



AIAA 97-2717

**An Improved Time-Dependent
Flowfield Solution
for Solid Rocket Motors**

Joseph Majdalani
Marquette University
Milwaukee, WI 53233

Conference and Exhibit
6–9 July 1997
Seattle, WA

AN IMPROVED TIME-DEPENDENT FLOWFIELD SOLUTION
FOR SOLID ROCKET MOTORS

J. Majdalani*

Marquette University, Milwaukee, WI 53233

and

W. K. Van Moorhem†

University of Utah, Salt Lake City, UT 84112

Abstract

A model of the time-dependent velocity field in solid rocket motors is derived analytically for an oscillatory field that is subject to steady sidewall injection. The oscillatory pressure amplitude is assumed to be small by comparison to the mean pressure. The mathematical approach includes solving the momentum equation governing the rotational flow using separation of variables and multiple-scales. This requires identifying scales at which unsteady inertia, convection, and diffusion are significant. A composite scale is obtained that combines three disparate scales. The time-dependent axisymmetric solution obtained incorporates the effects of unsteady inertia, viscous diffusion, and the radial and axial convection of unsteady vorticity by Culick's mean flow components. The resulting agreement with the numerical solution to the momentum equation is remarkable. The uncertainty in a short analytical expression is found to be smaller than the injection Mach number, which represents the error associated with the mathematical model itself. The multiple-scales solution agrees extremely well with Flandro's recent flowfield solution. The present solution has the advantage of being shorter, more manageable in extracting quantities of interest, and capable of showing the significance of physical parameters on the solution.

Nomenclature

a_0 = mean stagnation sound speed, $\sqrt{\gamma p_0 / \rho_0}$
 A_p = dimensional oscillatory pressure amplitude
 k_m = wave number, $m\pi R / L = \omega_0 R / a_0$
 L = internal chamber length
 M_b = wall injection Mach number, V_b / a_0
 p_0 = mean chamber pressure, $\rho_0 a_0^2 / \gamma$
 p = dimensionless pressure, p^* / p_0

r = radial position, $r^* / R = (1 - y)$
 R = dimensional effective radius
 Re_k = kinetic Reynolds no., $\omega_0 R^2 / \nu_0 = k_m / \delta^2$
 S_p = penetration no., $V_b^3 \omega_0^{-2} \nu_0^{-1} R^{-1} = M_b^3 k_m^{-2} \delta^{-2}$
 St = Strouhal number, $\omega_0 R / V_b = k_m / M_b$
 t = dimensionless time, $t^* a_0 / R = \bar{t} / k_m$
 \bar{t} = dimensionless time, $k_m t = 2\pi f_m t^*$
 U = Culick's mean flow velocity, (U_r, U_z)
 U_r = Radial mean flow velocity, $-r^{-1} \sin \theta$
 U_z = Axial mean flow velocity, $\pi z \cos \theta$
 \hat{u} = dimensionless acoustic velocity, \hat{u}^* / a_0
 \tilde{u} = dimensionless vortical velocity, \tilde{u}^* / a_0
 $\mathbf{u}^{(1)}$ = total unsteady velocity, $\mathbf{u}^{*(1)} / a_0 = (\hat{u} + \tilde{u})$
 $\bar{\mathbf{u}}(r, z)$ = part of the vortical velocity, $\tilde{\mathbf{u}}(r, z, t) e^{ik_m t}$
 V_b = radial injection speed at the wall
 y = distance from the wall, $y^* / R = (1 - r)$
 r_1 = modified scale in multiple-scales procedures
 z = axial distance from the head-end, z^* / R
 δ^2 = the inverse of Re , $\nu_0 / (a_0 R) = 1 / Re$
 δ_s = Stokes' characteristic thickness, $\sqrt{\nu_0 / \omega_0}$
 ε = small parameter, $Re_k^{-1} = \delta_s^2 / R^2$
 ε_w = pressure wave amplitude, A_p / p_0
 γ = mean ratio of specific heats
 λ_n = separation constant, $(2n + 1; n = 0, 1, \dots)$
 ν_0 = chamber fluid mean kinematic viscosity
 θ = characteristic variable, $\pi r^2 / 2$
 ρ_0 = chamber mean density
 ω_m = dimensionless circular frequency, $m\pi R / L$
 ω_0 = dimensional circular frequency, $m\pi a_0 / L$
 ξ = viscous parameter, $S_p^{-1} = k_m^2 \delta^2 / M_b^3$

Subscripts

0, 1, ... = denotes the order of approximation
 r, z = denotes a radial or axial component

*Assistant Professor, Department of Mechanical and Industrial Engineering. Member AIAA.

†Professor, Department of Mechanical Engineering. Senior Member AIAA.

Copyright © 1997 by J. Majdalani and W.K. Van Moorhem. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission.

