ME 563 - Intermediate Fluid Dynamics - Su
Lecture 17 - The Navier-Stokes equations: velocity gradient tensor

Reading: Acheson, §6.1, 6.2.

Up to now we've discussed the Euler equations that describe the motion of ideal fluids, and the Navier-Stokes equations for viscous, incompressible fluids, without really explaining where they came from. In fact the Navier-Stokes equations in particular took a while to be 'finalized,' and there are some important assumptions inherent in them. We're going to take some time to develop Navier-Stokes equations from the ground up. The discussion will *not* follow the book directly, but the notes and the text are intended to complement each other.

Near the beginning of the course we introduced the velocity gradient tensor, $\nabla \mathbf{u}$ –

$$\nabla \mathbf{u} = \frac{\partial u_j}{\partial x_i} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}. \tag{1}$$

At the time we mentioned this because it represents a vector (the ∇ operator) operating on another vector (the velocity, $\mathbf{u}$) and yielding a second-order tensor. It turns out the velocity gradient tensor is central to the derivation of the N-S equations. In this lecture we will discuss the physical implications of $\nabla \mathbf{u}$.

1 Deformation, strain, and rotation

As we've seen (and as you recall from your introductory fluids class), in a parallel flow of a Newtonian fluid, the velocity gradient normal to the flow direction is directly proportional to the imposed shear stress. That is, if the velocity is $\mathbf{u} = u(y)\hat{e}_x$, the shear stress $\tau = \mu(du/dy)$, where the proportionality constant μ we've called the dynamic viscosity. Figure 1 illustrates this for the case of a flow between two parallel plates.

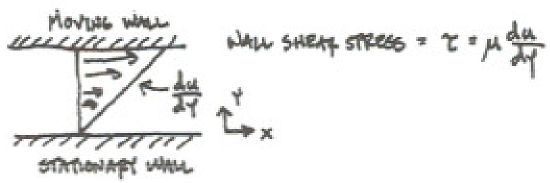

Figure 1: Shear stress in a parallel flow.

Physically (without being precise), we say that the viscosity of the fluid resists deformation because of internal friction. In this example, the 'deformation' takes the form of the shear term du/dy. What about the more general case of non-parallel flow?

For simplicity, we'll talk first about two-dimensional flow, where the velocity field is $\mathbf{u}(x, y) = u(x, y)\hat{e}_x + v(x, y)\hat{e}_y$, and the velocity gradient tensor is

$$\nabla \mathbf{u} = \frac{\partial u_j}{\partial x_i} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{2}$$

A priori we might expect that the off-diagonal terms of $\nabla \mathbf{u}$, $\partial v/\partial x$ and $\partial u/\partial y$, describe the shear stress, in analogy with the parallel flow case, but it's not quite so simple.

Consider a two-dimensional fluid element, with infinitesimal side lengths $dx = dy$ (Fig. 2a), in

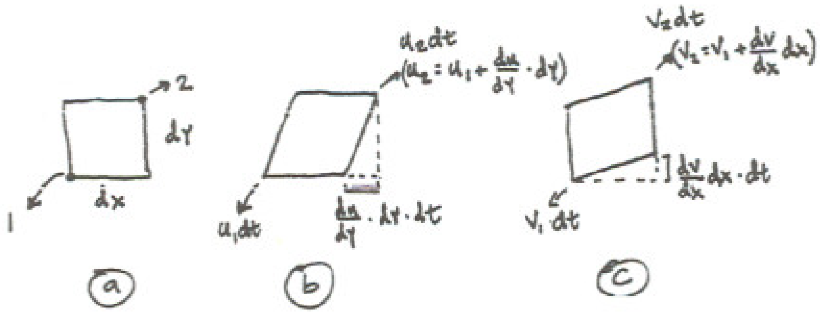

Figure 2: Shear deformation of a 2-D fluid element in parallel flow.

