Vector Derivative

The basic types of derivatives operating on a Vector Field are the Curl $\nabla\times$, Divergence $\nabla\cdot$, and Gradient ∇.

Vector derivative identities involving the Curl include

$$\nabla \times (k\mathbf{A}) = k\nabla \times \mathbf{A} \quad (1)$$

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) + (\nabla f) \times \mathbf{A} \quad (2)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \quad (3)$$

$$\nabla \times \left(\frac{\mathbf{A}}{f}\right) = \frac{f(\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla f)}{f^2} \quad (4)$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}. \quad (5)$$

In Spherical Coordinates,

$$\nabla \times \mathbf{r} = \mathbf{0} \quad (6)$$

$$\nabla \times \hat{\mathbf{r}} = \mathbf{0} \quad (7)$$

$$\nabla \times [\mathbf{r} f(r)] = f(r)(\nabla \times \mathbf{r}) + [\nabla f(r)] \times \mathbf{r} = f(r)(\mathbf{0}) + \frac{df}{dr}\hat{\mathbf{r}} \times \mathbf{r} = \mathbf{0} + \mathbf{0} = \mathbf{0}. \quad (8)$$

Vector derivative identities involving the Divergence include

$$\nabla \cdot (k\mathbf{A}) = k\nabla \cdot \mathbf{A} \quad (9)$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + (\nabla f) \cdot \mathbf{A} \quad (10)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (11)$$

$$\nabla \cdot \left(\frac{\mathbf{A}}{f}\right) = \frac{f(\nabla \cdot \mathbf{A}) - (\nabla f) \cdot \mathbf{A}}{f^2} \quad (12)$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \quad (13)$$

$$\nabla(\mathbf{uv}) = \mathbf{u}\nabla \cdot \mathbf{v} + (\nabla \mathbf{u}) \cdot \mathbf{v}. \quad (14)$$

In Spherical Coordinates,

$$\nabla \cdot \mathbf{r} = 3 \quad (15)$$

$$\nabla \cdot \hat{\mathbf{r}} = \frac{2}{r} \quad (16)$$

$$\nabla \cdot [\mathbf{r} f(r)] = \frac{\partial}{\partial x}[x f(r)] + \frac{\partial}{\partial y}[y f(r)] + \frac{\partial}{\partial z}[z f(r)] \quad (17)$$

$$\frac{\partial}{\partial x}[x f(r)] = x \frac{\partial f}{\partial x} + f = x \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + f \quad (18)$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} = x(x^2 + y^2 + z^2)^{-1/2} = \frac{x}{r} \quad (19)$$

$$\frac{\partial}{\partial x}[x f(r)] = \frac{x^2}{r} \frac{df}{dr} + f. \quad (20)$$

By symmetry,

$$\nabla \cdot [\mathbf{r} f(r)] = 3f(r) + \frac{1}{r}(x^2 + y^2 + z^2)\frac{df}{dr} = 3f(r) + r\frac{df}{dr} \quad (21)$$

$$\nabla \cdot (\hat{\mathbf{r}} f(r)) = \frac{3}{r} f(r) + \frac{df}{dr} \quad (22)$$

$$\nabla \cdot (\hat{\mathbf{r}} r^n) = 3r^{n-1} + (n-1)r^{n-1} = (n+2)r^{n-1}. \quad (23)$$

Vector derivative identities involving the Gradient include

$$\nabla(kf) = k\nabla f \quad (24)$$

$$\nabla(fg) = f\nabla g + g\nabla f \quad (25)$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \quad (26)$$

$$\nabla(\mathbf{A} \cdot \nabla f) = \mathbf{A} \times (\nabla \times \nabla f) + \nabla f \times (\nabla \times \mathbf{A}) + \mathbf{A} \cdot \nabla(\nabla f) + \nabla f \cdot \nabla \mathbf{A}$$

$$= \nabla f \times (\nabla \times \mathbf{A}) + \mathbf{A} \cdot \nabla(\nabla f) + \nabla f \cdot \nabla \mathbf{A} \quad (27)$$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \quad (28)$$

$$\nabla(f+g) = \nabla f + \nabla g \quad (29)$$

$$\nabla(\mathbf{A} \cdot \mathbf{A}) = 2\mathbf{A} \times (\nabla \times \mathbf{A}) + 2(\mathbf{A} \cdot \nabla)\mathbf{A} \quad (30)$$

$$(\mathbf{A} \cdot \nabla)\mathbf{A} = \nabla(\tfrac{1}{2}\mathbf{A}^2) - \mathbf{A} \times (\nabla \times \mathbf{A}). \quad (31)$$