Superscripts

- (0),(1),... = denotes the perturbation level
* = asterisk denotes a dimensional quantity
^ = circumflex denotes acoustic oscillations
~ = tilde denotes vortical oscillations

Introduction

The time-dependent velocity field plays an essential role in the accurate prediction of combustion instability in solid rocket motors. This is due to the mathematical nature of the stability integrals whose reliability depends on a precise representation of the velocity field.¹ This has instigated a number of researchers to investigate the true nature of the velocity field in order to improve the predictive capability of combustion stability programs. With such a goal in mind, Brown et al² designed an elegant experiment to study the character of an oscillatory flow including sidewall injection of Nitrogen through fine sintered bronze walls. Vuillot et al,³ Roach et al,^{4,5} and Sabnis et al,⁶ made good use of various computational tools to study and explain the nature of the internal flowfield. Recent theoretical and computational studies of the transient evolution of the velocity field prescribed by harmonic endwall⁷ and sidewall⁸ disturbances by Kassoy et al have also been undertaken. The above-mentioned studies concurred that flow rotationality was a significant element that must be accounted for in order to improve the widely used one-dimensional acoustic model.

Credit must be given to Flandro who was among the earliest investigators to realize that the simple one-dimensional irrotational solution⁹ used to model the time-dependent field did not incorporate the effects of three-dimensionality, mean flow interaction, convection of vorticity, and viscous diffusion, which could play significant roles in the prediction of combustion stability. He therefore attempted to solve the momentum equation governing the rotational time-dependent flowfield analytically. Due to mathematical complexities, Flandro^{10,11} realized at first that some terms needed to be sacrificed and others approximated in order to extract useful analytical solutions. His first approach¹¹ retained the radial convection of vorticity and viscous diffusion while ignoring the axial convection of vorticity. In order to solve the resulting equation exactly, Culick's radial mean flow component¹² had to be approximated to its constant value at the wall. The resulting solution provided much insight despite its limitation of being axially independent and applicable to a narrow region near the wall. As an improvement on Flandro's first solution,¹¹ the authors^{13,14} employed multiple-scales expansions^{15,16} to extract an axially independent analytical expression that utilized the exact mean flow profile. As a result, its

applicability was not limited to a thin region near the wall as with the previous solution. However, it was limited in that it did not incorporate the axial dependence. In his second attempt, Flandro¹⁷ neglected viscous diffusion and retained both radial and axial convection terms representing the transport of unsteady vorticity by Culick's mean flow component.¹² The resulting solution incorporated an axial dependence despite its limitation of being inviscid. Finally, in his third approach, Flandro¹⁸ extracted a rotational and viscous solution that incorporated all of the important mechanisms. The last two approaches used regular perturbations and the vorticity equation. In the process, velocity was derived from vorticity which was determined first. Flandro's last solution was very accurate since it depended on a perturbation parameter that is very small in rocket motors. The only undesirable feature of Flandro's last solution is that it resulted in a complicated expression that did not show readily how the solution depended on the various independent parameters involved.

In the following analysis, a multiple-scales procedure will be employed to derive the time-dependent axisymmetric velocity field directly from the momentum equation. Beginning with the full momentum equation, a perturbation solution is obtained to the order of the Mach number. The uniformly valid solution will be shown to be very accurate, matching the numerical solution everywhere. Its advantage relative to previous models is that it is short, accurate, capable of indicating the role of various physical parameters, and incorporating all of the intricate physical mechanisms involved in the problem, including the axial dependence.

Analysis

Reminiscent of a previously used approach,¹⁴ the analysis will follow typical assumptions employed when modeling the internal flowfield in rocket motors. In the process, the total velocity is separated into two parts, $\mathbf{u} = M_b \mathbf{U} + \mathbf{u}^{(1)}$: 1) a steady flow component ($M_b \mathbf{U}$) given by Culick's profile,¹² and a time-dependent component $\mathbf{u}^{(1)}$ that we propose to find.

Fundamental Equations

The first order time-dependent motion is governed by the viscous momentum equation^{10,17}

$$\frac{\partial \mathbf{u}^{(1)}}{\partial t} + M_b \left[\nabla (\mathbf{u}^{(1)} \cdot \mathbf{U}) - \mathbf{u}^{(1)} \times (\nabla \times \mathbf{U}) - \mathbf{U} \times (\nabla \times \mathbf{u}^{(1)}) \right] \\ = \delta^2 \left[\frac{4}{3} \nabla (\nabla \cdot \mathbf{u}^{(1)}) - \nabla \times (\nabla \times \mathbf{u}^{(1)}) \right] - \frac{\nabla p^{(1)}}{\gamma} \quad (1)$$

