Potter & Foss

for a longer time, resulting in a lower velocity. In a turbulent flow large-amplitude fluctuations do occur so that this phenomenon is often observed.

A second source of the time variation of p_0 is the integral quantity. The fluctuations in V are a result of the turbulent eddy structures in the flow, and these structures have characteristic length $(s_2 - s_1)$; this leads to a nonzero value for the integral.

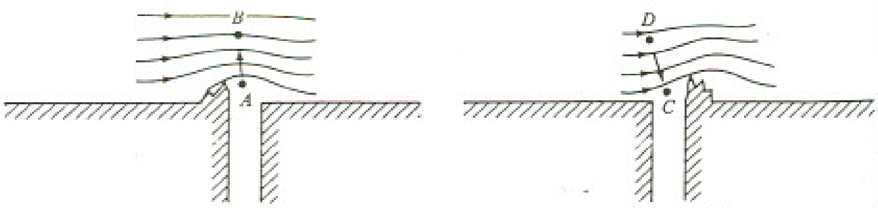
Example 8.2

Although the measurement of the pressure at a solid boundary is one of the most reliable measurements in fluid mechanics, it is subject to errors resulting from the presence of burrs at the lip of the hole. Explain the influence of a burr on the upstream and the downstream edges of a hole used for a static tap measurement.

Solution. The presence of a burr will cause the fluid to deflect vertically upward with a maximum point of the arch slightly downstream of the burr. This will cause a low pressure region in the separated wake behind the burr and a high pressure region in front of the burr. A characteristic streamline pattern is shown by the sketch. In Fig. E8.2(a) the streamline curvature indicates an outward-pointing normal vector so that from Eq. 8.11, for steady flow,

$$\frac{\partial p}{\partial n} = \rho \frac{V^2}{R} \quad \text{or} \quad \Delta p \cong \rho \frac{V^2}{R} \Delta n$$

From this we note that $p_A < p_B$. The reading would be low. From Fig. E8.2(b) the streamline curvature indicates a normal vector pointing toward point C so that $p_C > p_D$. A high reading would be recorded.



(a) (b)

Fig. E8.2

8.3 EQUATIONS OF POTENTIAL FLOW

Initially, nonvortical or irrotational fluid may become rotational under the direct action of viscous diffusion or noninertial acceleration effects. Consequently, there are entire flows, which are driven by

pressure or gravitational forces
irrotational, that is, a flow in wh
or deform but not rotate. In ge
where the viscous effects caus
approximated by an irrotational f
tion for identifying a flow as irro

$\nabla \times$

This means that the velocity field
by the gradient of a *scalar poten*

V

Note that the *vector* velocity is ob
function ϕ. In scalar form, Eq. 8.

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v

w

The continuity equation for an in

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x}$$

Using Eqs. 8.16 the continuity
potential, becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial}{\partial}$$

which is *Laplace's equation*.

The momentum equation (3.52
negligible in the inviscid flow, an

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right.$$

where h is the vertical dimension

velocity. In a turbulent flow large-
his phenomenon is often observed.
is the integral quantity. The
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the separated wake behind the burr
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$$\Delta p \simeq \rho \frac{V^2}{R} \Delta n$$

ling would be low. From Fig. E8.2(b)
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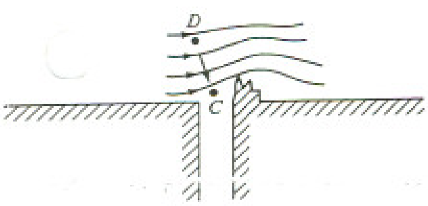

(b)

8.2

V

nal fluid may become rotational
iffusion or noninertial acceleration
ntire flows, which are driven by

pressure or gravitational forces, in which the bulk of the flow is irrotational, that is, a flow in which each fluid element may accelerate or deform but not rotate. In general, the fluid near a solid boundary, where the viscous effects cause a no-slip condition, will not be approximated by an irrotational flow. A necessary and sufficient condition for identifying a flow as irrotational is

$$\nabla \times \mathbf{V} = 0 \tag{8.14}$$

This means that the velocity field $\mathbf{V}$ is a conservative vector field given by the gradient of a *scalar potential function* ϕ (see Art. 1.3); that is,

$$\mathbf{V} = \nabla\phi \tag{8.15}$$

Note that the *vector* velocity is obtained from a knowledge of the *scalar* function ϕ. In scalar form, Eq. 8.15 includes the three equations

$$u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y} \qquad w = \frac{\partial\phi}{\partial z} \tag{8.16}$$

The continuity equation for an incompressible flow is

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8.17}$$

Using Eqs. 8.16 the continuity equation, in terms of the velocity potential, becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{8.18}$$

which is *Laplace's equation.*

The momentum equation (3.52), without the viscous term, which is negligible in the inviscid flow, and with the use of Eq. (1.61) is

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{V^2}{2}\right) = -\frac{\nabla p}{\rho} - g\nabla h \tag{8.19}$$

where h is the vertical dimension and we have used $\nabla \times \mathbf{V} = 0$. Using

$\mathbf{V} = \nabla\phi$, this becomes

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{V^2}{2} + \frac{p}{\rho} + gh\right] = 0 \tag{8.20}$$

at every point in the potential flow. This means that

$$\frac{\partial\phi}{\partial t} + \frac{V^2}{2} + \frac{p}{\rho} + gh = \text{const} \tag{8.21}$$

For steady flows the Bernoulli equation results,

$$\frac{V^2}{2} + \frac{p}{\rho} + gh = \text{const} \tag{8.22}$$

This equation is valid everywhere, not just along a streamline.

To solve for a potential flow around a body we must first determine ϕ such that Laplace's equation (8.18) is satisfied, then find the velocity field from Eqs. 8.16 and the pressure field from Eq. 8.22. The pressure field can then be integrated over the area of interest to give a force. This would be the technique followed to determine the lift on an airfoil.

The immediate problem is the determination of the potential function ϕ for a particular problem of interest. The general three-dimensional problem will not be studied in this text because conventional methods are restricted to either plane, two-dimensional flows or axisymmetric flows. For both of these special classes of flows it is possible to define a function $\psi(x, y)$ called a *stream function* which is constant along a streamline. Since the flow is tangential to a solid surface, ψ is constant along a body. For the two-dimensional plane problem, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8.23}$$

If we define

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x} \tag{8.24}$$

then continuity is automatically satisfied for an inviscid or a viscous flow. Using the equation for a streamline (Eq. 1.50) with $\mathbf{V} = u\,\hat{i} + v\,\hat{j}$ and $d\mathbf{R} = dx\,\hat{i} + dy\,\hat{j}$, we see that $v\,dx - u\,dy = 0$ along a streamline. This is exactly $d\psi = \dfrac{\partial\psi}{\partial x}\,dx + \dfrac{\partial\psi}{\partial y}\,dy = 0$, using Eqs. 8.24. Hence, we conclude that ψ is constant along a streamline. For a plane irrotational flow the first two components of vorticity, ξ and η, are identically zero; the third component gives

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

or

Consequently, the stream fur an irrotational flow; therefore the velocity can be obtained 8.22.

Two techniques may be er tion ϕ or the stream function Laplace's equation ... the either a numerical tec... method. The second and ofte simple functions that satisfy pose these simple functions, equation is linear, to provide second method will be empha procedure for potential-flow

It should be emphasized th ... either of these functions and pressure distribution.

A note on boundary con Laplace's equation is second- on the complete boundary enc is, the stream function (or the be known over the *entire* su shown in Fig. 8.6. The dotte surrounding the region of in from the body would be

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$$\frac{\partial\psi}{\partial x} \qquad (8.24)$$

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$= 0$, using Eqs. 8.24. Hence, we

reamline. For a plane irrotational
city, ξ and η, are identically zero;

$$\frac{\partial u}{\partial y} = 0$$

or

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad (8.25)$$

Consequently, the stream function also satisfies Laplace's equation in an irrotational flow; therefore, if we can determine the stream function, the velocity can be obtained from Eqs. 8.24 and the pressure from Eq. 8.22.

Two techniques may be employed to determine the potential function ϕ or the stream function ψ. The first technique is to solve directly Laplace's equation with the appropriate boundary conditions, using either a numerical technique or, possibly, the separation of variables method. The second and often utilized technique is to investigate some simple functions that satisfy Laplace's equation and then to superimpose these simple functions, which is allowable because Laplace's equation is linear, to provide the flow around the body of interest. This second method will be emphasized since it is the most commonly used procedure for potential-flow considerations.

It should be emphasized that we need only determine the ψ function or the ϕ function to within a constant, since a constant can be added to either of these functions and it will not effect the velocity field or the pressure distribution.

A note on boundary conditions for potential flows is in order. Laplace's equation is second-order and requires boundary conditions on the complete boundary enclosing a particular region of interest; that is, the stream function (or the velocity potential) or its derivative must be known over the *entire* surface. Consider a flow around a body shown in Fig. 8.6. The dotted surface *and* the body form the surface surrounding the region of interest. The condition at large distances from the body would be

$$u = U,\; v = 0 \qquad \text{or} \qquad \psi = Uy \qquad (8.26)$$

The no-slip condition is no longer required on the body's surface since the effects of viscosity are neglected. Hence, we need not require the tangential component on the surface to be zero. The body is a streamline and along a streamline the stream function is constant. Thus, on the body we can choose the constant to be zero (it is arbitrary) so that $\psi = 0$ on the body and the stream function is specified on the entire surface. The condition at the body for the velocity potential is more difficult to specify so the stream function is generally used.

Another observation for plane irrotational flows is that

$$\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}, \qquad \frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} \qquad (8.27)$$

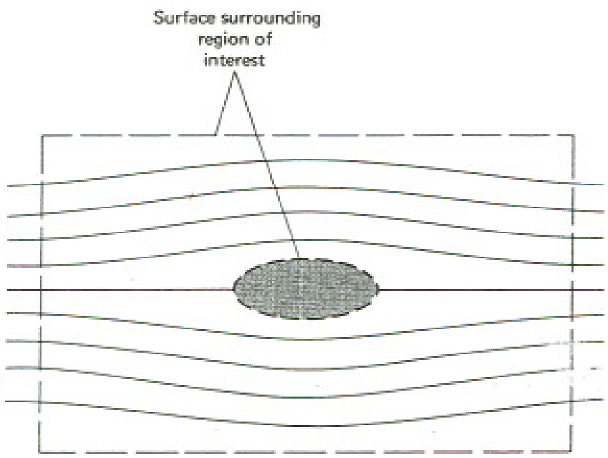


Fig. 8.6. Flow around a body.

These follow from Eqs. 8.16 and 8.24. They are the famous *Cauchy-Riemann equations* and enable us to use the theory of complex variables in our two-dimensional, plane problems. The functions ϕ and ψ are *harmonic functions* and form an *analytic complex function* ($\phi + i\psi$) called the *complex velocity potential*. Conformal transformations, along with all the complex variable theory, can thus be used for this class of problems. We will not use complex variables in this text but it is interesting to note the restricted class of problems for which it is useful in fluid mechanics, namely, plane, incompressible, irrotational flows.

Example 8.3

Show that in a two-dimensional incompressible flow the difference in the stream function between any two streamlines represents the flow rate per unit of depth between the two streamlines.

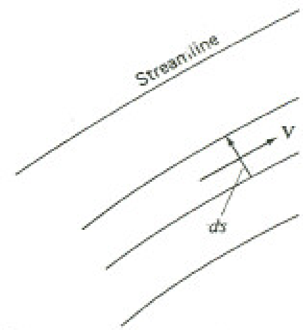
Solution. The infinitesimal flow rate per foot of depth flowing past the elemental distance dl (Fig. E8.3) is

$$dq = u\,dy - v\,dx$$

where the negative sign results since to go from ψ to $\psi + d\psi$ we must move in the negative x-direction. Substituting for u and v from eqs. (8.24) gives

$$dq = \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx$$

$$= d\psi$$



If we integrate this from streamlir

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Example 8.4

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flow irrotational? If so, calculate t

Solution. The velocity compon

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and

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$\xi = 0, \quad \eta =$

The flow is irrotational, since all
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The velocity potential is found a

$u =$

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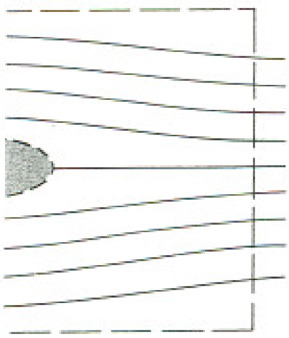
$\phi =$

Differentiating the above with res

$$\frac{\partial\phi}{\partial y}$$



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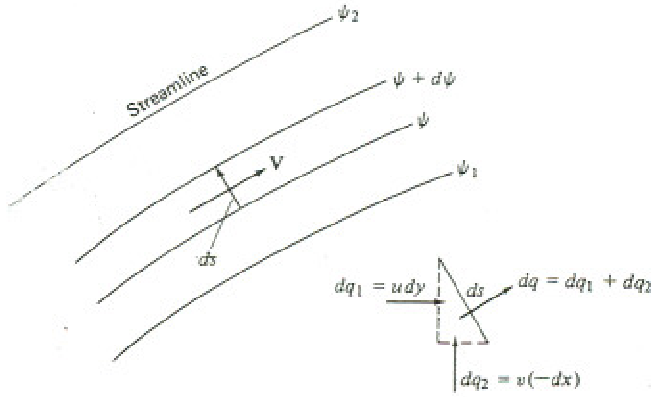

Fig. E8.3

If we integrate this from streamline (1) to streamline (2) we obtain

$$q = \psi_2 - \psi_1$$

for the incompressible, two-dimensional flow.