Vector second derivative identities include

$$\nabla^2 t \equiv \nabla \cdot (\nabla t) = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \quad (32)$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}). \tag{33}$$

This very important second derivative is known as the Laplacian.

$$\nabla \times (\nabla t) = \mathbf{0} \tag{34}$$

$$\nabla(\nabla \cdot \mathbf{A}) = \nabla^2 \mathbf{A} + \nabla \times (\nabla \times \mathbf{A}) \tag{35}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{36}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\begin{aligned} \nabla \times (\nabla^2 \mathbf{A}) &= \nabla \times [\nabla(\nabla \cdot \mathbf{A})] - \nabla \times [\nabla \times (\nabla \times \mathbf{A})] \\ &= -\nabla \times [\nabla \times (\nabla \times \mathbf{A})] \\ &= -\{\nabla[\nabla \cdot (\nabla \times \mathbf{A})] - \nabla^2 (\nabla \times \mathbf{A})]\} \\ &= \nabla^2 (\nabla \times \mathbf{A}) \end{aligned} \tag{37}$$

$$\begin{aligned} \nabla^2 (\nabla \cdot \mathbf{A}) &= \nabla \cdot [\nabla(\nabla \cdot \mathbf{A})] \\ &= \nabla \cdot [\nabla^2 \mathbf{A} + \nabla \times (\nabla \times \mathbf{A})] = \nabla \cdot (\nabla^2 \mathbf{A}) \end{aligned} \tag{38}$$

$$\begin{aligned} \nabla^2 [\nabla \times (\nabla \times \mathbf{A})] &= \nabla^2 [\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}] \\ &= \nabla^2 [\nabla(\nabla \cdot \mathbf{A})] - \nabla^4 \mathbf{A} \end{aligned} \tag{39}$$

$$\nabla \times [\nabla^2 (\nabla \times \mathbf{A})] = \nabla^2 [\nabla(\nabla \cdot \mathbf{A})] - \nabla^4 \mathbf{A} \tag{40}$$

$$\begin{aligned} \nabla^4 \mathbf{A} &= -\nabla^2 [\nabla \times (\nabla \times \mathbf{A})] + \nabla^2 [\nabla(\nabla \cdot \mathbf{A})] \\ &= \nabla \times [\nabla^2 (\nabla \times \mathbf{A})] - \nabla^2 [\nabla \times (\nabla \times \mathbf{A})]. \end{aligned} \tag{41}$$

Combination identities include

$$\mathbf{A} \times (\nabla \mathbf{A}) = \tfrac{1}{2} \nabla (\mathbf{A} \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla) \mathbf{A} \tag{42}$$

$$\nabla \times (\phi \nabla \phi) = \phi \nabla \times (\nabla \phi) + (\nabla \phi) \times (\nabla \phi) = \mathbf{0} \tag{43}$$

$$(\mathbf{A} \cdot \nabla) \hat{\mathbf{r}} = \frac{\mathbf{A} - \hat{\mathbf{r}}(\mathbf{A} \cdot \hat{\mathbf{r}})}{r} \tag{44}$$

$$\nabla f \cdot \mathbf{A} = \nabla \cdot (f \mathbf{A}) - f(\nabla \cdot \mathbf{A}) \tag{45}$$

$$f(\nabla \cdot \mathbf{A}) = \nabla \cdot (f \mathbf{A}) - \mathbf{A} \nabla f, \tag{46}$$

where (45) and (46) follow from divergence rule (2).

See also Curl, Divergence, Gradient, Laplacian, Vector Integral, Vector Quadruple Product, Vector