Spherical Geometry

Let $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ be the unit vectors in Cartesian coordinates in the x, y, and z directions and $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ be the unit vectors in the r, θ, and ϕ directions in spherical coordinates. Thus, an arbitrary vector $\mathbf{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = r\hat{\mathbf{r}} + \theta\hat{\theta} + \phi\hat{\phi}$. Figure 1 helps define the relationship between the components of a vector in the two coordinate systems:

$$x = r \sin\theta \cos\phi \quad (1)$$

$$y = r \sin\theta \sin\phi \quad (2)$$

$$z = r \cos\theta \quad (3)$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad (4)$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} \quad (5)$$

$$\phi = \tan^{-1} \frac{y}{x}. \quad (6)$$

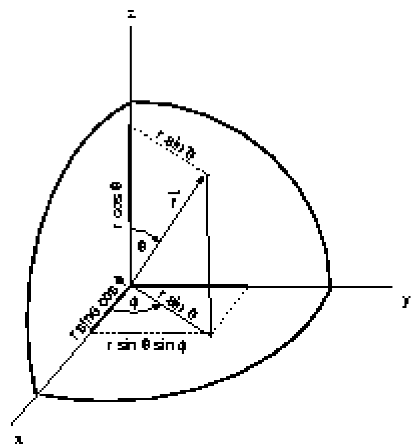

Figure 1: Relationship between vector components in spherical and Cartesian coordinates.

Figures 2 - 4 help with finding the relationship between the unit vectors. From Figure 2 it should be apparent that $\hat{\mathbf{r}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta$. From Figure 3 we see that $\hat{\theta} = \hat{\mathbf{x}} \cos\theta \cos\phi + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta$ and in Figure 4 we see that $\hat{\phi} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$:

$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta \quad (7)$$

$$\hat{\theta} = \hat{\mathbf{x}} \cos\theta \cos\phi + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta \quad (8)$$

$$\hat{\phi} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi \quad (9)$$

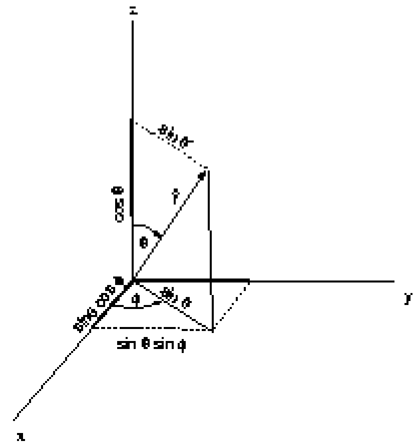

Figure 2: Drawing to illustrate Cartesian components of $\hat{\mathbf{r}}$.

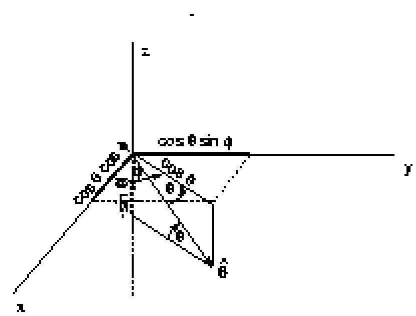

Figure 3: Drawing to illustrate Cartesian components of $\hat{\theta}$

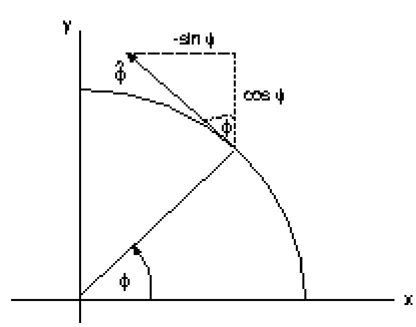

Figure 4: Drawing to illustrate Cartesian components of $\hat{\phi}$