Example 8.4

The streamfunction for a particular flow is given as $\psi(x, y) = x^2 - y^2$. Is this flow irrotational? If so, calculate the velocity potential.

Solution. The velocity components are

$$u = \frac{\partial \psi}{\partial y} = -2y$$

and

$$v = -\frac{\partial \psi}{\partial x} = -2x$$

The vorticity components are then

$$\xi = 0, \qquad \eta = 0, \qquad \zeta = -2 + 2 = 0$$

The flow is irrotational, since all vorticity components are zero. A particle would not rotate, it would only deform.

The velocity potential is found as follows. From the first equation,

$$u = \frac{\partial \phi}{\partial x} = -2y$$

so that

$$\phi = -2xy + f(y)$$

Differentiating the above with respect to y gives

$$\frac{\partial \phi}{\partial y} = -2x + \frac{\partial f}{\partial y}$$

Equating this to $v = -2x$ gives $f = C$ where C is a constant. The velocity potential is then

$$\phi = -2xy + C$$

The constant C is not important since it does not affect the velocity or pressure fields. Hence, it is often set equal to zero.

Example 8.5

Show that the potential lines and the streamlines for a two-dimensional, incompressible, inviscid flow intersect one another at right angles, with the result that a curvilinear grid is formed.

Solution. Two contours intersect at right angles if their slopes form negative reciprocals at each point in the flow field. The local slope of a constant-ψ line can be expressed in terms of the ratio of the velocity components; that is

$$\text{Slope of constant-}\psi\text{ line} = v/u$$

Along a line of constant ϕ, $d\phi = 0$, so that

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = 0$$

Hence, the slope $\dfrac{dy}{dx}$ of a constant-ϕ line is given as

$$\left.\frac{dy}{dx}\right|_{\phi=\text{const}} = \frac{-\partial\phi/\partial x}{\partial\phi/\partial y} = -u/v$$

This slope of a constant-ϕ line is seen to be the negative reciprocal of the slope of a constant-ψ line. Hence, the two lines are orthogonal everywhere their slope is defined. This feature of the ϕ- and ψ-lines results in the formation of an "orthogonal grid" of curvilinear squares. Such a grid is shown in Fig. E8.5.

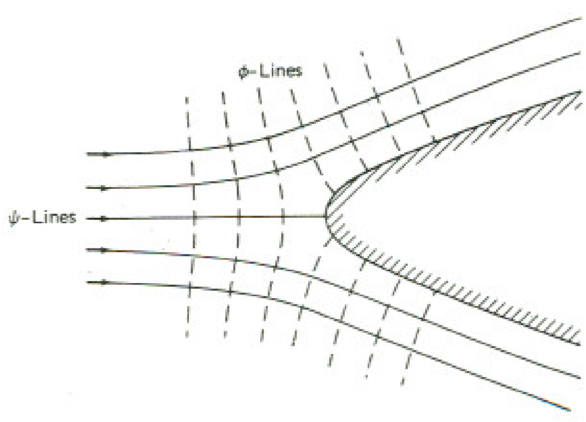

Fig. E8.5

Example 8.6

Write the viscous term for a N in terms of the vorticity and sho effects. Then show that for a pote

Solution. The viscous term fo $\mu\nabla^2\mathbf{V}$. We can use a vector identi

$$\nabla \times (\nabla \times \mathbf{V}$$

This can be verified by expanding
The quantity $\nabla \cdot \mathbf{V}$ is zero beca

$$\nabla^2\mathbf{V} =$$

Thus, the viscous term can altern term is not zero then the vorticit effects are non-zero only in regio not necessarily, true; that is if non-zero (unless $\Delta \times \omega \simeq 0$). Usi

we see that if ω is everywhere zer

Extension 8.6.1. Assume that around a body. Viscous effects surrounding the body. Does the Navier-Stokes equations for the f

Extension 8.6.2. Identify wheth would or would not be adequate exiting from a bellows, the flow airplane, the flow which escapes f

8.4 SOME SIMPLE PLANE POTEN

We will now investigate som Laplace's equation, $\nabla^2\psi = 0$. represents a potential flow. W engineer depends on the stre constant; that is, whether such of interest. Some functions wh are considered in the following where C is a constant. The velocity

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$\frac{\partial\phi}{\partial y}\,dy = 0$

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$\frac{\phi/\partial x}{/\partial y} = -u/v$

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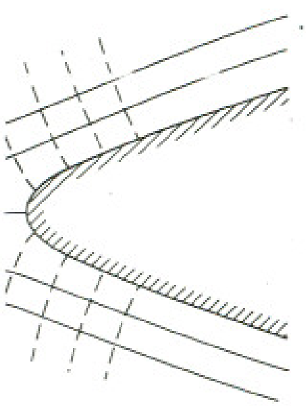


3.5

Example 8.6

Write the viscous term for a Newtonian incompressible, homogeneous flow in terms of the vorticity and show that vorticity always accompanies viscous effects. Then show that for a potential flow the viscous term vanishes.

Solution. The viscous term for an incompressible, homogeneous fluid is $\mu\,\nabla^2\mathbf{V}$. We can use a vector identity which states that

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - (\nabla \cdot \nabla)\mathbf{V}$$

This can be verified by expanding both sides in cartesian coordinates.

The quantity $\nabla \cdot \mathbf{V}$ is zero because of continuity; hence,

$$\nabla^2\mathbf{V} = -\nabla \times (\nabla \times \mathbf{V})$$

Thus, the viscous term can alternately be written as $-\mu\Delta \times \omega$, so that if this term is not zero then the vorticity cannot be zero. We see then, that viscous effects are non-zero only in regions of vorticity. The converse is usually, but not necessarily, true; that is if vorticity is non-zero the viscous term is non-zero (unless $\Delta \times \omega \cong 0$). Using

$$\mu\,\nabla^2\mathbf{V} = -\mu\nabla \times \omega$$

we see that if ω is everywhere zero then the viscous term vanishes.

Extension 8.6.1. Assume that a potential-flow solution exists for flow around a body. Viscous effects are confined to a thin boundary layer surrounding the body. Does the potential flow solution satisfy the complete Navier-Stokes equations for the flow external to the boundary layer?

Ans. Yes

Extension 8.6.2. Identify whether the bulk flow in the following problems would or would not be adequately discribed as a potential flow: the flow exiting from a bellows, the flow inside a journal bearing, the flow over an airplane, the flow which escapes from a closet as the door is closed.

8.4 SOME SIMPLE PLANE POTENTIAL FLOWS

We will now investigate some rather simple functions which satisfy Laplace's equation, $\nabla^2\psi = 0$. Any function satisfying this equation represents a potential flow. Whether it is of particular interest to the engineer depends on the streamline pattern represented by $\psi(x, y)$ constant; that is, whether such a function includes a form of an object of interest. Some functions which give streamline patterns of interest are considered in the following sections of this article.

1. Uniform flow

Since Laplace's equation is second-order, a first-order dependence of ψ on x and y represents a possible stream function. Specifically,

$$\psi = Ax + By \tag{8.28}$$

The velocity components are

$$u = \frac{\partial \psi}{\partial y} = B \tag{8.29}$$

and

$$v = -\frac{\partial \psi}{\partial x} = -A \tag{8.30}$$

This represents a uniform flow as shown in Fig. 8.7. If $A = 0$, the uniform flow is only in the x-direction.

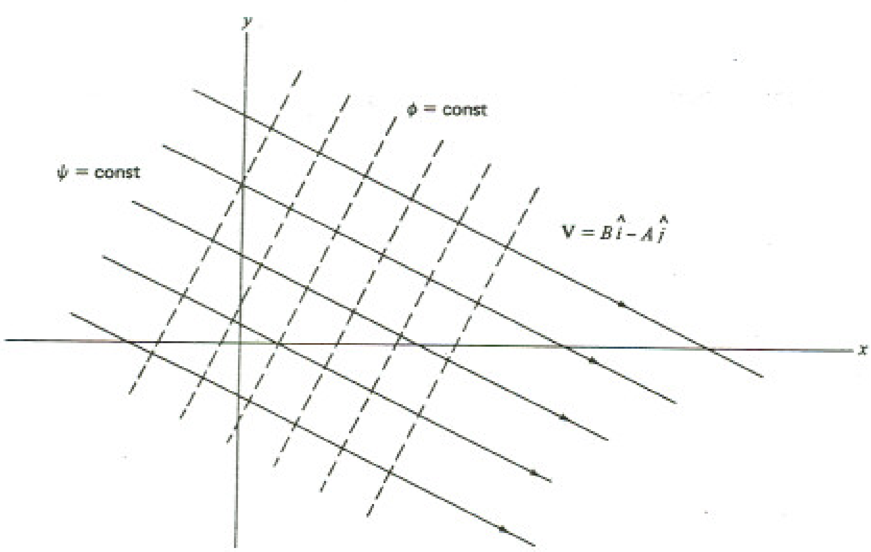


Fig. 8.7. Uniform flow.

The velocity potential for this uniform flow can be shown to be

$$\phi = Bx - Ay \tag{8.31}$$

2. Stagnation Flow

Another simple function which satisfies Laplace's equation is

$$\psi = Axy \tag{8.32}$$

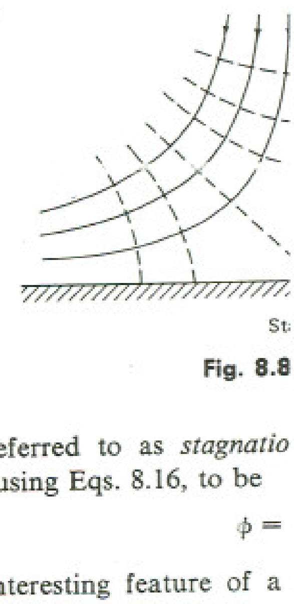
The velocity components are

$u =$

and

$v = -$

The streamlines represented b
Since any streamline can be
consider this to be flow in a co
Fig. 8.8. Both velocity compon



St

Fig. 8.8

often referred to as *stagnatio*
found, using Eqs. 8.16, to be

$\phi =$

An interesting feature of a
passing through the stagnation
flow proceeds in one direction
airfoil the dividing streamline s
top of the airfoil from that whi

3. Sources and Sinks

For many applications it
ordinates, shown as r and θ in
coordinates, is

$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r}$

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+ By (8.28)

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= −A (8.30)

shown in Fig. 8.7. If $A = 0$, the
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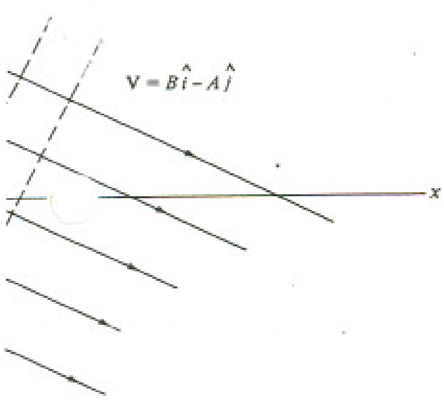

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x − Ay (8.31)

satisfies Laplace's equation is

Axy (8.32)

The velocity components are

$$u = \frac{\partial \psi}{\partial y} = Ax \tag{8.33}$$

and

$$v = -\frac{\partial \psi}{\partial x} = -Ay \tag{8.34}$$

The streamlines represented by $\psi = 0$ are the x-axis and the y-axis. Since any streamline can be replaced by a solid boundary we can consider this to be flow in a corner or flow against a wall, as shown in Fig. 8.8. Both velocity components are zero at the origin; hence this is

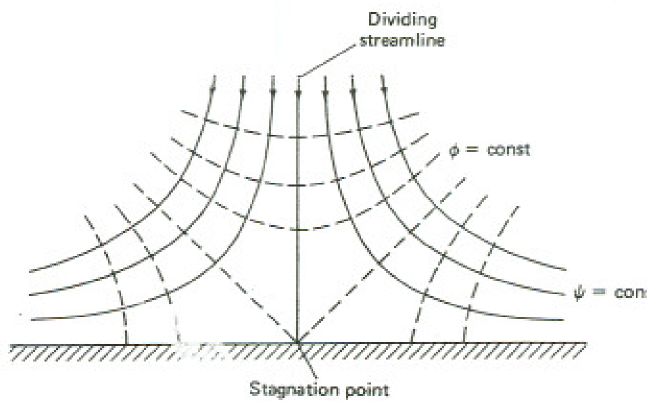

Fig. 8.8 Stagnation flow.

often referred to as *stagnation flow*. The velocity potential can be found, using Eqs. 8.16, to be

$$\phi = \frac{A}{2}(x^2 - y^2) \tag{8.35}$$

An interesting feature of a stagnation flow is that the streamline passing through the stagnation point divides the flow so that part of the flow proceeds in one direction and part in another. For flow around an airfoil the dividing streamline separates the flow that proceeds over the top of the airfoil from that which travels underneath the airfoil.

3. Sources and Sinks

For many applications it is more convenient to use polar coordinates, shown as r and θ in Fig. 8.9. Laplace's equation, in polar coordinates, is

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{8.36}$$

$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \qquad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$

The velocity components are

$$u = v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v = v_\theta = -\frac{\partial \psi}{\partial r} \tag{8.37}$$

which follow from the continuity equation,

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{8.38}$$

Consider the simple harmonic function

$$\psi = A\theta \tag{8.39}$$

Using Eqs. 8.37, the velocity components are

$$v_r = \frac{A}{r} \qquad v_\theta = 0 \tag{8.40}$$

Since v_θ is everywhere zero, the streamline pattern must be represented by radial lines emanating from the origin, as shown in Fig. 8.9. If A is positive a source is represented; if A is negative, a sink results.

