\psi}y+\left(\frac{\mu}{2\psi}\right)^2=\left(\frac{\mu}{2\psi}\right)^2, \tag{7}$$

or

$$x^2+\left(y+\frac{\mu}{2\psi}\right)^2=\left(\frac{\mu}{2\psi}\right)^2, \tag{8}$$

which is a family of circles of radius of $\mu/(2\psi)$ with its center at $(x,y)=(0,-\mu/(2\psi))$. The streamlines are shown in figure 1b.

As an *exercise* show that the velocity in polar coordinates is given by,

$$v_r=-\frac{\mu}{R^2}\cos\theta,\ v_\theta=-\frac{\mu}{R^2}\sin\theta. \tag{9}$$

Potential flow past a circular cylinder

Consider the superposition of a uniform flow and a doublet,

$$F(z) = Uz + \frac{\mu}{z}, \tag{10}$$

With $z = Re^{i\theta}$ we rewrite this potential as

$$\begin{aligned} F(z) &= URe^{i\theta} + \frac{\mu}{R}e^{-i\theta} \\ &= \left(UR + \frac{\mu}{R}\right)\cos\theta + i\left(UR - \frac{\mu}{R}\right)\sin\theta. \end{aligned} \tag{11}$$

Therefore the streamlines are given by

$$\psi = \left(UR - \frac{\mu}{R}\right)\sin\theta. \tag{12}$$

Suppose we pick μ such that on a circle with a given radius, $R = a$, is guaranteed to be a streamline. Notice that if we pick the streamline to be $\psi = 0$, on $R = a$ we have the condition that

$$\psi = \left(Ua - \frac{\mu}{a}\right)\sin\theta = 0, \tag{13}$$

or we must pick μ so that

$$\mu = Ua^2. \tag{14}$$

Thus the potential for flow past a cylinder of radius a is

$$F = U\left(z + \frac{a^2}{z}\right). \tag{15}$$

a plot of the streamlines is shown in Figure 2. Note that for large distances from the cylinder, $R \to \infty$ and the potential reduces to that for a uniform flow.

Now let's calculate the velocity field, the speed and in turn the pressure which comes from Bernoulli's equation. The complex velocity is simply given by

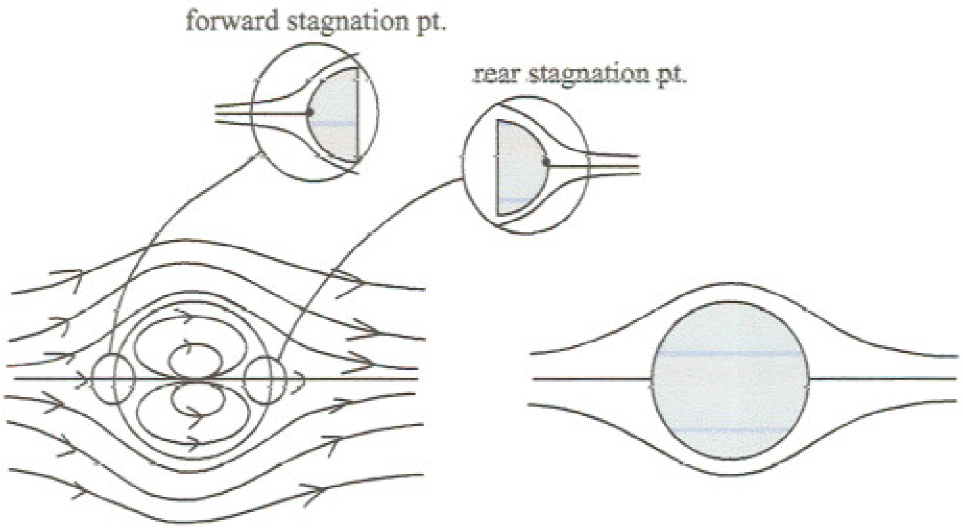

Figure 2: Potential flow past a cylinder

$$W = U\left(1 - \frac{a^2}{z^2}\right) = U\left(1 - \frac{a^2}{R^2}e^{-i\,2\theta}\right) = (v_r - i\,v_\theta)e^{-i\,\theta}. \tag{16}$$

As an exercise, show that

$$v_\theta = -U\left(1 + \frac{a^2}{R^2}\right)\sin\theta,\; v_r = U\left(1 - \frac{a^2}{R^2}\right)\cos\theta. \tag{17}$$

