The Dynamics of Fluids

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Lecture 18

Exact solutions of the Navier Stokes equations

In the next part of the course we will consider the Navier-Stokes equations for an incompressible fluid. Thus we must now include the effects of viscosity. We will also assume that the bulk viscosity is zero. The Navier-Stokes equations are the momentum equations and they are

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f}. \tag{1}$$

In addition, we have the continuity equations, the statement of incompressibility and the energy equation for an ideal equation of state. They are

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$\frac{D\rho}{Dt} = 0, \tag{3}$$

$$\rho C_v \frac{DT}{dt} = \Phi + k\nabla^2 T,\ \Phi \equiv 2\mu E_{ij} E_{ij}. \tag{4}$$

These are six equations for six unknowns $\rho, \mathbf{v}, P$, and T. There are only a limited number of exact solutions to the Navier-Stokes equations (and

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attendant equations), due to the non-linearity of the terms $\mathbf{v} \cdot \nabla()$. The exact solutions that are known are either for unidirectional flow or for flows in special geometries that allow for simplification of the equations due to symmetries. We will explore some of these next and in particular examine the consequences of viscosity in the flow.

Steady uni-directional flow

Suppose that the velocity is unidirectional and therefore has only one component

$$\mathbf{v} = u(x, y, z, t)\hat{i}, \tag{5}$$

Then by substitution into the continuity equation it follows that

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} = 0, \tag{6}$$

with the implication that

$$u = u(y, z, t) \tag{7}$$

Thus we see that in the flow direction that velocity component cannot depend on the flow direction. See Figure 1.

As an *exercise* verify that it follows that the convective acceleration term vanishes, i.e.

$$\mathbf{v} \cdot \nabla \mathbf{v} = 0. \tag{8}$$

The Navier-Stokes equations in the y and z - directions show that the pressure can only depend on the x - direction, i.e.

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0, \tag{9}$$

so that

$$P = P(x, t). \tag{10}$$

The x-component of the Navier-Stokes equation then reduces to

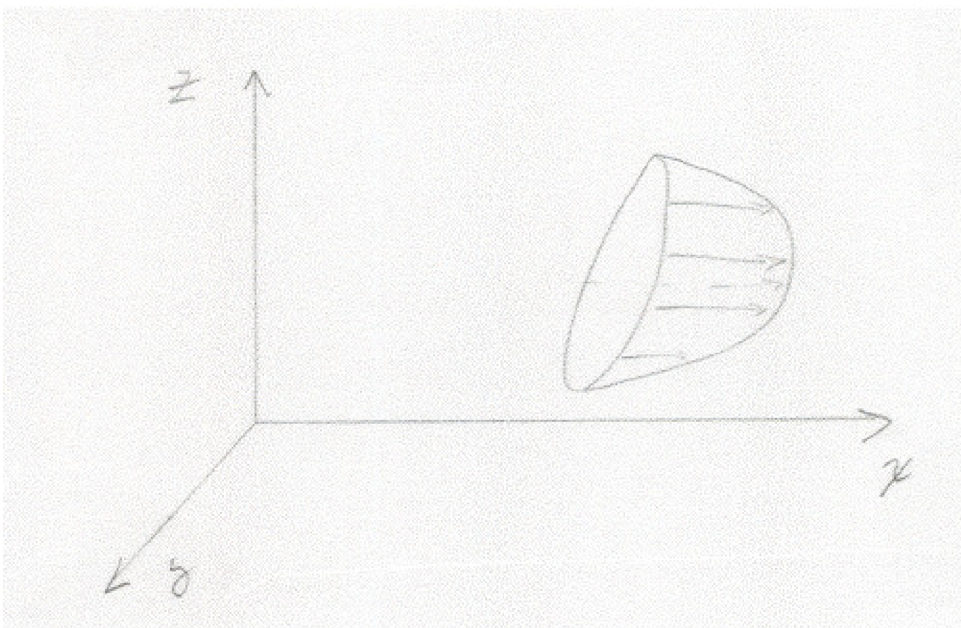

Figure 1: A sketch of a unidirectional flow.

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right). \tag{11}$$

Since $u = u(y, z, t)$ it follows from consistency that the pressure gradient can be at most a function of time, i.e.

$$\frac{\partial P}{\partial x} = F(t). \tag{12}$$

Then it follows that the problem of incompressible, unidirectional flow reduces to

$$\rho\frac{\partial u}{\partial t} = -F(t) + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right). \tag{13}$$

And the boundary condition is simply to specify the surface velocity, U_s at various surfaces $y = constant$.

