GEFD SUMMER SCHOOL

Some basic dimensionless parameters and scales

The table summarizes (using standard notation as far as it exists) some of the basic quantities encountered in the lectures, whose order of magnitude is usually the first consideration when assessing a fluid-dynamical situation.

In fluid dynamics, even more than in other branches of physics, quantities like these have more than one meaning and more than one mode of use. For instance, the nominal length and velocity scales L, U may or may not be true scales in the sense that we can estimate the order of magnitude of $\partial/\partial x$ as L^{-1}, of $\nabla\mathbf{u}$ as U/L, of $\nu\nabla^2\mathbf{u}$ as $\nu U/L^2$, and so on. In a thin boundary layer with thickness ℓ and downstream lengthscale L, for instance, we might have $\mathbf{u}\cdot\nabla\mathbf{u} \sim U^2/L$ and $\nu\nabla^2\mathbf{u} \sim \nu U/\ell^2$. Then it is $U\ell^2/\nu L$ that needs to be of order unity, not the ordinary Reynolds number* UL/ν, if viscous forces are to balance typical accelerations. Again, it is often relevant to consider L and U to be *external* parameters (such as pipe radius and volume flux/pipe area in the Reynolds experiment), especially when interested in 'scaling up' or 'scaling down' in the sense of identifying the class of problems that reduce to the same problem when suitably nondimensionalized (e.g. half the pipe radius and twice the volume flux requires four times the viscosity, in order to get the same pipe-flow problem). In ordinary low-Mach-number flows, the timescale quite often $\sim L/U$, and this is often tacitly assumed when estimating typical material rates of change $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla \sim U/L$.

Dimensionless parameter	Symbol and formula	Interpretation†
Mach number	$M = \dfrac{U}{c_{\text{sound waves}}}$	$\dfrac{\text{typical advective acceleration}}{\text{wave-induced particle accel.}}$
External **Froude number**	$Fr_{\text{e}} = \dfrac{U}{c_{\text{ext. gravity waves}}} = \dfrac{U}{(gH)^{1/2}}$	"
Internal **Froude number**	$Fr_{\text{i}} = \dfrac{U}{c_{\text{int. gravity waves}}} = \dfrac{U}{NH}$	"

Small values of $M,\ Fr_{\text{e}},\ Fr_{\text{i}}$ imply the possibility of 'balanced' or 'adjusted' flows that do not self-excite the waves in question (cf. mass on stiff spring moved gently), e.g. nearly-incompressible flow when M is small enough, or layerwise-2D stratified flow (e.g. 'Los Angeles smog') when Fr_{i} is small enough. (Beware: governing equations are 'stiff'.) N = buoyancy frequency of stable stratification: $N^2 = g\,\partial\ln(\text{potential density})/\partial z$.

Reynolds number	$R = Re = UL/\nu$	$\dfrac{\text{typical advective acceleration}}{\text{typical viscous force/mass}}$
Boundary-layer thickness (for flow past obstacle)	$\left(\dfrac{\nu L}{U}\right)^{1/2} = Re^{-1/2}L$	= diffusion length for time L/U

Kolmogorov microscale $\quad \ell_{\text{K}} = (\nu^3/\epsilon)^{1/4}$

(Nominal length scale at which viscous dissipation becomes important in three-dimensional turbulence that is dissipating energy at rate ϵ per unit mass; ϵ has dimensions length^2 time^{-3}.)

Associated velocity and time scales $U_{\text{K}} \sim (\nu\epsilon)^{1/4}$, $\quad t_{\text{K}} \sim (\nu/\epsilon)^{1/2}$

[*Consistency checks:* $U_{\text{K}}\ell_{\text{K}}/\nu \sim 1$, and $t_{\text{K}} \sim \ell_{\text{K}}^2/\nu$ (viscous diffusion time)]

* not Reynold's. After Osborne Reynolds who in a famous experiment (MEM, Lecture 1) showed its relevance to whether pipe flow is laminar or turbulent.

† when L, U etc are true scales in a flow with simple structure

Dimensionless parameter	Symbol and formula	Interpretation†
(gradient) **Richardson number**	$Ri = N^2/(U_z)^2$ ($= {Fr_\mathrm{i}}^{-2}$ if $U_z = U/H$)	As for ${Fr_\mathrm{i}}^{-2}$; U_z = vertical shear. (Note that U_zH *is*, quite often, the relevant velocity scale.)
Péclet number	$Pe = UL/\kappa$ (κ = heat diffusivity)	$\dfrac{\text{typ. advective rate of change of temp.}}{\text{typ. diffusive rate of change of temp.}}$
Prandtl number	ν/κ	$\dfrac{\text{momentum diffusivity}}{\text{heat diffusivity}}$
Schmidt number	ν/κ_s	$\dfrac{\text{momentum diffusivity}}{\text{solute diffusivity}}$

Rossby number (**Kibel' number** in Soviet literature)	$Ro = U/\Omega L \sim U/2\Omega L$ $\sim U/c_{\text{inertia waves}}$	$\dfrac{\text{typical relative advective accel.}}{\text{typical Coriolis accel.}}$ ($Ro \ll 1 \Rightarrow$ rotationally stiff)
Ekman number	$E = \nu/\Omega L^2$	$\dfrac{\text{typical viscous force/mass}}{\text{typical Coriolis accel.}}$

(For relevance to spindown time $\Omega^{-1}E^{-1/2}$, L needs to be a scale in the Ω direction)

Ekman-layer thickness scale	$(\nu/\Omega)^{1/2}$	Diffusion length for time Ω^{-1}
Prandtl's ratio of scales	$H/L \sim f/N$ $f = 2\Omega \sin(\text{latitude})$	

(Natural vertical-to-horizontal aspect ratio for stratified, rotating flow at low Fr_i and Ro)

Associated quantities:

Rossby length	$L \sim NH/f$	(also 'Rossby radius'; no standard symbol)
Rossby height	$H \sim fL/N$	(no standard symbol)
Burger number	$Bu = N^2H^2/f^2L^2$	(sometimes defined the other way up)

(H and L are vertical and horizontal length scales; $H \ll L$ in atmosphere and ocean)

Rayleigh number $\quad Ra = \dfrac{g'H^3}{\nu k} \qquad (g' = g\dfrac{\Delta\rho}{\rho})$

Ra is the product $RePe$ for vigorous thermal convection (assuming that the velocity scale U is such that vertical advective acceleration $U^2/H \sim$ buoyancy acceleration $\sim g'$).

Nusselt number $\quad \dfrac{\text{total vertical heat or buoyancy flux in a thermally convecting layer}}{\text{conductive heat or buoyancy flux if convection suppressed}}$

Flux Richardson number $\quad Ri_\mathrm{f} = \dfrac{\text{vertical eddy buoyancy flux}}{U_z \times \text{eddy momentum flux}}$

Ri_f arises from the turbulent energy equation for stratified shear flows. It compares the rate at which eddies do work against gravity (in reducing the stable stratification) with the rate at which they acquire energy from the mean shear U_z.

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† when L, U etc are true scales in a flow with simple structure