So that on the surface $R = a$ we have,

$$v_r = 0, v_\theta = -2U\sin\theta. \tag{18}$$

If the cylinder is placed in such a way that surfaces of constant z, where z measures the distance upward, intersect the circular cross-section, then Bernoulli's equation reduces to

$$P_s = P + \frac{1}{2}\rho|\mathbf{v}|^2, \tag{19}$$

where $|\mathbf{v}|^2$ is the speed squared and P_s is the stagnation point pressure. On the surface of the cylinder $v_r = 0$ and the equation for the pressure reduces to

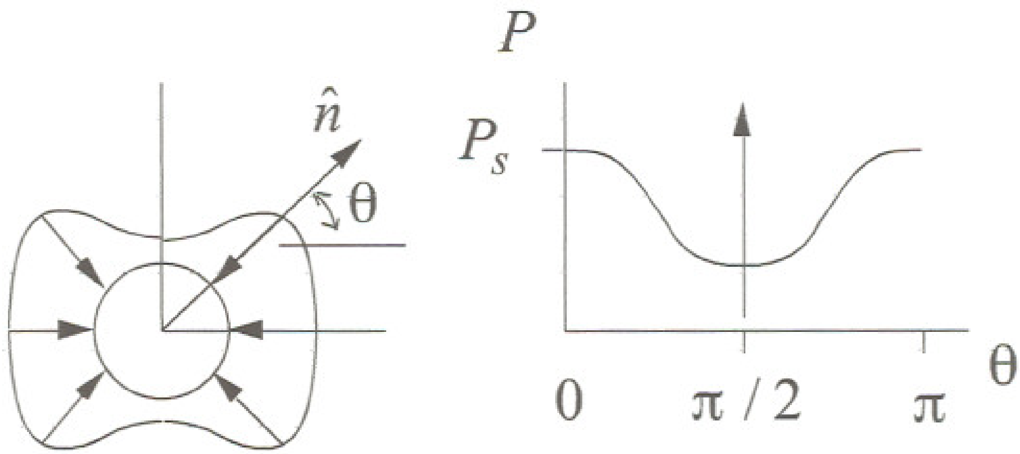

Figure 3: The pressure distribution over a circular cylinder

$$P = P_s - 2\rho U^2 sin^2(\theta). \tag{20}$$

Note that there is a *forward and a rear stagnation point* on the cylinder since both components of velocity vanish there and the pressure is P_s. Notice also that plotting the pressure distribution as a function of θ say, shows that it is symmetric with respect to $\theta = \pi/2$.

The pressure distribution on the cylinder is shown in Figure 3. and note that the pressure acts perpendicular to the cylinder, normal to the surface. Now let's calculate the total resultant force from the pressure distribution acting on the cylinder's surface. Then the differential force is given by

$$d\mathbf{F} = -P\mathbf{n}Rd\theta, \text{ where, } \mathbf{n} = -cos(\theta)\mathbf{i} - sin(\theta)\mathbf{j}, \tag{21}$$

and

$$\mathbf{F_r} = \int_0^{2\pi} (P_s - 2\rho U^2 sin^2(\theta))(cos(\theta)\mathbf{i} + sin(\theta)\mathbf{j})Rd\theta \tag{22}$$

Thus the net resultant force is zero, even though there is a nonzero pressure distribution acting on the cylinder. This is called D'Alembert's paradox. But the paradox is simply resolved by noticing 1) that ideal flow is inviscid, so that there are no viscous forces that could act on the body and that 2) the pressure distribution is perfectly symmetric due to the symmetric flow field generated by the cylinder.

Flow past a circular cylinder with circulation

Here we consider the potential that is created by adding a potential vortex to the potential that describes a uniform flow past a cylinder, i.e.

$$F = U\left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln z/a. \tag{23}$$

On the surface $z = ae^{i\theta}$ we have that the streamfunction is given by

$$\psi = -\frac{\Gamma}{2\pi}\ln R/a, \tag{24}$$

so that on the cylinder surface where $R = a$, $\psi = 0$ and is seen as a closed streamline. However the $x - axis$ is no longer a streamline in general.