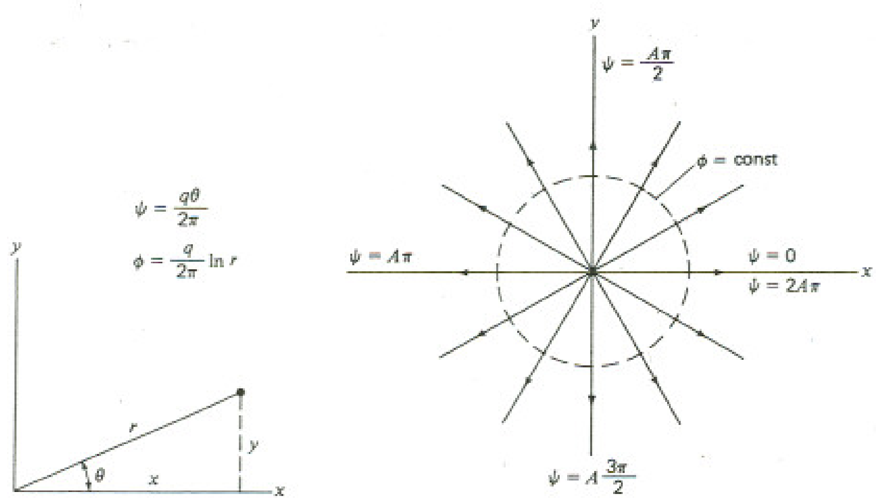

Fig. 8.9. Source flow.

The velocity at a particular r is constant for all θ. Hence we can integrate around a circle enclosing the origin to obtain the flow rate q as

$$q = \int_0^{2\pi} v_r r \, d\theta$$
$$= 2\pi A \tag{8.41}$$

In terms of the *source strength*

where q is measured in ft^3/se

In cartesian coordinates th

ψ

and the velocity components

$u = \frac{Ax}{x^2 +}$

The associated velocity poten

$\phi =$

in cartesian coordinates. In p

4. An Irrotational Vor

Another simple function o line pattern, is

This satisfies Laplace's equa $r = 0$. Hence the flow must b at the origin. The velocity co

$v_r =$

and obviously represent circu Fig. 8.10. The velocity increa is a good example of this typ

The *circulation* Γ is definec

For the specific case of the

$$v_\theta \quad - \frac{\partial \psi}{\partial r} \tag{8.37}$$

uation,

$$\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{8.38}$$

on

$$\theta \tag{8.39}$$

ients are

$$v_\theta = 0 \tag{8.40}$$

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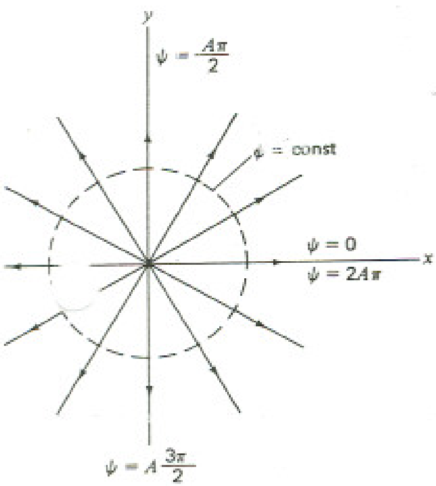
is negative, a sink results.



ource flow.

constant for all θ. Hence we can
the origin to obtain the flow rate q

$$\int^{\pi} v_r r \, d\theta$$

$$A \tag{8.41}$$

In terms of the *source strength* q, the streamfunction is

$$\psi = \frac{q}{2\pi}\theta \tag{8.42}$$

where q is measured in $ft^3/sec/ft$ of depth.

In cartesian coordinates the streamfunction is

$$\psi = A \tan^{-1}\frac{y}{x} \tag{8.43}$$

and the velocity components are

$$u = \frac{Ax}{x^2 + y^2}, \qquad v = \frac{Ay}{x^2 + y^2} \tag{8.44}$$

The associated velocity potential would be

$$\phi = A \ln \sqrt{x^2 + y^2} \tag{8.45}$$

in cartesian coordinates. In polar coordinates, ϕ would be

$$\phi = A \ln r \tag{8.46}$$

4. An Irrotational Vortex

Another simple function of interest, because of its resulting streamline pattern, is

$$\psi = A \ln r \tag{8.47}$$

This satisfies Laplace's equation everywhere but at the origin, where $r = 0$. Hence the flow must be irrotational everywhere, except possibly at the origin. The velocity components are

$$v_r = 0, \qquad v_\theta = -\frac{A}{r} \tag{8.48}$$

and obviously represent circular streamlines about the origin, shown in Fig. 8.10. The velocity increases as the origin is approached; a tornado is a good example of this type of motion.

The *circulation* Γ is defined as (counterclockwise is positive)

$$\Gamma = \oint \mathbf{V} \cdot d\mathbf{s} \tag{8.49}$$

For the specific case of the irrotational vortex and for the contour

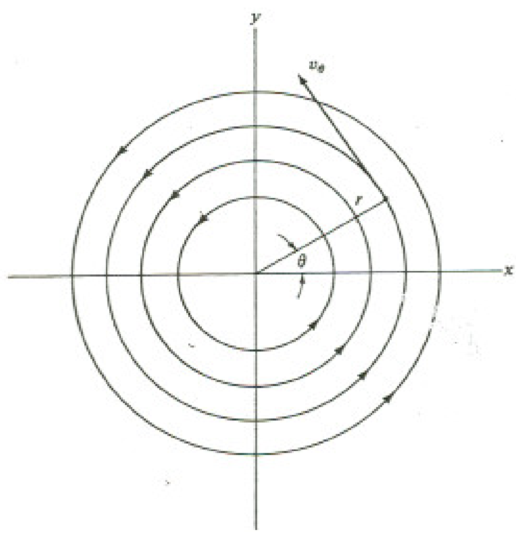


Fig. 8.10. An irrotational vortex.

formed by a circle around the origin, Γ may be expressed as

$$\Gamma = \int_0^{2\pi} v_\theta (r\, d\theta) = -2\pi A \tag{8.50}$$

The stream function, in terms of the *vortex strength* Γ is

$$\psi = -\frac{\Gamma}{2\pi} \ln r \tag{8.51}$$

The reason why circulation exists in an irrotational flow is that we have integrated around a singularity. If any path had been chosen which did not enclose the origin, Γ would have been zero. At the origin there exists an infinite vorticity, with the vorticity zero everywhere else.

In cartesian coordinates the streamfunction is

$$\psi = A \ln \sqrt{x^2 + y^2} \tag{8.52}$$

and the velocity components are

$$u = \frac{Ay}{x^2 + y^2} \qquad v = -\frac{Ax}{x^2 + y^2} \tag{8.53}$$

The corresponding potential function would be

$$\phi = -A \tan^{-1} \frac{y}{x} \tag{8.54}$$

or

5. A Doublet

We can create another sim
equation by the method of supe
a sink at $x = +\epsilon$, where ϵ is a s
function is

$\psi = A \tan^{-1}$

Remembering that

$\partial f/\partial x = \lim_{\epsilon \to 0}$

we may put Eq. 8.56 in the for

$\psi = 2\epsilon$

$= 2\epsilon$

where $\epsilon \to 0$ and $A \to \infty$ so t
flow is called a *doublet*. Defini
the streamfunction is

$\psi =$

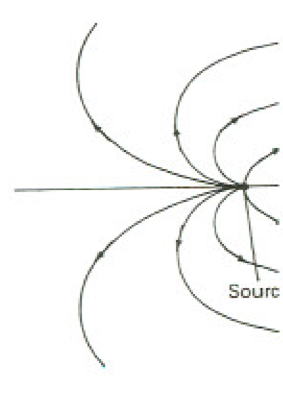
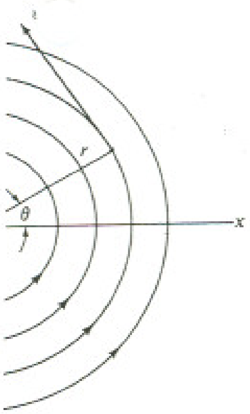


Fig. 8.11. Superposition of a so



ational vortex.

Γ may be expressed as

$) = -2\pi A$ (8.50)

vortex strength Γ is

ln r (8.51)

n irrotational flow is that we have
/ path had been chosen which did
e been zero. At the origin there
orticity zero everywhere else.
nfunction is

$^2 + y^2$ (8.52)

$= -\dfrac{Ax}{x^2 + y^2}$ (8.53)

would be

$^{-1}\dfrac{y}{x}$ (8.54)

or

$$\phi = -A\theta \tag{8.55}$$

5. A Doublet

We can create another simple function which satisfies Laplace's equation by the method of superposition. Place a source at $x = -\epsilon$ and a sink at $x = +\epsilon$, where ϵ is a small quantity (see Fig. 8.11). The stream function is

$$\psi = A\tan^{-1}\frac{y}{x+\epsilon} - A\tan^{-1}\frac{y}{x-\epsilon} \tag{8.56}$$

Remembering that

$$\partial f/\partial x = \lim_{\epsilon\to 0}\frac{f(x+\epsilon, y) - f(x-\epsilon, y)}{2\epsilon}$$

we may put Eq. 8.56 in the form

$$\psi = 2\epsilon A\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right)$$

$$= 2\epsilon A\left[-\frac{y}{x^2+y^2}\right] \tag{8.57}$$

where $\epsilon \to 0$ and $A \to \infty$ so that ϵA remains constant. The resultant flow is called a *doublet*. Defining the *doublet strength* μ to be $\mu = 2\epsilon A$, the streamfunction is

$$\psi = -\frac{\mu y}{x^2+y^2} \tag{8.58}$$

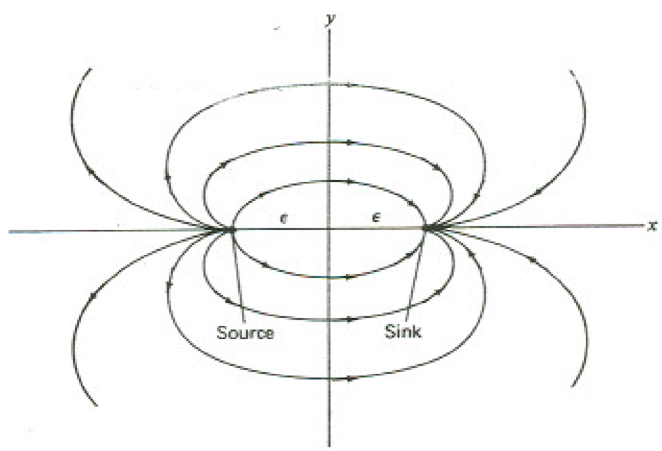


Fig. 8.11. Superposition of a source and sink. If $\epsilon = 0$, a doublet is formed.

This doublet is oriented in the negative x-direction. It could have been directed differently if the original source and sink had been placed along a different line; however, it is most common to orient the doublet as represented in Eq. 8.58.

In polar coordinates, the streamfunction is

$$\psi = -\frac{\mu \sin \theta}{r} \tag{8.59}$$

with the velocity components

$$v_r = -\frac{\mu \cos \theta}{r^2} \qquad v_\theta = -\frac{\mu \sin \theta}{r^2} \tag{8.60}$$

The cartesian velocity components are

$$u = -\frac{\mu(x^2 - y^2)}{(x^2 + y^2)^2} \qquad v = -\frac{2\mu xy}{(x^2 + y^2)^2} \tag{8.61}$$

The corresponding velocity potential for the doublet is

$$\phi = \frac{\mu x}{x^2 + y^2} \tag{8.62}$$

or, in polar coordinates,

$$\phi = \frac{\mu \cos \theta}{r} \tag{8.63}$$

The resulting doublet flow is shown in Fig. 8.12 with both streamlines and potential lines shown. For ϕ and ψ constant, Eqs. 8.58 and 8.62

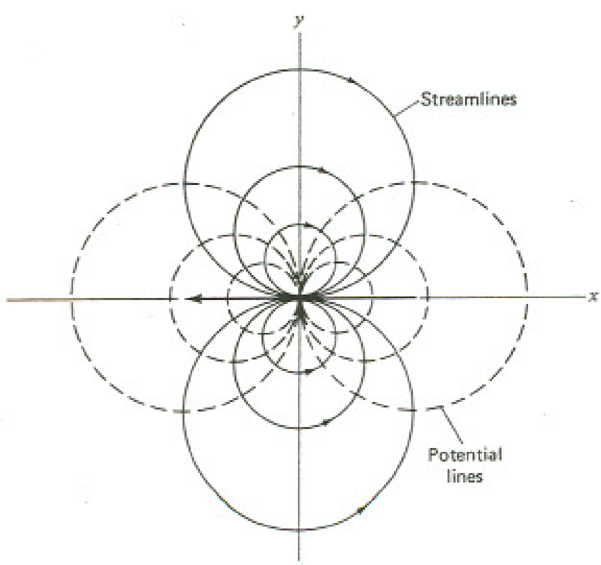


Fig. 8.12. The doublet oriented along the x-axis.

show that two families of circl
Note the orthogonality betweer

Example 8.7

Determine the velocity potential

Solution. Using polar coordina
the velocity potential with

$$v_r = \frac{\partial \phi}{\partial r}$$

Hence, from eq. (8.40),

giving

$$\phi =$$

Then, since $v_\theta = 0$ for a source,

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Thus, $f(\theta)$ is at most a constant;
within a constant we simply let th
velocity field or the pressure field.

