Divergence and Curl of a Vector Field

Recall that a vector field $\mathbf{F}$ on $\mathbf{R}^3$ can be expressed as

$$\mathbf{F}(x, y, z) =< F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) >,$$

where the component functions are scalar functions from $\mathbf{R}^3$ to $\mathbf{R}$. For the sake of brevity, we will often use $\mathbf{x}$ to denote the vector $< x, y, z >$ and write $\mathbf{F}(\mathbf{x}) =< F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}) >$ or even drop the variables altogether and write $\mathbf{F} =< F_1, F_2, F_3 >$.

Let $\mathbf{F} =< F_1, F_2, F_3 >: \mathbf{R}^3 \to \mathbf{R}^3$ be a differentiable vector field. The **divergence** of $\mathbf{F}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

and the **curl** of $\mathbf{F}$ is the vector field

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

Note that the divergence can be defined for any vector field $\mathbf{F}: \mathbf{R}^n \to \mathbf{R}^n$, while the curl is only defined on vector fields in 3-space.

There is a nice second way to deal with these functions. Define the operator ∇, called the "del" operator, by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

The action of the del operator on a scalar function $f: \mathbf{R}^3 \to \mathbf{R}$ is just the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \operatorname{grad}(f)$$

The divergence and the curl of a vector field are given by the two vector products that we have: the curl is the dot product and the curl is the cross product.

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} \qquad \operatorname{curl}(F) = \nabla \times \mathbf{F}.$$

Theorem 1 *Let f be a differentiable scalar function and $\mathbf{F}$ be a differentiable vector field on $\mathbf{R}^3$.*

1. *curl(grad f) $= 0$ or $\nabla \times (\nabla f) = 0$.*
2. *div(curl $\mathbf{F}$) $= 0$ or $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.*

The *Laplacian operator* on a scalar function is defined to be

$$\triangle f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot \nabla f = \nabla^2 f.$$

The Laplacian of a vector field **F** is

$$\triangle \mathbf{F} = \langle \triangle F_1, \triangle F_2, \triangle F_3 \rangle .$$

Geometrically, the curl **F** is a measurment of the tendency of a fluid to swirl around an axis. A field **F** with curl **F** $= 0$ at a point P is called *irrotational at P*. The divergence of a vector field measures the *incompressibility* of a vector field and we can often interpret the divergence of a vector field as the total outflow per unit area. The Laplacian operator helps to describe the diffusion process that is going on in that particular vector field, such as the concentration of a liquid changing as a chemical is dissolved in it or the heat of a solid diffusing from warmer to cooler regions.

Theorem 2 *The operators described above are linear operators:*

$$\begin{aligned}
grad(f+g) &= grad\, f + grad\, g, & grad(cf) &= c\, grad\, f \\
div(\mathbf{F}+\mathbf{G}) &= div\,\mathbf{F} + div\,\mathbf{G}, & div(c\mathbf{F}0 &= c\, div\,\mathbf{F} \\
curl(\mathbf{F}+\mathbf{G})(&= curl\,\mathbf{F} + curl\,\mathbf{G}, & curl(c\mathbf{F}) &= c\, curl\,\mathbf{F} \\
\triangle(f+g) &= \triangle(f) + \triangle(g), & \triangle(cf) &= c\triangle(f) \\
\triangle(\mathbf{F}+\mathbf{G}) &= \triangle(\mathbf{F}) + \triangle(\mathbf{G}), & \triangle(c\mathbf{F}) &= c\triangle(\mathbf{F})
\end{aligned}$$

There are some very intriquing properties of the operators. We see a few in the following theorems. The first theorem we have already covered.

Theorem 3

$$grad(fg)(\mathbf{x}) = g(\mathbf{x})\, grad(f)(\mathbf{x}) + f(\mathbf{x})\, grad(g)(\mathbf{x}) \tag{1}$$

$$grad(\frac{f}{g})(\mathbf{x}) = \frac{g(\mathbf{x})\, grad(f)(\mathbf{x}) - g(\mathbf{x})\, grad(f)(\mathbf{x})}{g(\mathbf{x})^2} \qquad if\; g(\mathbf{x}) \neq 0 \tag{2}$$

Given two vector fields **F** and **G**, we will let the expression $(F \cdot \nabla)(G)$ denote the vecor field whose components are

$$(F \cdot \nabla)(G) =< F \cdot \text{grad}\, G_1, F \cdot \text{grad}\, G_2, F \cdot \text{grad}\, G_3 > .$$

Theorem 4

$$grad(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)(\mathbf{G}) + (\mathbf{G}\cdot\nabla)(\mathbf{F}) + \mathbf{F} \times curl\,\mathbf{G} + \mathbf{G} \times curl\,\mathbf{F} \tag{3}$$

$$div(f\mathbf{F}) = f\, div(\mathbf{F}) + \mathbf{F} \cdot grad(f) \tag{4}$$

$$div(\mathbf{F}\times\mathbf{G}) = \mathbf{G} \cdot curl(\mathbf{F}) - \mathbf{F} \cdot curl(\mathbf{G}) \tag{5}$$

$$curl(f\mathbf{F}) = f\, curl(\mathbf{F}) + grad(f) \times \mathbf{F} \tag{6}$$

$$curl(F \times \mathbf{G}) = \mathbf{F}\, div(\mathbf{G}) - \mathbf{G}\, div(\mathbf{F}) + (\mathbf{G}\cdot\nabla)(\mathbf{F}) - (\mathbf{F}\cdot\nabla)(\mathbf{G}) \tag{7}$$

$$\triangle(fg) = g\triangle f + f\triangle g + 2\, grad(f) \cdot grad(g) \tag{8}$$

$$\triangle(\mathbf{F}\cdot\mathbf{G}) = \mathbf{G}\cdot\triangle\mathbf{F} + \mathbf{F}\triangle\mathbf{G} + 2\, grad\,\mathbf{F} \cdot grad\,\mathbf{G} \tag{9}$$

where

$$grad\,\mathbf{F}\cdot grad\,\mathbf{G} = \sum_{i=1}^{3} grad(F_i)\cdot grad(G_i).$$

How do these differential operators behave in different curvilinear coordinate systems? I will just present the results here. It is not particularly hard, but time-consuming to work throught the details.

Let's consider only cylindrical and spherical coordinates.

In cylindrical coordinate systems, with $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$ we have

$$\begin{aligned}
\operatorname{grad} f &= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \\
\operatorname{div}(\mathbf{F}) &= \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \\
\operatorname{curl}(\mathbf{F}) &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix} \\
\triangle f &= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right) \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

For spherical coordinates, the expressions for the different differential operators are (with $\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$):

$$\begin{aligned}
\operatorname{grad} f &= \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin(\phi)}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \\
\operatorname{div}(\mathbf{F}) &= \frac{1}{\rho^2\sin(\phi)}\left(\frac{\partial}{\partial \rho}(\rho^2\sin(\phi)F_\rho) + \frac{\partial}{\partial \theta}(\rho\mathbf{F}_\theta) + \frac{\partial}{\partial \phi}(\rho\sin(\phi)F_\phi)\right) \\
&= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \theta}(F_\theta) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(\sin(\phi)F_\phi) \\
\operatorname{curl}(\mathbf{F}) &= \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \rho\sin\phi\mathbf{e}_\phi \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_\rho & \rho F_\phi & \rho\sin\phi F_\theta \end{vmatrix} \\
\triangle f &= \frac{1}{\rho^2\sin\phi}\left(\frac{\partial}{\partial \rho}\left(\rho^2\sin\phi\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right)\right) \\
&= \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial f}{\partial \phi}
\end{aligned}$$