Once again (see Currie) one calculates that velocity for this potential by taking the derivative of the potential and then identifying individual velocity components using the definition of W. Generally one find that

$$v_r = U\left(1 - \frac{a^2}{R^2}\right)cos(\theta), v_\theta = -U\left(1 + \frac{a^2}{R^2}\right)sin(\theta) - \frac{\Gamma}{2\pi R}. \tag{25}$$

On the surface of the cylinder $R = a$ one obtains

$$v_r = 0, v_\theta = -2U sin(\theta) - \frac{\Gamma}{2\pi a}. \tag{26}$$

Stagnation points in the flow are found by setting both velocity components identically equal to zero, thus doing so we find that the stagnation points are found on the surface of the cylinder when θ takes on values

$$sin(\theta_s) = -\frac{\Gamma}{4\pi a U} \tag{27}$$

This shows that finding the stagnation points breaks down neatly into three separate cases based on dimensionless circulation parameter defined by the above equation namely,

$$0 < \frac{\Gamma}{4\pi U a} < 1, \frac{\Gamma}{4\pi U a} = 1, \frac{\Gamma}{4\pi U a} > 1., \tag{28}$$

In the first two cases, the stagnation point actually lies on the cylinder, but in the last the stagnation point does not and lies out in the inerior of the fluid. This is clear since $\sin(\theta)$ can not take on values that are greater (or less) than one. Then the location of the stagnation point is found from solving both equation for $v_r, v_\theta = 0$, simultaneously. For very small values of the vortex circulation the flow looks quite similar to that of flow past a cylinder, but as the circulation increases the stagnation points move closer together on the cylinder, eventually merging and finally lifting off the cylinder to move into the fluid interior. The corresponding streamlines are deformed due to this addition of the vortex which induces a swirling motion in the flow. Figure 4. shows representative streamline patterns for the cases discussed above. Unlike flow past a cylinder, the streamlines are not symmetric about the $x - axis$ and consequently the pressure distribution isn't either. It follows that the pressure distribution has created a net result force sometimes called the *Magnus force* which is recognized as a lifting force. We will say more about this below.

As an *exercise* find the general equation for the stagnation point for large values of the circulation and show that the stagnation point lies in the interior of the fluid.

Calculation of the total resultant for closed bodies: Blausius' Theorem

There are many cases in the study of aerodynamics, like the very last example in which we want to calculate the resultant force due to a certain complex potential that has its singularities *inside* a closed streamline. The closed streamline, of course then represents a body, and if the body in asymmetric or the potential is such that an asymmetric pressure distribution is created, then there is a nonzero resultant force acting on the body. We can calculate this force resultant in a simple way by means of *Blausius' theorem*

First let's give the result and calculate its consequence for the last example. If X (*the drag*) and Y (*the lift*) are taken to be the force resultants created by the pressure distribution in the $x-$direction and $y-$direction respectively and M is taken to be the resultant moment about the origin created by the pressure distribution, and if $W = F'(z)$ is the complex velocity, then Blausius' theorem states that