The constant-ϕ lines are circles at
the streamlines.

Extension 8.7.1. The function
that

$$\phi =$$

Why can we choose $C = 0$? If
components, and stresses change?

8.5 SUPERPOSITION

The flows that are presented
"simple flows"; their flow pa
mathematical descriptions are u
define a flow field of engineeri
the construction of more com
superposition, that is, by adding
simple flows are the basic buil

egative x-direction. It could have igi… source and sink had been ɜr, … most common to orient the

nction is

$$\iota \frac{\sin\theta}{r} \tag{8.59}$$

$$v_\theta = -\frac{\mu \sin\theta}{r^2} \tag{8.60}$$

ıre

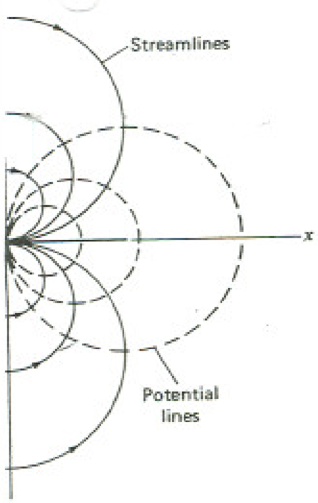
$$v = -\frac{2\mu xy}{(x^2+y^2)^2} \tag{8.61}$$

ıl for the doublet is

$$\frac{\mu x}{\ + y^2} \tag{8.62}$$

$$\frac{\cos\theta}{r} \tag{8.63}$$

ı in Fig. 8.12 with both streamlines ınd ψ constant, Eqs. 8.58 and 8.62



oriented along the x-axis.

show that two families of circles result, all passing through the origin. Note the orthogonality between the streamlines and potential lines.

Example 8.7

Determine the velocity potential ϕ for a source flow.

Solution. Using polar coordinates the velocity components are related to the velocity potential with

$$v_r = \frac{\partial\phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

Hence, from eq. (8.40),

$$\frac{\partial\phi}{\partial r} = \frac{A}{r}$$

giving

$$\phi = A \ln r + f(\theta)$$

Then, since $v_\theta = 0$ for a source,

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\frac{\partial f}{\partial\theta} = 0$$

Thus, $f(\theta)$ is at most a constant; but, since we only wish to determine ϕ to within a constant we simply let the constant be zero. This will not affect the velocity field or the pressure field. Finally,

$$\phi = A \ln r$$

The constant-ϕ lines are circles about the origin. They are always normal to the streamlines.

Extension 8.7.1. The function ϕ, in general, should include a constant so that

$$\phi = A \ln r + C$$

Why can we choose $C = 0$? If $C = 10$, how would the pressure, velocity components, and stresses change?

8.5 SUPERPOSITION

The flows that are presented in the previous article are referred to as "simple flows"; their flow patterns are easily perceived and their mathematical descriptions are uncomplicated. These simple flows may define a flow field of engineering interest but their principal use is in the construction of more complicated flow fields by the process of *superposition*, that is, by adding two or more flows. In this sense, the simple flows are the basic building blocks for two-dimensional plane flows. We can now superimpose the ψ-functions to create the flows of interest. This superposition is allowable because the governing equations for the ϕ- and ψ-functions are linear; specifically, $\nabla^2\phi = \nabla^2\psi = 0$. We simply add any combination of the stream functions together and we are assured that the new function satisfies all the basic equations. The purpose is to create a ψ or ϕ-function which represents a flow field of interest; numerous examples are given below. It will become clear that if one streamline can be identified which has a desired geometric shape, then this streamline will be designated as the "body" and, in general, the streamlines beyond this region taken as the flow around the body. Note that, since no flow crosses a streamline, its role is identical to that of a solid surface.

1. Flow Past a Half-body

If a uniform flow in the x-direction is combined with a source flow, the stream function and velocity potential are

$$\psi = Uy + \frac{q}{2\pi}\tan^{-1}\frac{y}{x} \tag{8.64}$$

$$\phi = Ux + \frac{q}{2\pi}\ln\sqrt{x^2+y^2} \tag{8.65}$$

The streamline which divides the source flow from the external flow forms a half-body shown in Fig. 8.13. The value of the stream function on the body would be $\psi = q/2$ since $\psi = 0$ on the positive x-axis. On

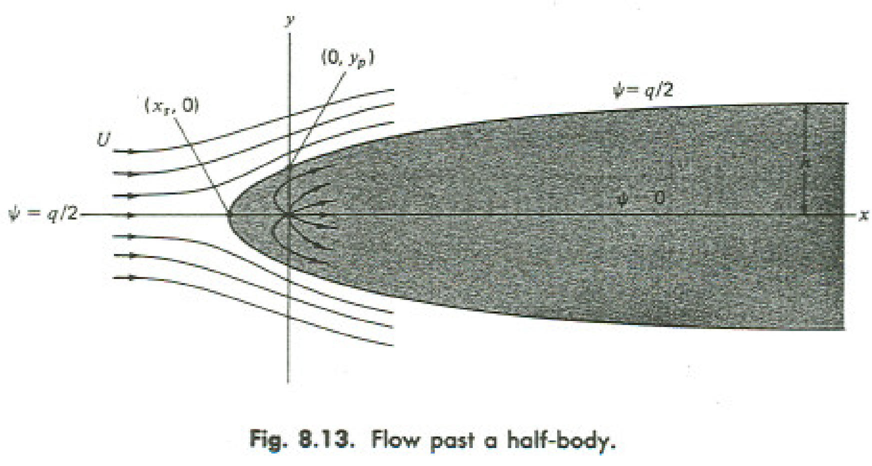


Fig. 8.13. Flow past a half-body.

the negative x-axis, $\tan^{-1} y/$
function as $\psi = q/2$.

It will be instructive to find t
the asymptotic dimension of the
on the x-axis at the point where
8.64 with respect to y, we have,

$u = U$ ·

Setting $u = 0$ the x-coordinate

x_s

The y-intercept occurs where
y-axis. The value $q/2$ is chose
of fluid flowing between the x-
surface is half of the fluid emi
Example 8.3. At the y-intercep

$\psi = Uy_p +$

giving

y

The asymptotic dimension h
$y = h$, namely,

$\psi = Uh +$

so that

2. Flow Past a Cylinde

The combination of a unifo
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:an$^{-1}\ \dfrac{y}{x}$ (8.64)

ln $\sqrt{x^2 + y^2}$ (8.65)

ource flow from the external flow
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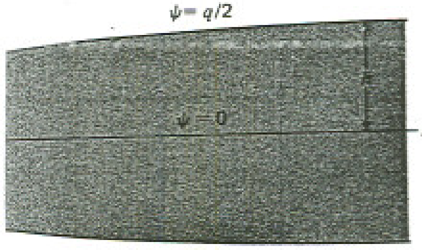
e $\psi = 0$ on the positive x-axis. On



ast a half-body.

the negative x-axis, $\tan^{-1} y/x = \pi$ and Eq. 8.64 gives the stream function as $\psi = q/2$.

It will be instructive to find the stagnation point, the y-intercept, and the asymptotic dimension of the half-body. The stagnation point occurs on the x-axis at the point where $u = 0$ and $y = 0$. By differentiating Eq. 8.64 with respect to y, we have, along the x-axis,

$$u = U + \frac{qx}{2\pi(x^2 + y^2)} \tag{8.66}$$

Setting $u = 0$ the x-coordinate of the stagnation point is given by

$$x_s = -\frac{q}{2\pi U} \tag{8.67}$$

The y-intercept occurs where the $\psi = q/2$ streamline intersects the y-axis. The value $q/2$ is chosen for ψ since we know that the quantity of fluid flowing between the x-axis and the curve designating the body surface is half of the fluid emitting from the source, namely, $q/2$. See Example 8.3. At the y-intercept point $x = 0$, so that

$$\psi = Uy_P + \frac{q}{2\pi}\tan^{-1}\frac{y_P}{0} = q/2 \tag{8.68}$$

giving

$$y_P = q/4U \tag{8.69}$$

The asymptotic dimension h would be found from letting $x \to \infty$ and $y = h$, namely,

$$\psi = Uh + \frac{q}{2\pi}\tan^{-1}\frac{h}{\infty} = q/2 \tag{8.70}$$

so that

$$h = \frac{q}{2U} \tag{8.71}$$

2. Flow Past a Cylinder

The combination of a uniform flow in the x-direction and a doublet oriented in the negative x direction will result in flow past a cylinder, shown in Fig. 8.14. The stream function and velocity potential are

$$\psi = Ur \sin\theta - \frac{\mu \sin\theta}{r} \tag{8.72}$$

$$\phi = Ur \cos\theta + \frac{\mu \cos\theta}{r} \tag{8.73}$$

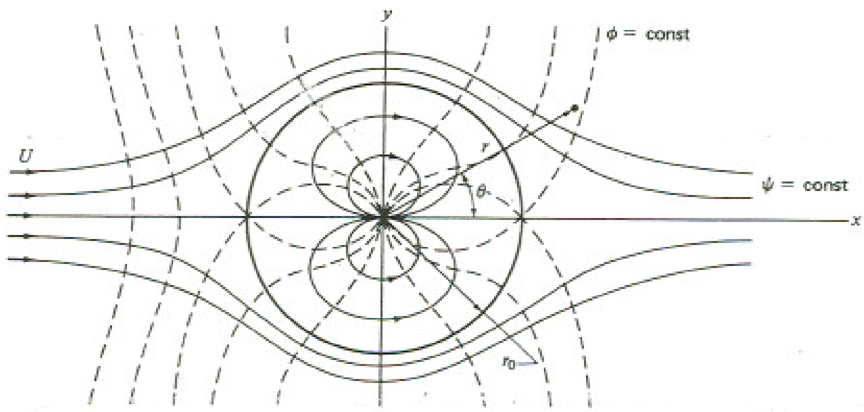


Fig. 8.14. Flow past a cylinder, showing orthogonality between streamlines and potential lines.

The radius of the cylinder is determined by locating the radius at which the radial component of velocity $v_r = 0$. The r-component of velocity, using Eq. 8.37, is

$$v_r = \left(U - \frac{\mu}{r^2}\right)\cos\theta \tag{8.74}$$

Setting $v_r = 0$ gives the radius of the cylinder, as

$$r_0 = \sqrt{\frac{\mu}{U}} \tag{8.75}$$

The stagnation points are located by setting $v_\theta = 0$ with $r = r_0$. Making use of Eq. 8.37, the θ-component of velocity is

$$v_\theta = -\left(\frac{\mu}{r^2} + U\right)\sin\theta \tag{8.76}$$

The stagnation points on the cylinder are thus located at $\theta_s = 0°$ and $\theta_s = 180°$. At these angles and $r = r_0$ both v_θ and v_r are zero.

The streamfunction, in terr

$\psi = U$

The velocity on the cylinder
8.75 in 8.76, namely,

v_c

with $v_r = 0$. The pressure dist
force effects, is found by usir

$p_c =$

where p_0 is the pressure at t
zero.

We observe that the press
decreases to a minimum valu
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tional flow around a body, t
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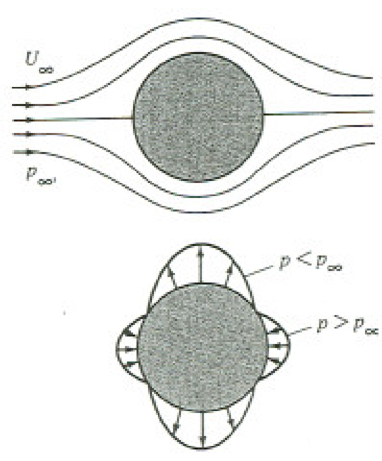


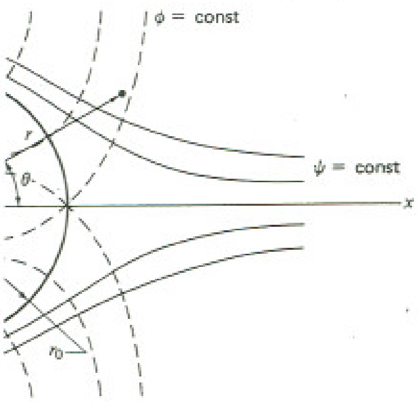
(a) Inviscid flow solution (zero drag)

Fig. 8.15. Pre

on and velocity potential are

$$- \; \cdots \frac{\sin\theta}{} \qquad (8.72)$$

$$+ \frac{\mu \cos\theta}{r} \qquad (8.73)$$



ng orthogonality between streamlines

rmined by locating the radius at
city $v_r = 0$. The r-component of

$$\left. \cdots \right) \cos\theta \qquad (8.74)$$

cylinder, as

$$\frac{\mu}{U} \qquad (8.75)$$

setting $v_\theta = 0$ with $r = r_0$. Making
velocity is

$$+ U\Big) \sin\theta \qquad (8.76)$$

r are thus located at $\theta_s = 0°$ and
both v_θ and v_r are zero.