a two-dimensional, steady (not time-varying) flow field. We are not interested now in expansion or compression terms, only shearing terms, so $u(x, y) = u(y)$ and $v(x, y) = v(x)$. (This is equivalent to looking only at the off-diagonal terms of the velocity gradient tensor.) First, consider the parallel velocity field $\mathbf{u} = u(y)\hat{e}_x$. If the fluid element is square at time $t = 0$, as in Fig. 2a, then some (infinitesimal) time later ($t = dt$) point 1 will have moved a distance $u_1\,dt$, and point 2 a distance $(u_1 + (du/dy)\,dy)dt$, and the fluid element gets deformed as in Fig. 2b. (The side lengths $dx = dy$ are infinitesimal, so we only have to worry about linear velocity variations.)

The fluid element is deformed similarly if the velocity field is parallel in the y-direction, $\mathbf{u} = v(x)\hat{e}_y$. In that case, the fluid element will look like Fig. 2c at time $t = dt$. Since the u and v velocity components are orthogonal, for the 2-D velocity field $\mathbf{u} = u(y)\hat{e}_x + v(x)\hat{e}_y$ we can superpose the two deformations to find the shape of the fluid element at time $t = dt$, as in Fig. 3. In the

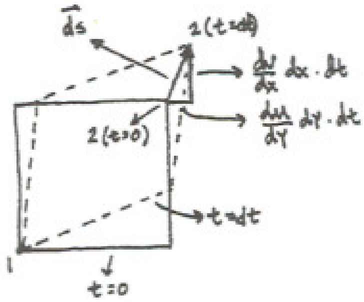

Figure 3: General shear deformation of a 2-D fluid element.

figure, the displacement vector for point 2 is labeled $\mathbf{ds}$, and can be written

$$\mathbf{ds} = \left(\frac{du}{dy}dy\,\hat{e}_x + \frac{dv}{dx}dx\,\hat{e}_y\right)dt = \left(\frac{du}{dy}\hat{e}_x + \frac{dv}{dx}\hat{e}_y\right)dt\,dx$$

making use of $dx = dy$.

There are two special cases of this shear deformation. If $du/dy = -dv/dx$, then the fluid element rotates, but remains square (Fig. 4). This corresponds to solid body rotation. In this case, there is actually no 'deformation' of the fluid element *and no viscous stress.*

The second special case is where $du/dy = dv/dx$ (Fig. 5). In this case the diagonals of the fluid element maintain their angular orientation. This is called pure shear strain, in which there *is no rotation.*

(Acheson has a useful definition of rotation, namely that there is local rotation in a flow if two initially perpendicular fluid line elements have a non-zero average angular velocity. For the cases in Figs. 4 and 5, this means that we just have to see if the diagonals in the fluid element are rotating.)

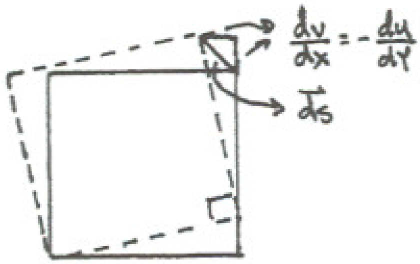

Figure 4: Pure rotation of a 2-D fluid element.

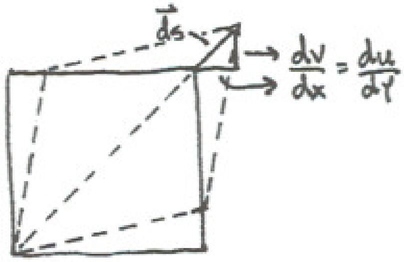

Figure 5: Pure strain of a 2-D fluid element.