Steady flow: flow between parallel plates

If we consider flow between parallel plates, we further restrict our geometry and we set $\partial/\partial x = 0$. So the x- momentum equation becomes

$$\frac{\partial^2 u}{\partial y^2} = \frac{F}{\mu}, \tag{14}$$

with the solution

$$u = \frac{1}{2}\frac{F}{\mu}y^2 + Ay + B. \tag{15}$$

Without loss of generality we can suppose that the bottom boundary is stationary and that the top boundary is moving to give the velocity boundary conditions

$$u(0) = 0,\ u(h) = U. \tag{16}$$

These boundary conditions then determine the constants A, B to that we may write the final solution as

$$u + U\frac{y}{h} + \frac{1}{2}\frac{F}{\mu}h^2\left[\left(\frac{y^2}{h} - \frac{y}{h}\right)\right]. \tag{17}$$

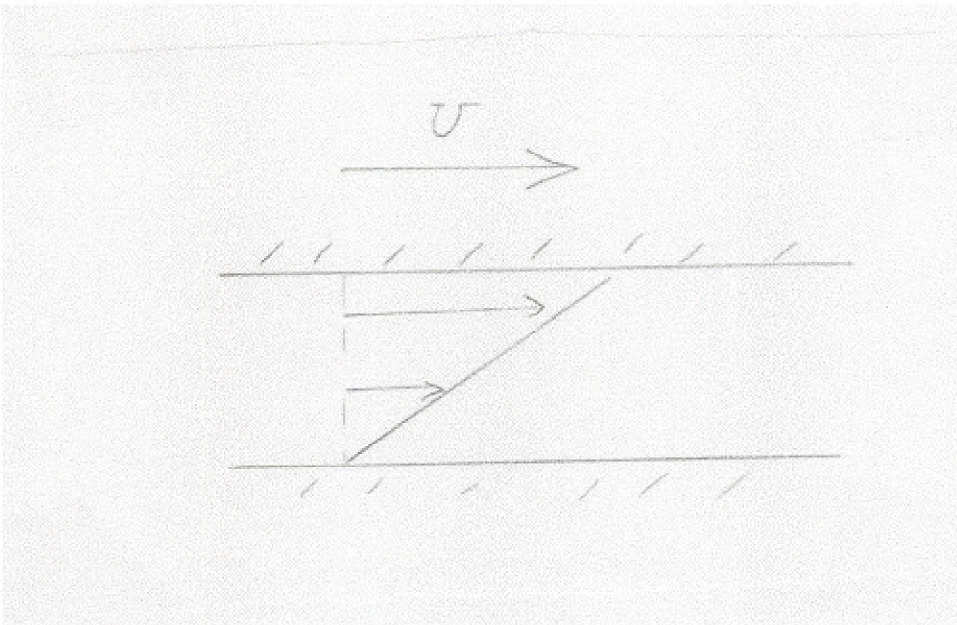

Figure 2: Sketch of Couette flow.

For zero pressure gradient, $F = 0$, the solution is simply a straight line and is called *Couette flow.*
For nonzero pressure gradient, $F < 0$, (say), but with both walls stationary, i.e. $U = 0$, then the solution for the velocity profile looks like a parabola and is called *Poiseuille flow.*
For nonzero pressure gradient and a moving wall, the flow is called *generalized Couette flow* and the a sketch is shown in a figure.

Notice that we have solved for the velocity field independently of the energy equation. In this case, we can solve for the temperature (form the energy) equation in a successive fashion. As an *exercise* show that the energy equation reduces to the temperature equation, for $T(y)$

$$k\,\frac{\partial^2 T}{\partial y^2} = -\mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{18}$$

with representative constant temperature boundary conditions

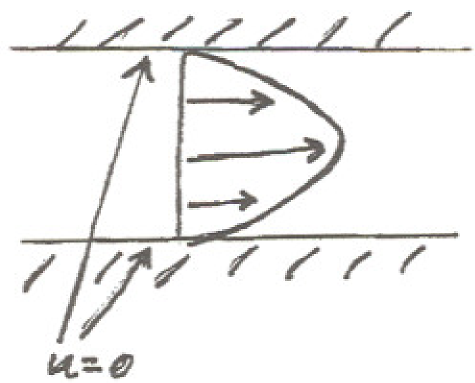

Figure 3: Sketch of Poiseuille flow.

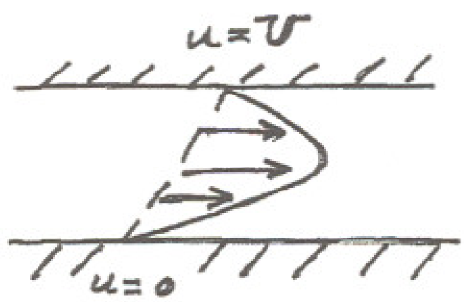

Figure 4: A sketch of generalized Couette flow.

$$T(0) = T_0,\ T(h) = T_1. \tag{19}$$

For the example of plane Couette flow, $F = 0$ and inserting the velocity field to evaluate the dissipation function for the energy equation, and then solving subject to the constant temperature boundary condition shows that

$$T = T_0 + \frac{T_1 + T_0}{2}\left(1 + \frac{y-h}{h}\right) + \frac{\mu U^2}{8k}\left(1 - \frac{(y-h)^2}{h^2}\right). \tag{20}$$

As an *exercise* consider the unidirectional flow in a pipe (in polar coordinates, by representing the flow field as

$$\mathbf{v} = u(r)\mathbf{e}_z, \tag{21}$$

where the $z-$ *direction* is the principal flow direction and the velocity in that direction can at most depend on the lateral radius from the centerline r. Note that we assume that $\partial/\partial\theta = 0$. If one applies the no slip boundary condition on the walls of a pipe at $r = R$, show that

$$u = u_{max}\left(1 - \frac{r^2}{R^2}\right), \tag{22}$$

where

$$u_{max} = -\frac{\partial p}{\partial z}\frac{R^2}{4\mu}. \tag{23}$$