The streamfunction, in terms of the cylinder radius, is then

$$\psi = Ur \sin\theta - \frac{r_0^2 U \sin\theta}{r} \qquad (8.77)$$

The velocity on the cylinder where $r = r_0$ is found by substituting Eq. 8.75 in 8.76, namely,

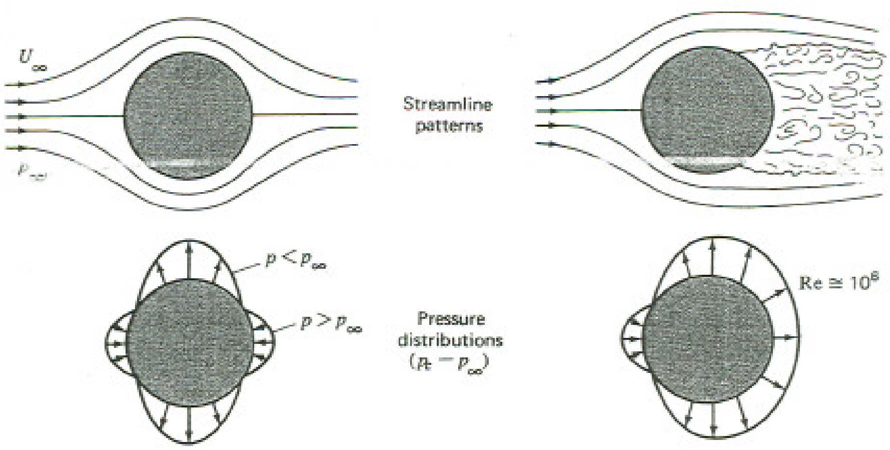
$$v_c = -2U \sin\theta \qquad (8.78)$$

with $v_r = 0$. The pressure distribution on the cylinder, neglecting body-force effects, is found by using Bernoulli's equation and is

$$p_c = p_0 - 2\rho U^2 \sin^2\theta \qquad (8.79)$$

where p_0 is the pressure at the stagnation point where the velocity is zero.

We observe that the pressure is maximum at the stagnation point, decreases to a minimum value on the top and bottom of the cylinder, and reaches the same maximum value at the opposite stagnation point. Because of this symmetrical distribution no drag would result. This simple result has been obtained using a circular cylinder; however, it is applicable to all non-circular cylinders in potential flows. *In an irrotational flow around a body, the drag is always zero.* For the circular cylinder, the pressure distribution is shown in Fig. 8.15 for an irrotational flow, (part a) and for a real flow with separation (part b). The



(a) Inviscid flow solution (zero drag)

(b) Flow with separation (large "pressure" drag)

Fig. 8.15. Pressure distribution on a cylinder.

pressure distribution predicted from the potential flow analysis is a good approximation to the real pressure distribution close to the point of separation. In the rotational wake region the irrotational solution is no longer an acceptable approximation to the actual flow. If the Reynolds number associated with the flow is low enough, no separation will occur; however, then viscous effects cannot be neglected for this situation and again the potential flow theory is not an acceptable approximation to the flow. The fact that the potential flow solution is an acceptable approximation to the flow up to the separation point makes it important to engineers solving for flows around bodies. In fact, for streamlined bodies the flow may not separate even at large Reynolds numbers; for these aerodynamic bodies the potential flow approximates the flow over the whole body. The low-pressure region of a separated flow is, in general, the dominant contributor to the aerodynamic drag of bluff bodies. Flow around bodies is considered in Chapter 10.

Flow around a circular cylinder is often used to illustrate a graphical method which gives an approximate solution to an irrotational plane flow. The method is based on the orthogonality which exists between the streamlines and potential lines. To utilize the method, streamlines are sketched so that they are equally spaced in regions of uniform flow; then potential lines are sketched in, equally spaced in regions of uniform flow and with the same spacing as the streamlines. Near the body the potential lines are sketched in with the same approximate spacing as the streamlines. The resulting pattern of streamlines and potential lines is a *flow net*, shown in Fig. 8.16. Potential flow theory states that streamlines and potential lines intersect at right angles,

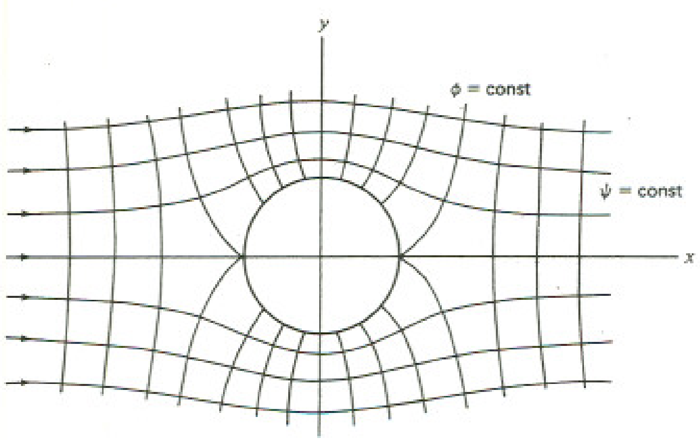


Fig. 8.16. Flow net for flow around a cylinder.

hence following the foregoing p
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should be squares. An initial ske
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that either the streamlines or the
together. A second attempt at th
are more nearly squares, and a fl
Iterations are continued until al
closely as possible. Usually a co
After several iterations a finer gr
results. After the flow net is con
approximated by observing the c
vicinity of the point. The velc
distance between the streamlines
two streamlines in the vicinity
between the same two streamli
velocity is U, then continuity ca
the point would be approximate
be approximated with the use of

3. Flow Around a Cylind

If an irrotational vortex is ad
there results

$$\psi = Ur \sin \theta -$$

The velocity distribution on th
given by

$$v_c = -2U$$

The pressure distribution on th

$$p_c = p_0 - \rho \frac{U}{2}$$

where p_0 is the stagnation poir
ing the vertical component of

*Alternatively, the velocity can be ol
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the potential flow analysis is a
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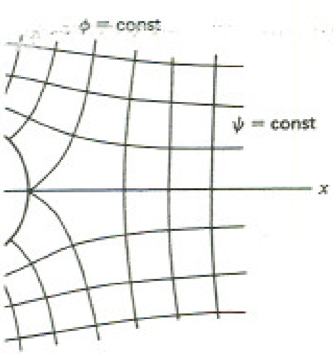

low around a cylinder.

hence following the foregoing procedure all spaces in the flow net should approximate squares, and in regions of uniform flow, the spaces should be squares. An initial sketch of the flow net usually results in regions in which the spaces are elongated or rectangular. This indicates that either the streamlines or the potential lines are sketched too close together. A second attempt at the sketch should produce spaces that are more nearly squares, and a flow pattern that is more nearly correct. Iterations are continued until all the spaces approximate squares as closely as possible. Usually a coarse grid is used to start the process. After several iterations a finer grid is introduced to give more accurate results. After the flow net is completed, the velocity at a point can be approximated by observing the distance between the streamlines in the vicinity of the point. The velocity is inversely proportional to the distance between the streamlines. For example, if the distance between two streamlines in the vicinity of a point is one half of the distance between the same two streamlines in the region where the uniform velocity is U, then continuity can be used to show that the velocity at the point would be approximately $2U$.* The pressure change can then be approximated with the use of Bernoulli's equation.

3. Flow Around a Cylinder with Circulation

If an irrotational vortex is added to the streamfunction of Eq. (8.77) there results

$$\psi = Ur\sin\theta - \frac{r_0^2 U\sin\theta}{r} - \frac{\Gamma}{2\pi}\ln r \qquad (8.80)$$

The velocity distribution on the cylinder with radius $r_0 = \sqrt{\mu/U}$ is given by

$$v_c = -2U\sin\theta + \frac{\Gamma}{2\pi}\sqrt{\frac{U}{\mu}} \qquad (8.81)$$

The pressure distribution on the cylinder is

$$p_c = p_0 - \rho\frac{U^2}{2}\left[2\sin\theta - \frac{\Gamma}{2\pi r_0 U}\right]^2 \qquad (8.82)$$

where p_0 is the stagnation point pressure. The lift is found by integrating the vertical component of the pressure force, shown in Fig. 8.17,

*Alternatively, the velocity can be obtained from the approximation $\Delta\phi/\Delta s$, where Δs is the streamwise spacing of the curvilinear square.

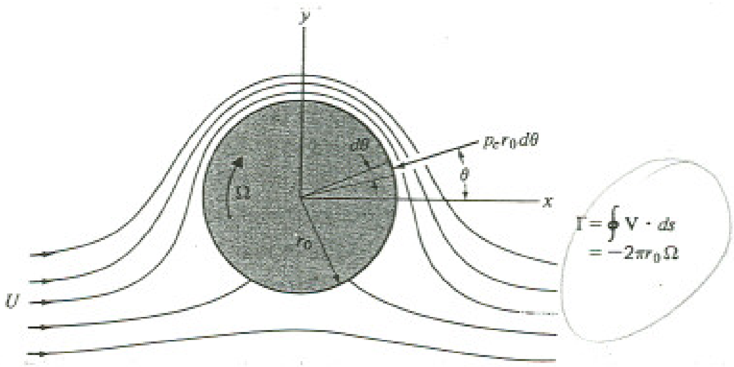

Fig. 8.17. Flow around a cylinder with circulation.

and is

$$L = -\int_0^{2\pi} p_c \sin\theta \, r_0 d\theta \tag{8.83}$$

With the expression for the pressure p_c from Eq. 8.82, this may be integrated to give

$$L = -\rho U \Gamma \quad = \rho U 2\pi r_0 \Omega \tag{8.84}$$

This simple expression for the lift is also applicable to all non-circular cylinders. It and the zero drag conclusion form the *Kutta-Joukowsky theorem*.

4. Series of Sources and Sinks

A series of sources and sinks could be situated on the x-axis and superimposed on a uniform flow to create flow around a streamlined body, shown in Fig. 8.18a. A series of sources and sinks of various strengths would be combined with a uniform flow as

$$\psi = Uy + \sum_{i=1}^{N} \frac{q_i}{2\pi} \tan^{-1} \frac{y}{x - x_i} \tag{8.85}$$

So that no net flow emanates from the body, we require that

$$\sum_{i=1}^{N} q_i = 0 \tag{8.86}$$

By letting $q_i \to 0$ at either end of the symmetrical body we can force the body to be pointed; if q_i is finite at an end it will be blunt.

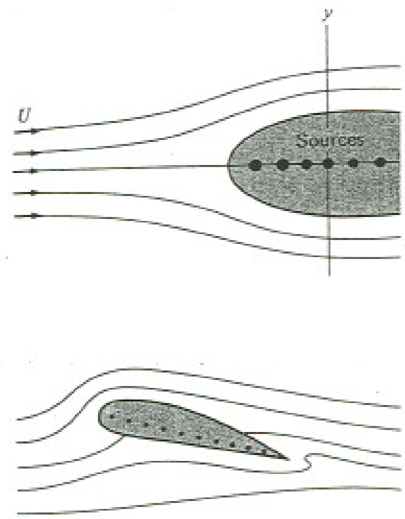

Fig. 8.18. Flow

The sources and sinks could
Fig. 8.16b, to give the airfoil a
result in a flow as shown. The f
edge with a stagnation point lo
implies a tremendously large r
edge since $V \neq 0$ as $R \to 0$ in ∂
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8.16c. In order that our potenti
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be located at the trailing edge
on the airfoil is $-\rho U\Gamma$, and is
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The technique of superimpo
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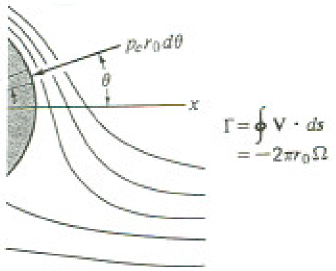

cylinder with circulation.

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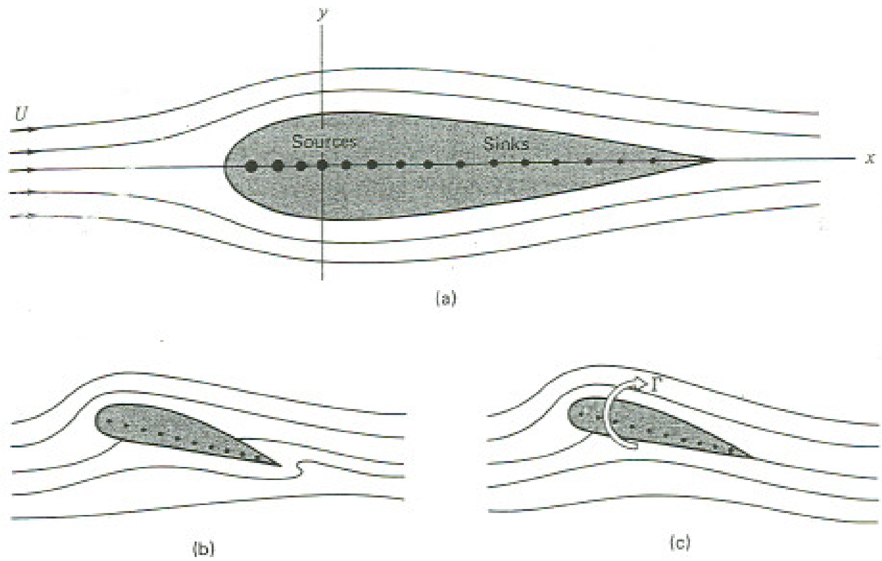

Fig. 8.18. Flow around a streamlined body.