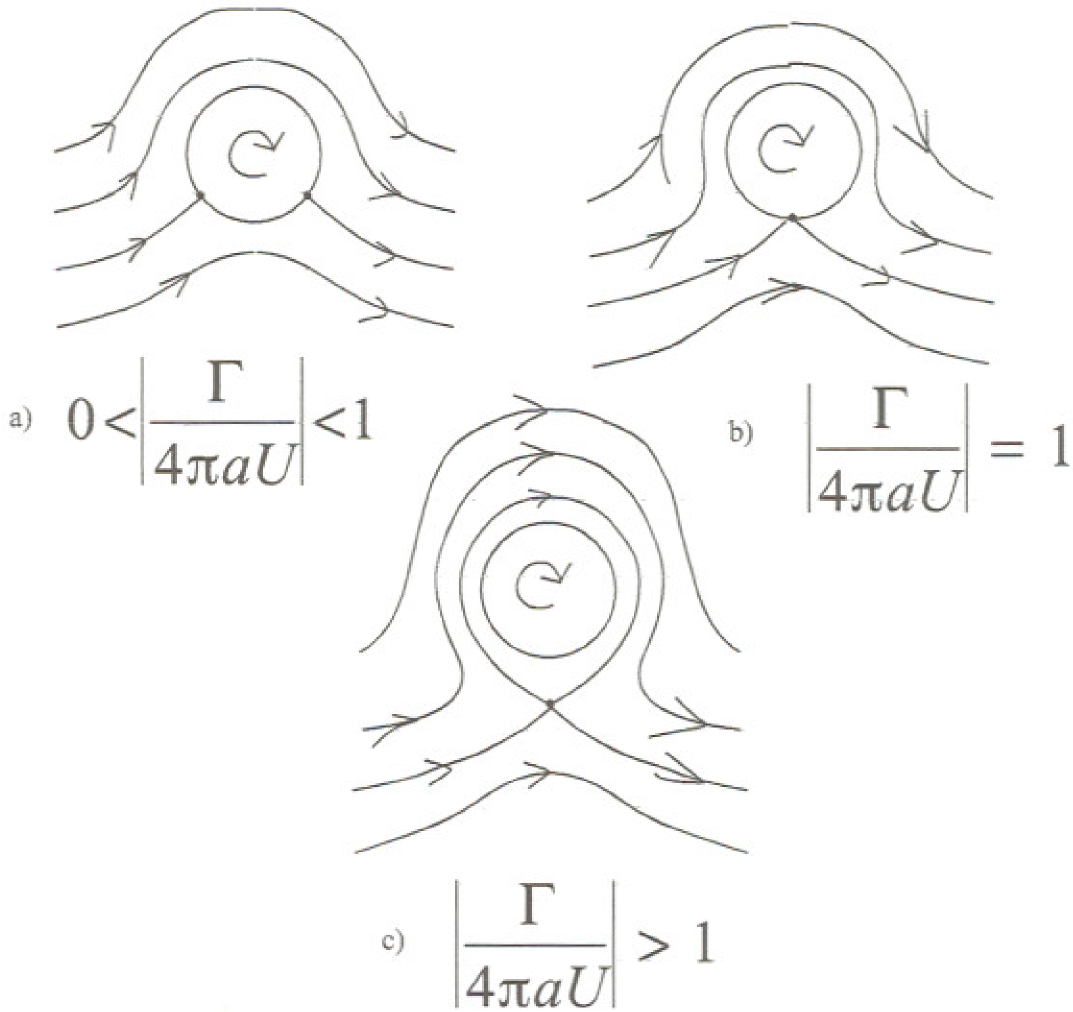

Figure 4: Streamlines for various values of the circulation

$$X - iY = \frac{i\rho}{2}\int_C W^2 dz,\ M = -\frac{\rho}{2}\Re\left\{\int_C zW^2 dz\right\}. \tag{29}$$

where C is any simple, close contour that encloses that body.

Now let's apply this theorem to the last example which is flow past a circular cylinder with circulation. Note that to use the theorem, our task is to

- First evaluate W^2
- Carry out the complex contour integral, which means that we will need to evaluate the residues of the singularities of the argument of the integrals, within C.

Circular cylinder with circulation

By taking the derivative of equation (23) and squaring the result we obtain

$$W^2 = U^2\left(1 - \frac{a^2}{z^2}\right) + 2U\left(1 - \frac{a^2}{z^2}\right)\frac{i\,\Gamma}{2\pi z} - \frac{\Gamma^2}{4\pi^2 z^2}, \tag{30}$$

Note that if we calculate the resultant forces of the pressure distribution on the cylinder, then we must calculate the contour integral $\int_C W^2 dz$ where C can be chosen to be the circular cylinder. To do this contour integral we must find the residues of W^2, then multiply the sum of the residues by $2\pi\,i$. Note that from equation (30) that W^2 is only singular at $z = 0$ and that it is a simply matter to pick of the coefficient of the z^{-1} term in the powers series expansion of W^2 about $z = 0$, (which by (30) is actually a finite expansion). The *residue* of W^2 near $z = 0$ is clearly,

$$i\,\frac{\Gamma U}{\pi}, \tag{31}$$

hence, the integral is computed as

$$\int_C W^2 dz = 2\pi\,i\left(\frac{i\,\Gamma U}{\pi}\right) = -2U\Gamma. \tag{32}$$

Consequently, we have for X, Y that

$$X - iY = \frac{i\rho}{2}(-2U\Gamma) = -i\rho U\Gamma, \tag{33}$$

or

$$X = 0, \text{ and } Y = \rho U\Gamma. \tag{34}$$

So we find that for a circular cylinder with circulation that the *drag* i.e X, is zero and that the *lift*, i.e. Y is related and proportional to the circulation Γ. Note that the lift force is also a function of density, being larger for larger densities. If you had an airplane with an engine with a certain power rating, would it be easier to take off in the summer or winter?

As an *exercise* find the moment M for the circular cylinder with circulation.