Now let's go back and look at only the on-diagonal terms in the 2-D velocity gradient tensor (Eq. 2). Let $\mathbf{u} = u(x)\hat{e}_x + v(y)\hat{e}_y$. The effect of the velocity gradient terms du/dx and dv/dy on the square fluid element of Fig. 2 is obvious; positive du/dx stretches the element in the x-direction, and positive dv/dy stretches the element in the y-direction. Similarly, negative du/dx and dv/dy represent compressions in the x- and y-directions. These are called *normal* strains, and obviously involve no rotation. Viscous fluids resist these normal strains also, but not in quite as intuitive a way as the shear strains (more about this later).

We have now resolved the general 'deformation' of a fluid element under the action of velocity gradients, and have shown that viscosity is relevant to those terms that are independent of rotation. For convenience, then, we want to separate the velocity gradient tensor into parts that describe the strain and rotation separately.

The way to do this suggests itself if we appeal to the above discussion on shear deformations. We saw that there is no viscous stress for $du/dy = -dv/dx$, and no rotation for $du/dy = dv/dx$. The logical thing to do, then, is to make the viscous shear stress proportional to $(du/dy) + (dv/dx)$, and the rotation proportional to $(du/dy) - (dv/dx)$. This is the same thing as separating the tensor $\nabla\mathbf{u}$ into its symmetric and anti-symmetric parts. We'll call the symmetric part $\mathbf{e}$ and the anti-symmetric part Ω –

$$\mathbf{e} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \ \text{"rate-of-strain tensor"}$$
$$\Omega = \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right) \ \text{"rotation rate tensor"} \tag{3}$$

The factor of 1/2 is incorporated so $\mathbf{e} + \Omega = \nabla\mathbf{u}$. These tensors are called 'rates' of strain and rotation because the terms have units $(\text{time})^{-1}$.

Going back to three dimensions, $\mathbf{e}$ can be written (using symmetry)

$$\mathbf{e} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}, \ \text{where } e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{4}$$

and Ω can be written (using anti-symmetry)

$$\Omega = \begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{pmatrix}, \text{ where } \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) \tag{5}$$

The tensor $\mathbf{e}$ has six independent components, and Ω three (not surprising, since $\nabla \mathbf{u}$ has nine). Since Ω only has three independent components, we might expect that the rotation rate can be expressed as a vector. Recalling the definition of a curl from the first lecture, we write

$$\begin{aligned} \omega = \nabla \times \mathbf{u} &= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\hat{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\hat{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\hat{e}_3 \\ &= \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} \end{aligned} \tag{6}$$

Comparing (6) with (5), we see that $2\Omega_{ij} = \omega_k \epsilon_{ijk}$, or

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{7}$$

The quantity $\omega = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ is just the vorticity. We already discussed a little how the vorticity is analogous physically to the angular velocity, or rotation rate. Equation (7) tells us the same thing mathematically.

2 Normal strains and viscosity

Let's look back at the on-diagonal terms of (4):

$$\begin{aligned} e_{11} &= \frac{\partial u_1}{\partial x_1} \\ e_{22} &= \frac{\partial u_2}{\partial x_2} \\ e_{33} &= \frac{\partial u_3}{\partial x_3}. \end{aligned}$$

In Cartesian coordinates, these are just $\partial u/\partial x$, $\partial v/\partial y$, and $\partial w/\partial z$, describing expansion and compression. Notice that, for incompressible flows,

$$e_{ii} = e_{11} + e_{22} + e_{33} = \nabla \cdot \mathbf{u} = 0.$$

The sum of the diagonal terms of a tensor is known as its *trace*. For incompressible flows, then, the trace of the rate-of-strain tensor is zero. (This will become interesting later.)

To summarize: the physical reason for separating $\nabla \mathbf{u}$ into the rate-of-strain and rotation rate tensors in (3) is because of the effects of viscosity. The components of the rate-of-strain tensor $\mathbf{e}$ describe motions that are resisted by viscosity; shearing, compression and expansion. The components of the rotation rate tensor Ω describe motions that are not resisted by viscosity. We will incorporate this knowledge into the N-S equations in the next lecture.