The sources and sinks could be distributed along a line, as shown in Fig. 8.16b, to give the airfoil an angle of attack. This would, however, result in a flow as shown. The flow would turn the corner at the trailing edge with a stagnation point located on the top surface. Of course, this implies a tremendously large normal pressure gradient at the trailing edge since $V \neq 0$ as $R \rightarrow 0$ in $\partial p / \partial n = \rho V^2 / R$; hence the fluid would not turn the corner at the trailing edge but would be as shown in Fig. 8.16c. In order that our potential flow be a good approximation to the real flow a vortex is superimposed as shown. The strength Γ of the vortex is chosen such that the dividing streamline leaves the rear of the airfoil at the trailing edge. The condition that the dividing streamline be located at the trailing edge is the *Kutta condition*. The resulting lift on the airfoil is $-\rho U \Gamma$, and is a good approximation to the lift on the actual airfoil.

The technique of superimposing sources and sinks is difficult if we wish to form a predetermined body. The more common technique used today is to distribute sources and sinks along the surface of a known body, for example an airfoil, and then to determine the source and sink strengths by requiring the normal component of velocity on the body to vanish. The digital computer is obviously very handy in problems where a large number of sources and sinks are involved.

5. Image Flow

Some interesting flows can be generated by the *method of images*. For example, if a source flow next to a plane wall a distance d away were desired we could generate this flow by placing a source at the position $(d, 0)$ and its image, a source of equal strength, at the position $(-d, 0)$. This is shown in Fig. 8.19. Because of symmetry the y-axis becomes a streamline and hence can be replaced with a solid boundary. The combined velocity potential is

$$\phi = \frac{q}{2\pi} \ln \sqrt{(x-d)^2 + y^2} + \frac{q}{2\pi} \ln \sqrt{(x+d)^2 + y^2} \qquad (8.87)$$

resulting in velocity components

$$u = \frac{q}{2\pi}\left[\frac{x-d}{(x-d)^2 + y^2} + \frac{x+d}{(x+d)^2 + y^2}\right]$$

$$v = \frac{q}{2\pi}\left[\frac{y}{(x-d)^2 + y^2} + \frac{y}{(x+d)^2 + y^2}\right] \qquad (8.88)$$

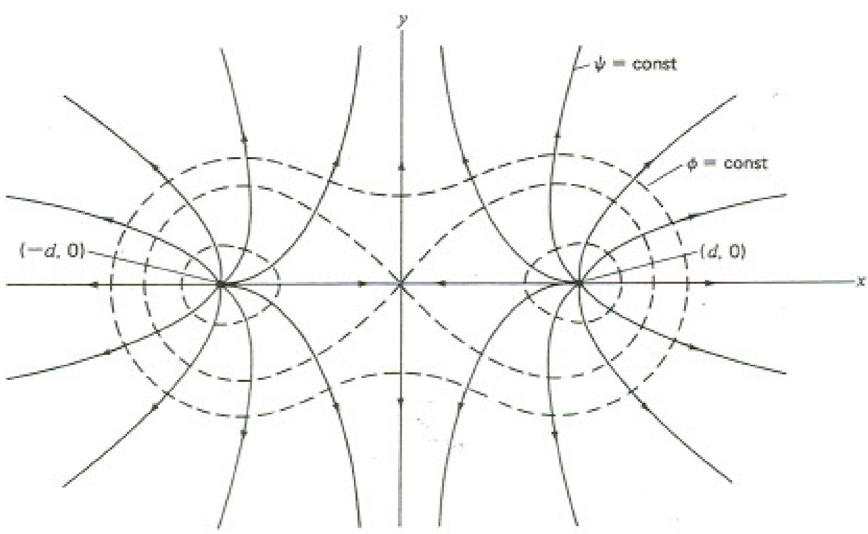

Fig. 8.19. Source flow near a plane wall.

On the y-axis, $x = 0$ and from t
$u = 0$. Only v is non-zero along
tangent to the y-axis, and the y-
the right of the y-axis or the lef
wall. The origin is a stagnation
of the origin would be quite simi
the first section of Art. 8.4.

Various other flows of import
images. A source or sink flow in
by superimposing an infinite ser
An infinite series of doublets dis
with a uniform flow could re
channel.

Example 8.8

Determine the angles which loca
cylinder. In particular, if an 8-in.
free stream, locate the two stagnati

Solution. The velocities, calcul
are

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U$$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U$$

On the cylinder $r = r_0$ and v_r is
where $v_\theta = 0$, the stagnation point

sin

The angles which locate the two
expression

$$\theta_s = $$

The circulation is calculated by
Fig. 8.10) at $r = r_0$ to have a veloc
the cylinder. The circulation is th

$$\Gamma = \oint \mathbf{V}\cdot d\mathbf{s} = -r_0\Omega(2\pi r_0)$$

rate y the *method of images*. For
lane wall a distance d away were
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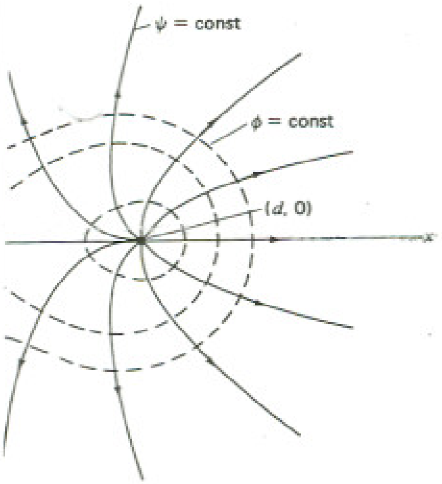
l strength, at the position $(-d, 0)$.
of symmetry the y-axis becomes a
ced with a solid boundary. The

$$+ \frac{q}{2\pi} \ln \sqrt{(x+d)^2 + y^2} \qquad (8.87)$$

$$\ldots + \frac{x+d}{(x+d)^2 + y^2} \Bigg]$$

$$(8.88)$$

$$\ldots + \frac{y}{(x+d)^2 + y^2} \Bigg]$$



near a plane wall.

On the y-axis, $x = 0$ and from the above expression for u we see that $u = 0$. Only v is non-zero along the y-axis. Thus, the velocity vector is tangent to the y-axis, and the y-axis must be a streamline. The flow on the right of the y-axis or the left then represents a source flow near a wall. The origin is a stagnation point and the flow in the near vicinity of the origin would be quite similar to that near the stagnation point of the first section of Art. 8.4.

Various other flows of importance can be formed by the method of images. A source or sink flow in the end of a channel can be generated by superimposing an infinite series of sources or sinks along the y-axis. An infinite series of doublets distributed along the y-axis superimposed with a uniform flow could result in flow around a cylinder in a channel.

Example 8.8

Determine the angles which locate the two stagnation points on a rotating cylinder. In particular, if an 8-in. dia. cylinder rotates at 200 rpm in a 10-fps free stream, locate the two stagnation points.

Solution. The velocities, calculated from the streamfunction in eq. (8.80) are

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U \sin\theta - \frac{r_0^2}{r^2} U \sin\theta + \frac{\Gamma}{2\pi r}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta - \frac{r_0^2}{r^2} U \cos\theta$$

On the cylinder $r = r_0$ and v_r is obviously zero. We wish to find the point where $v_\theta = 0$, the stagnation point. Setting $v_\theta = 0$ gives

$$\sin\theta_s = \frac{\Gamma}{4\pi r_0 U}$$

The angles which locate the two stagnation points are thus given from the expression

$$\theta_s = \sin^{-1}\left(\frac{\Gamma}{4\pi r_0 U}\right)$$

The circulation is calculated by considering the streamline of the vortex (see Fig. 8.10) at $r = r_0$ to have a velocity of $r_0\Omega$, where Ω is the angular velocity of the cylinder. The circulation is then (counterclockwise is positive)

$$\Gamma = \oint \mathbf{V} \cdot d\mathbf{s} = -r_0\Omega(2\pi r_0) = -\frac{200 \times 2\pi}{60} \times \frac{2\pi}{9} = -14.6 \text{ ft}^2/\text{sec}$$

so that

$$\theta_s = \sin^{-1}\left(-\frac{14.6}{40\pi/3}\right) = \sin^{-1}(-0.348)$$

Hence,

$$\theta_s = 200.4°, 339.6°$$

Extension 8.8.1. Determine the Ω in rpm which would cause only one stagnation point to exist on the cylinder. *Ans.* 60 rad/sec

Extension 8.8.2. Determine the pressure distribution on the cylinder of Example 8.8 and calculate the maximum and minimum values of $(p_c - p_\infty)$. Water is flowing. *Ans.* 97 psf, −609 psf

Extension 8.8.3. An approximation to the flow described in this example can be created by the physical situation shown in Fig. E8.8. A flat ribbon, wrapped tightly around the cylinder from a roll of kitchen towels works well. Do the experiment. Explain why viscosity is *necessary* to approximate this inviscid flow problem. Explain why the cylinder executes a looping motion.

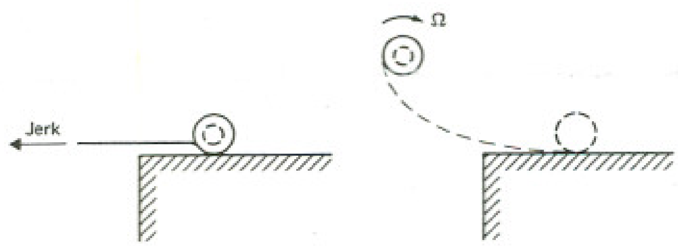


Fig. E8.8

8.6 AXISYMMETRIC FLOWS

Laplace's equation in spherical coordinates for the axisymmetric case is

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0 \quad (8.89)$$

The simple potential function

$$\phi = \frac{A}{r} \quad (8.90)$$

satisfies Laplace's equation and represents a *point source* with streamlines emanating along radial lines in all directions from the origin of r.

This point source with fluid e
different from the plane source
integration of the velocity arc
introduce the source strength q

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for a source.

By combining a source at x
$x = +\epsilon$ the axisymmetric doubl

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If we combine a uniform flo
flow around a sphere. The velo

$\phi = \frac{\mu}{}$

This flow is shown in Fig. 8.20
the coordinates r, θ, and x.

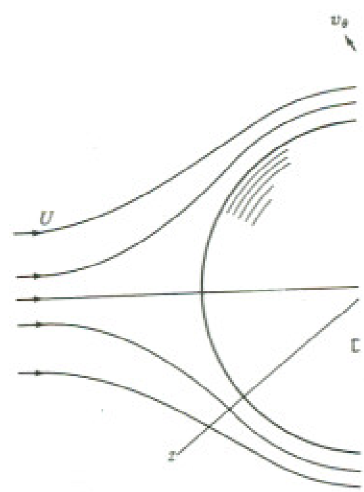

Fig. 8.20.

The velocity vector would b

$\mathbf{V} = \frac{\partial}{\partial}$

$) \quad \text{in}^{-1}(-0.348)$

1°, 339.6°

in rpm which would cause only one
Ans. 60 rad/sec

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Ans. 97 psf, −609 psf

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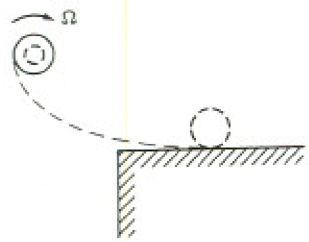


E8.8

oordinates for the axisymmetric case

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) = 0 \qquad (8.89)$$

$$= \frac{A}{r} \qquad (8.90)$$

epresents a *point source* with stream-
in all directions from the origin of r.

This point source with fluid emanating from a single point is quite different from the plane source with fluid emanating from a line. An integration of the velocity around a spherical surface allows us to introduce the source strength q and write

$$\phi = -\frac{q}{4\pi r} \qquad (8.91)$$

for a source.

By combining a source at $x = -\epsilon$ with a sink of equal strength at $x = +\epsilon$ the axisymmetric doublet results as $\epsilon \to 0$ with velocity potential

$$\phi = \frac{\mu \cos\theta}{r^2} \qquad (8.92)$$

If we combine a uniform flow and this doublet flow we would have flow around a sphere. The velocity potential would be

$$\phi = \frac{\mu \cos\theta}{r^2} + Ur\cos\theta \qquad (8.93)$$

This flow is shown in Fig. 8.20 along with a graphical presentation of the coordinates r, θ, and x.

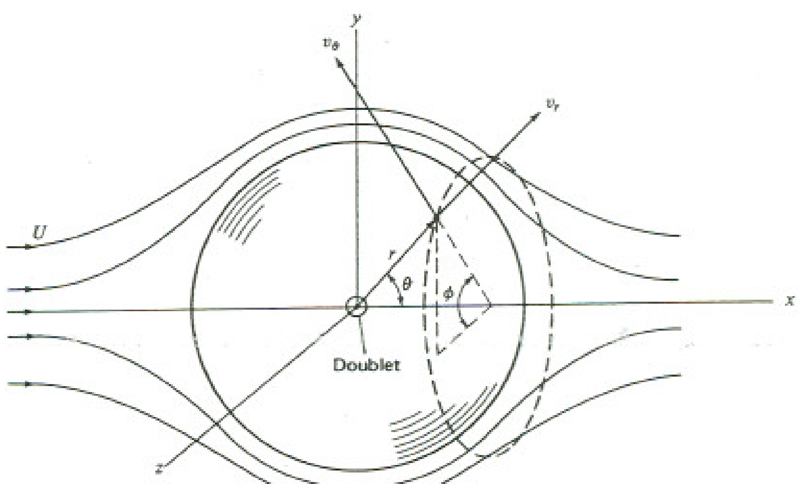


Fig. 8.20. Flow around a sphere.

The velocity vector would be found from

$$\mathbf{V} = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta \qquad (8.94)$$

where θ is measured from the positive x-axis. For flow around a sphere

$$v_r = \frac{\partial \phi}{\partial r} = U \cos \theta - \frac{2\mu \cos \theta}{r^3} \tag{8.95}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U \sin \theta - \frac{\mu \sin \theta}{r^3} \tag{8.96}$$

The radius of the sphere is found to be

$$r_0 = \left(\frac{2\mu}{U}\right)^{1/3} \tag{8.97}$$

A uniform flow could be superimposed with a series of sources and sinks or a series of doublets placed along a straight line segment to create flow around an axisymmetric body, as was shown in Fig. 8.18, for the plane flow.

It should be pointed out here that a stream function for the axisymmetric flow could have been defined with the aid of the continuity equation in spherical coordinates. But the velocity potential and streamfunction no longer satisfy the Cauchy-Riemann equations so that complex variables is not of use in axisymmetric flows. It may be useful, however, to determine the streamfunction since it is constant along a solid boundary. The continuity equation, in spherical coordinates, for axisymmetric flows is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(v_\theta \sin \theta) = 0 \tag{8.98}$$

The axisymmetric stream function would then be defined from

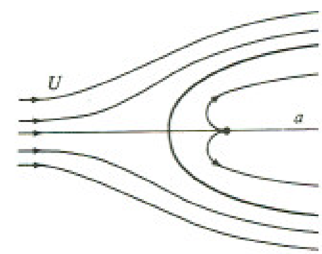
$$v_r = \frac{1}{r^2 \sin \theta}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{1}{r \sin \theta}\frac{\partial \psi}{\partial r} \tag{8.99}$$

Using Eqs. 8.95 and 8.96 the streamfunction for flow around a sphere would be

$$\psi = \tfrac{1}{2} U r^2 \sin^2 \theta - \frac{\mu \sin^2 \theta}{r} \tag{8.100}$$

Example 8.9

A point source is placed at $x = -a$ and an equal strength sink at $x = +a$ (Fig. E8.9). Determine the maximum radius r_0 of the axisymmetric body formed if the source and sink are placed in a uniform flow.



Solution. Using Eq. 8.91 in car

$$\phi = Ux - \frac{q}{4\pi\sqrt{(x+a)^2 + y}}$$

Because of symmetry, the body wi shown. One may find the body ra that the total mass flux across the a velocity is

$$u = \frac{\partial \phi}{\partial x} = U + \frac{(x+a)}{4\pi[(x+a)^2 + y}$$

Along the $x = 0$ plane

$$u = U + -$$

$$= U + -$$

The flow rate q through a circle of

$$q = \int_0^{r_0} \Big[U +$$

$$= U\pi r_0^2 - qa$$

For a particular set of flow param

x-axis. For flow around a sphere

$$- \frac{2\mu \cos\theta}{r^3} \quad (8.95)$$

$$\text{in } \theta - \frac{\mu \sin\theta}{r^3} \quad (8.96)$$

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$$\Big)^{1/3} \quad (8.97)$$

ɔsed with a series of sources and along a straight line segment to body, as was shown in Fig. 8.18,

t a stream function for the axi- ied with the aid of the continuity But the velocity potential and Cauchy-Riemann equations so n axisymmetric flows. It may be eamfunction since it is constant tinuity equation, in spherical

$$\frac{\partial}{\partial\theta}(v_\theta \sin\theta) = 0 \quad (8.98)$$

uld then be defined from

$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \quad (8.99)$$

unction for flow around a sphere

$$- \frac{\mu \sin^2\theta}{r} \quad (8.100)$$

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d an equal strength sink at $x = +a$ adius r_0 of the axisymmetric body a uniform flow.

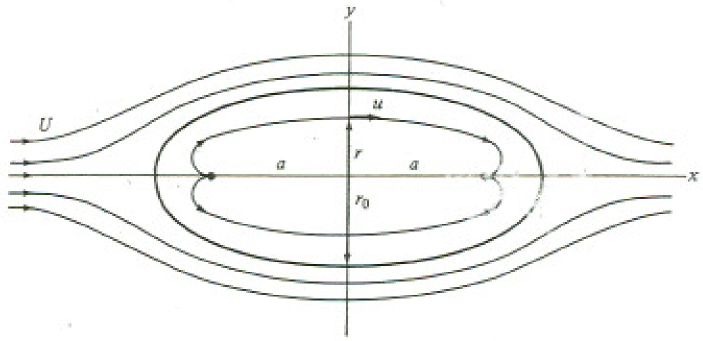

Fig. E8.9

Solution. Using Eq. 8.91 in cartesian coordinates, the velocity potential is

$$\phi = Ux - \frac{q}{4\pi\sqrt{(x+a)^2 + y^2 + z^2}} + \frac{q}{4\pi\sqrt{(x-a)^2 + y^2 + z^2}}$$

Because of symmetry, the body will have its maximum thickness at $x = 0$, as shown. One may find the body radius by integrating from $r = 0$ to $r = r_0$ so that the total mass flux across the area of integration is q. The x-component of velocity is

$$u = \frac{\partial\phi}{\partial x} = U + \frac{(x+a)q}{4\pi\left[(x+a)^2 + y^2 + z^2\right]^{3/2}} - \frac{(x-a)q}{4\pi\left[(x-a)^2 + y^2 + z^2\right]^{3/2}}$$

Along the $x = 0$ plane

$$u = U + \frac{2qa}{4\pi(a^2 + y^2 + z^2)^{3/2}}$$

$$= U + \frac{qa}{2\pi(a^2 + r^2)^{3/2}}$$

The flow rate q through a circle of radius r_0 is

$$q = \int_0^{r_0}\left[U + \frac{qa}{2\pi(a^2 + r^2)^{3/2}}\right]2\pi r\,dr$$

$$= U\pi r_0^2 - qa\left[\frac{1}{\sqrt{a^2 + r_0^2}} - \frac{1}{a}\right]$$

For a particular set of flow parameters r_0 could be determined.

Extension 8.9.1. Determine the maximum body radius r_0 if $U = 10$ fps, $a = 4$ ft and $q = 100$ ft^3/sec. *Ans.* 1.71 ft

Extension 8.9.2. Determine the length of the body for these parameters. *Ans.* 9.82 ft

---

8.7 ROTATIONAL INVISCID FLOWS

In the preceding articles the vorticity was considered zero. It is appropriate to discuss briefly the governing equations for inviscid flows which have vorticity. The differential momentum equation, in vector form, for an inviscid flow, is

$$\frac{\partial \mathbf{V}}{\partial t} + \tfrac{1}{2}\nabla V^2 + \omega \times \mathbf{V} = -\frac{\nabla p}{\rho} - g\,\nabla h \tag{8.101}$$

By taking the curl of both sides of this equation we obtain, for a constant-density flow,

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{V}) + \nabla \times (\omega \times \mathbf{V}) = 0 \tag{8.102}$$

recalling that the curl of the gradient of a scalar quantity is zero. It can be shown that

$$\nabla \times (\omega \times \mathbf{V}) = (\mathbf{V}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{V} \tag{8.103}$$

(This can be verified by expanding both sides.) The momentum equation, in terms of vorticity, is then

$$\frac{\partial \omega}{\partial t} + (\mathbf{V}\cdot\nabla)\omega = (\omega\cdot\nabla)\mathbf{V} \tag{8.104}$$

or

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\mathbf{V} \tag{8.105}$$

This equation is the inviscid counterpart of the vorticity transport equation (5.14).

Equation 8.105 allows us to state that *if an inviscid flow is irrotational*, that is ω is everywhere zero, *then it must remain irrotational*; for if $D\omega/Dt = 0$ everywhere in the flow, then ω at the next instant must be zero and hence ω must remain at the constant zero value. This may be referred to as the *persistence of irrotationality*.

For a two-dimensional plane f
($\xi = \eta = 0$), and one velocity
8.105 reduces to

allowing us to state that the v
inviscid flow cannot change as
Of course, viscosity can creat
flow, and can cause vorticity
foregoing, viscous effects have
Returning to Eq. 8.101, if a c
steady,

$$\nabla\left[\frac{V^2}{2} + \ldots\right.$$

Let $V^2/2 + p/\rho + gh = \Phi$; the
normal to both a streamline and
normal to constant-Φ lines; henc

$$\frac{V^2}{2} + \ldots$$

along a streamline *or a vortex*
defined by the family of vor

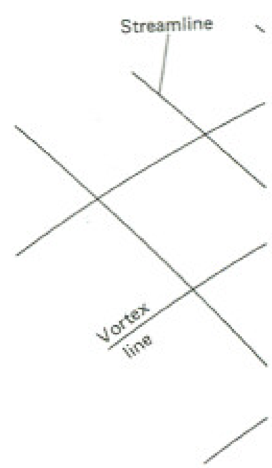

Fig. 8.21. Str

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Ans. 1.71 ft

th the body for these parameters.
Ans. 9.82 ft

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al momentum equation, in vector

$$\mathbf{V} = -\frac{\nabla p}{\rho} - g\,\nabla h \qquad (8.101)$$

of this equation we obtain, for a

$$\times\ (\omega \times \mathbf{V}) = 0 \qquad (8.102)$$

t of a scalar quantity is zero. It can

$$\mathbf{V}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{V} \qquad (8.103)$$

both sides.) The momentum equa-

$$\omega = (\omega\cdot\nabla)\mathbf{V} \qquad (8.104)$$

$$\omega\cdot\nabla)\mathbf{V} \qquad (8.105)$$

iterpart of the vorticity transport

hat *if an inviscid flow is irrotational,*
it must remain irrotational; for if
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e constant zero value. This may be
ationality.

For a two-dimensional plane flow, two vorticity components are zero ($\xi = \eta = 0$), and one velocity component is zero ($w = 0$). Equation 8.105 reduces to

$$\frac{D\zeta}{Dt} = 0 \qquad (8.106)$$

allowing us to state that the vorticity of a fluid particle in a plane inviscid flow cannot change as the particle moves along.

Of course, viscosity can create vorticity in an otherwise irrotational flow, and can cause vorticity to change in a plane flow; however, in the foregoing, viscous effects have been neglected.

Returning to Eq. 8.101, if a constant-density flow is restricted to be steady,

$$\nabla\left[\frac{V^2}{2} + \frac{p}{\rho} + gh\right] = \mathbf{V} \times \omega \qquad (8.107)$$

Let $V^2/2 + p/\rho + gh = \Phi$; then, from Fig. 8.21, we see that $\nabla\Phi$ is normal to both a streamline and a vortex line. But, we recall that $\nabla\Phi$ is normal to constant-Φ lines; hence we conclude that

$$\frac{V^2}{2} + \frac{p}{\rho} + gh = \text{const} \qquad (8.108)$$

along a streamline *or a vortex line* and, more generally, to the plane defined by the family of vortex lines that intersects a family of

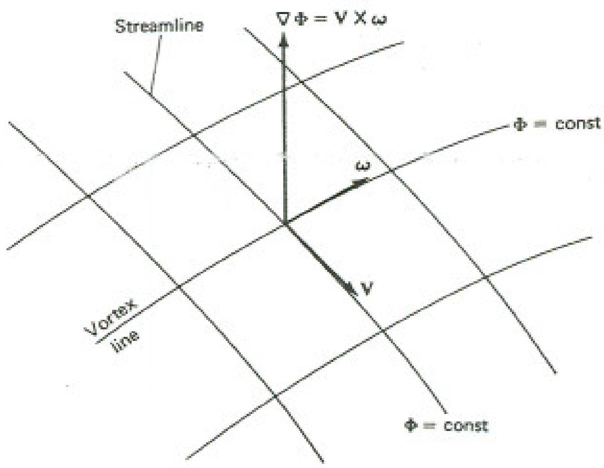


Fig. 8.21. Streamlines and vortex lines.

streamlines. Note that Bernoulli's equation for rotational flows is therefore more restrictive than for irrotational flows where Eq. 8.108 may be applied to all points in the flow.

8.8 HELE-SHAW FLOW

The Hele-Shaw apparatus uses a viscous-dominated flow between parallel plates to provide a visualization technique for the study of inviscid flows. This seeming paradox is at least potentially resolved when one realizes that the viscous effects are dominant for the flow between the plates (with respect to z) whereas the "irrotational" motion is in a direction parallel to the plates. The relationship of the two motions will be shown by the Navier-Stokes equations.

Consider a flow system as shown in Fig. 8.22. The velocity has only two components because of the small and constant spacing between the plates. Hence, the Navier-Stokes equations in the x- and y-directions are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (8.109)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (8.110)$$

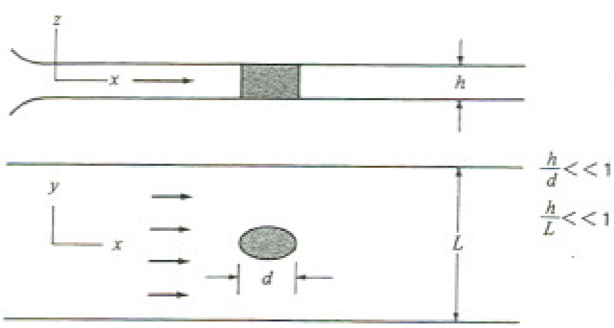


Fig. 8.22. Hele-Shaw apparatus.

For a viscous dominated flow, involving very slow motion, the obvious restrictions and assumptions lead to

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial z^2} \quad \text{and} \quad \frac{\partial p}{\partial y} = \mu\frac{\partial^2 v}{\partial z^2} \quad (8.111)$$

Because the z-spacing is small, the second derivative with respect to z

dominates the x- and y-derivati
giving

$$u = \frac{1}{2\mu}$$

and

$$v = \frac{1}{2\mu}$$

Let $u = U(x, y)$ and $v = V(x,$

$$-\frac{\partial p}{\partial x}$$

and

$$-\frac{\partial p}{\partial y}$$

Differentiate Eq. 8.114 with res
x; subtract, and there results

$$\frac{\partial V}{\partial x}$$

But this is precisely the con
z-component vorticity since

$$(\nabla \times$$

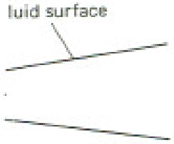
and since the flow between the
U- and V-components, the m
plane!

Hence, the Hele-Shaw appa
tion which can be used to visu
is often simply a large, flat hori
water flows at very low speed
free surface. Various objects
potassium permanganate are
object. Potential flow streamlin

8.1 Give several examples o
considered inviscid. Also give exa
inviscid-flow approximation woul

to gravity in the Euler-s equation and ompressible, steady flow. State some :e ... would be significant and some

area, forms at the outlet of a sharp- vn in the sketch. (a) Based on the starting from point A, describe why Will an air jet exhausting into the



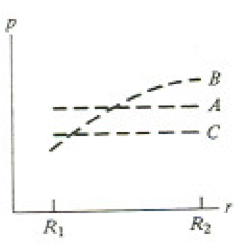
8.3

atomizer works, using an appropriate

8.4

a right-handed pitcher can most easily

a which involves the pressure distribu- appropriate forms of the Euler equa-



. 8.6

tions, explain the observed form of the pressure distributions. Is the velocity around the bend higher on the inside or the outside?

8.7 Given: the velocity potential $\phi = x^2 - 2xy - y^2$. Can this represent an incompressible-fluid flow? If so, find the stream function ψ.

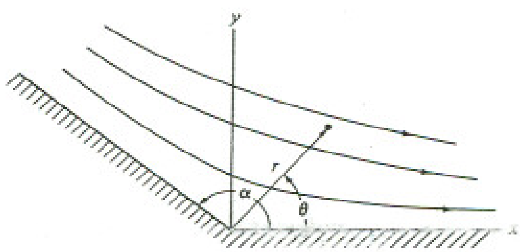
8.8 An aircraft is moving at a speed of 300 mph at an elevation where the pressure is 10 psi. Determine the expected pressure at the stagnation point and at a point on the upper surface of the wing where the velocity, from potential-flow theory, is calculated to be 400 mph. Assume incompressible flow for a first approximation.

8.9 Sketch regions of vorticity and regions of zero vorticity for the following: (a) entrance flow in a pipe; (b) flow over two-dimensional airfoil; (c) flow over a sphere, with a separated region; and (d) flow around a Greyhound bus.

8.10 It is proposed that, to find the potential flow of fluid around a cylinder of radius R located in the center of a channel of depth h, the governing equations be solved using a finite difference scheme. State the equation to be solved and the necessary boundary conditions. The flow rate is q ft³/sec/ft. What is the velocity profile upstream from the cylinder?

8.11 Can the stream function $\psi = A(x^2 - y^2)$ be used for a potential flow? If so, sketch the streamline pattern. Also, determine the potential function and sketch several potential lines.

8.12 Show that $\psi = Ar^{\pi/\alpha} \sin(\pi\theta/\alpha)$ represents flow in a corner, as shown. Show also the following: (a) if $\alpha = \pi/2$, stagnation flow results; (b) if $\alpha = \pi$, uniform flow results; and (c) if $\alpha = 2\pi$, flow around a semi-infinite flat plate results. Sketch some streamlines for the flow of part c.



Prob. 8.12

8.13 A velocity potential function is given by $\phi = 10x + 40x/(x^2 + y^2)$. (a) Verify that ϕ satisfies Laplace's equation, $\nabla^2\phi = 0$. (b) Determine the stream function and sketch the streamline corresponding to $\psi = 0$. (c) Find the pressure distribution in the water along the x-axis, assuming the pressure at $x = -\infty$ is 10 psi. (d) Locate any stagnation points.

8.14 Derive the expression for the velocity potential for (a) a plane vortex and (b) a plane doublet.

8.15 Using Eqs. 8.53, show that, for a vortex flow, the velocity vector is normal to a radius vector and hence in the θ-direction. Determine the magnitude of the velocity vector and compare with Eq. 8.48.

8.16 Show that the vorticity of an irrotational vortex is everywhere zero except at the origin, where it is infinite.

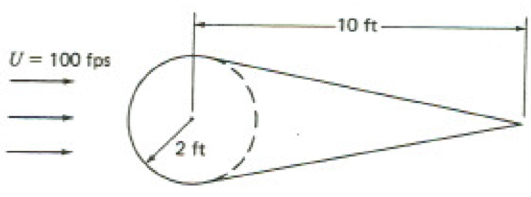
8.17 Superimpose a source of strength $q = 8$ at $x = -2$, a sink of equal strength at $x = 2$, and a uniform flow in the x-direction of $U = 10$. Does this represent flow around a closed body? If so, determine the maximum thickness of the body. What is the value of the stream function all along the x-axis and on the body?

8.18 Place a source at $(0, -\epsilon)$ and a sink of equal strength at $(0, \epsilon)$; let ϵ approach 0 and the source and sink strengths approach infinity. Derive an expression for the stream function of a doublet oriented along the y-axis.

8.19 A uniform flow of 20 fps is combined with a doublet of strength $\mu = 80\ \text{ft}^3/\text{sec}$ situated at the origin to give flow around a cylinder. (a) Sketch the velocity along the y-axis. (b) Determine the deceleration at the point $(-4, 0)$. (c) Find the force of the water on the front half of the 20-ft-long cylinder if the stagnation-point pressure is 5 psi.

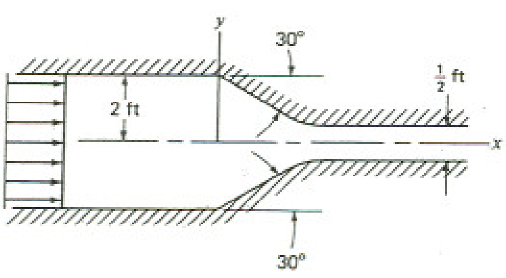
8.20 Air flows around a 2-ft-dia., 100-ft-high pole at 50 fps. Assuming a potential flow over the front half and a separated flow at constant pressure over the rear half, approximate the pressure drag on the pole. See Fig. 8.15b.

8.21 Air flows over the symmetrical plane body shown. Using a flow net, approximate the streamline pattern, assuming no separation. Estimate the minimum pressure on the body if the pressure in the free stream is 15 psi.



Prob. 8.21

8.22 Sketch the streamline pattern with the use of a flow net for the plane contraction shown. Assume the streamlines are equally spaced downstream of the contraction. Estimate the pressure at the point (1, 1) if the pressure upstream is 20 psi and the flow rate is $100\ \text{ft}^3/\text{sec}/\text{ft}$ of water.

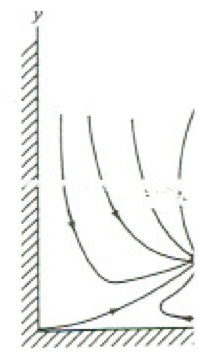


Prob. 8.22

8.23 Determine the rotational sp have to rotate in a 20-fps free-strea would exist on the cylinder. Sketch

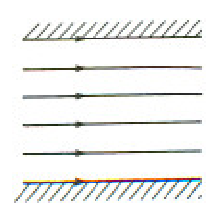
8.24 A 2-ft.-dia. cylinder is rotat Locate the stagnation point (or pc *Hint*: Be careful; the stagnation poi

8.25 Gas is stored in an undergı The streamline pattern in porous m flow. A well is placed in the plar impervious rock layers which interse uniformly throughout the layer. Fo mine the velocity that would be e method images.)



8.26 We wish to determine the channel, with the channel height cylinder. Superimpose a uniform f show with the use of a digital comp

8.27 A potential flow into a s Superimpose a large number of sinl along the x-axis.



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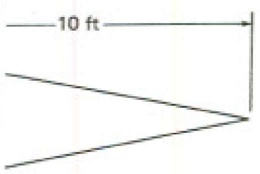

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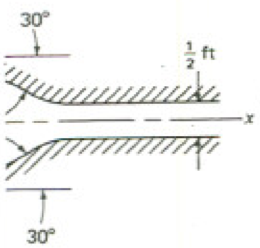
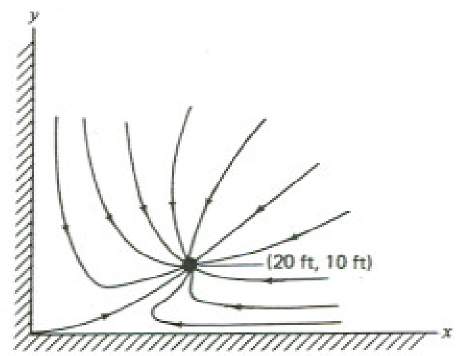


22

8.23 Determine the rotational speed Ω at which a 2-in.-dia. cylinder would have to rotate in a 20-fps free-stream flow so that only one stagnation point would exist on the cylinder. Sketch the streamline pattern.

8.24 A 2-ft.-dia. cylinder is rotated at 400 rpm in the flow moving at 10 fps. Locate the stagnation point (or points), and sketch the streamline pattern. *Hint*: Be careful; the stagnation point may not be on the cylinder.

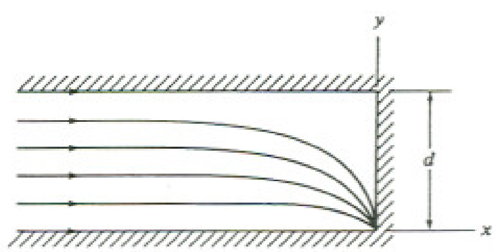
8.25 Gas is stored in an underground storage and pumped out as needed. The streamline pattern in porous media can be approximated by a potential flow. A well is placed in the plane layer containing the gas, next to two impervious rock layers which intersect at right angles. The well extracts the gas uniformly throughout the layer. For an extraction rate of 5 $ft^3/sec/ft$, determine the velocity that would be expected along the x-axis. (*Hint*: Use the method images.)



Prob. 8.25

8.26 We wish to determine the potential-flow field around a cylinder in a channel, with the channel height large compared with the diameter of the cylinder. Superimpose a uniform flow and a large number of doublets and show with the use of a digital computer that such a flow results.

8.27 A potential flow into a sink in the end of a channel is desired. Superimpose a large number of sinks to give the desired flow. Determine $u(x)$ along the x-axis.



Prob. 8.27

8.28 Derive an expression for the stream function for a point source and sketch a streamtube containing two neighboring streamlines. Also sketch a potential surface for the point source.

8.29 Determine the velocity at distances of 2 ft and 2 inches from a point sink if fluid is being withdrawn from a reservoir at a rate of 10 ft^3/sec. The sink is removed from any solid boundaries.

8.30 Determine the cartesian components u, v, w of the velocity vector for a point source. Show that they satisfy the continutiy equation $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$.

8.31 A point sink is placed 2 ft from an impervious boundary. Determine the velocity distribution along the stagnation line from the sink to the boundary. The extraction rate is 6 ft^3/sec.

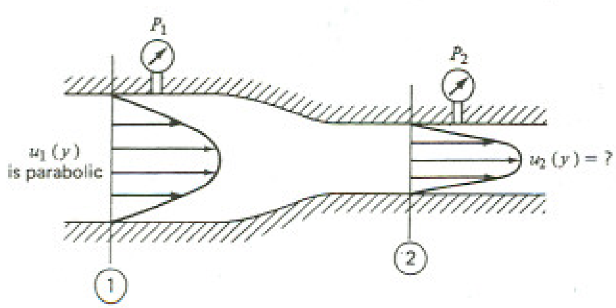
8.32 Verify the expression (8.100) for the streamfunction ψ for axisymmetric potential flow around a sphere. Determine the maximum velocity on a 8-in.-dia. sphere placed in a 20-fps uniform flow.

8.33 A point source of strength q is placed in a uniform flow. Determine the y-intercept of the half-body formed, the location of the stagnation point, and the asymptotic dimension of the axisymmetric body. Use $U = 10$ fps and $q = 200$ cfs.

8.34 Air flows over a large spherical weather balloon. Estimate the pressure drag if the flow can be approximated by a potential flow on the front half. The flow separates at the position of lowest pressure, and this low pressure is assumed to exist over the entire rear area of the sphere. See Fig. 8.15b for a similar type of flow. Use $U = 50$ mph, $r_0 = 10$ ft, and $\rho = 0.0024$ slug/ft^3.

8.35 Start with the differential momentum equation written in component form and, using rectangular cartesian coordinates, show that for a two-dimensional plane flow $D\zeta/Dt = 0$. (Differentiate the x-equation with respect to y and the y-equation with respect to x and subtract the resulting equations).

8.36 Using Eq. 8.106, argue that inviscid flow after a two-dimensional contraction cannot exist as shown. What's wrong with it? Assume that the parabolic profile is generated by viscosity but that through the short contraction viscous effects are negligible. Sketch the probable velocity distribution at section 2.



Prob. 8.36

8.37 For the contraction sho
profile with a linear distribution
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find the downstream location of
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