The total unsteady velocity vector can be broken up into acoustic (compressible, inviscid, irrotational) and solenoidal (incompressible, viscous, rotational) parts

$$\mathbf{u}^{(1)} = \hat{\mathbf{u}} + \tilde{\mathbf{u}} \quad (2)$$

Substituting Eq. (2) into Eq. (1), two equations result

$$\begin{aligned} \frac{\partial \hat{\mathbf{u}}}{\partial t} = & -\frac{\nabla \hat{p}}{\gamma} - M_b [\nabla(\hat{\mathbf{u}} \cdot \mathbf{U}) - \hat{\mathbf{u}} \times (\nabla \times \mathbf{U})] \\ & + \frac{4}{3} \delta^2 \nabla(\nabla \cdot \hat{\mathbf{u}}) \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}}{\partial t} + M_b [\nabla(\tilde{\mathbf{u}} \cdot \mathbf{U}) - \tilde{\mathbf{u}} \times (\nabla \times \mathbf{U}) - \mathbf{U} \times (\nabla \times \tilde{\mathbf{u}})] \\ = -\delta^2 [\nabla \times (\nabla \times \tilde{\mathbf{u}})] \end{aligned} \quad (4)$$

Equation (3) yields the first order acoustic term $\hat{\mathbf{u}}$ that is given by the plane wave solution^{5,11,17}

$$\hat{\mathbf{u}}(z, t) = i \frac{\varepsilon_w}{\gamma} \sin(k_m z) \exp(-ik_m t) \mathbf{e}_z \quad (5)$$

$$\hat{p}(z, t) = \varepsilon_w \cos(k_m z) \exp(-ik_m t) \quad (6)$$

Equation (4), which we propose to solve, governs the rotational component $\tilde{\mathbf{u}}$, where

$$\tilde{\mathbf{u}} = \tilde{u}_r \mathbf{e}_r + \tilde{u}_z \mathbf{e}_z \quad (7)$$

The Unsteady Rotational Momentum Equation

When Eq. (7) is substituted back into Eq. (4), the radial and axial components of the rotational momentum equation are obtained

$$\begin{aligned} \frac{\partial \tilde{u}_r}{\partial t} = & -M_b \left[\frac{\partial}{\partial r} (\tilde{u}_r U_r) + U_z \frac{\partial \tilde{u}_r}{\partial z} + \tilde{u}_z \frac{\partial U_r}{\partial z} \right] \\ & + \frac{\delta^2}{r} \frac{\partial}{\partial z} \left[r \left(\frac{\partial \tilde{u}_r}{\partial z} - \frac{\partial \tilde{u}_z}{\partial r} \right) \right] \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\partial \tilde{u}_z}{\partial t} = & -M_b \left[\frac{\partial}{\partial z} (\tilde{u}_z U_z) + U_r \frac{\partial \tilde{u}_z}{\partial r} + \tilde{u}_r \frac{\partial U_z}{\partial r} \right] \\ & + \delta^2 \left(\frac{\partial^2 \tilde{u}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{u}_z}{\partial r} - \frac{1}{r} \frac{\partial \tilde{u}_r}{\partial z} - \frac{\partial^2 \tilde{u}_r}{\partial r \partial z} \right) \end{aligned} \quad (9)$$

Subject to later theoretical verification, the magnitude of \tilde{u}_r is assumed to be of order M_b . Full numerical solutions and experimental measurements of the radial component also support this notion.²⁻⁶ Being a small quantity, ignoring this component at the first perturbation expansion level of the rotational velocity will not affect the solution, which is only accurate to the first order in the Mach number. Dropping \tilde{u}_r from Eq. (9), the rotational momentum equation governing the axial velocity component \tilde{u}_z becomes

$$\begin{aligned} \frac{\partial \tilde{u}_z}{\partial t} = & -M_b \left[\frac{\partial}{\partial z} (\tilde{u}_z U_z) + U_r \frac{\partial \tilde{u}_z}{\partial r} \right] \\ & + \delta^2 \left(\frac{\partial^2 \tilde{u}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{u}_z}{\partial r} \right) \end{aligned} \quad (10)$$

Solution by Separation of Variables

Letting $\tilde{u}_z = \bar{u}_z(r, z) \exp(-ik_m t)$, Eq. (10) becomes

$$\begin{aligned} ik_m \bar{u}_z(r, z) = & M_b \left[\frac{\partial}{\partial z} (\bar{u}_z U_z) + U_r \frac{\partial \bar{u}_z}{\partial r} \right] \\ & - \frac{\delta^2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}_z}{\partial r} \right) \end{aligned} \quad (11)$$

Rearranging Eq. (11) and expressing the resulting constants in terms of the nondimensional Strouhal and kinetic Reynolds numbers, the result is

$$\begin{aligned} i \bar{u}_z(r, z) = & \frac{1}{St} \left[\bar{u}_z \frac{\partial U_z}{\partial z} + U_z \frac{\partial \bar{u}_z}{\partial z} + U_r \frac{\partial \bar{u}_z}{\partial r} \right] \\ & - \frac{1}{Re_k} \left(\frac{\partial^2 \bar{u}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_z}{\partial r} \right) \end{aligned} \quad (12)$$

Substituting Culick's steady flow velocity,¹² using ε instead of the inverse of Re_k , and rearranging so that z -dependent terms are isolated, one gets

$$\begin{aligned} z \frac{\partial \bar{u}_z}{\partial z} = & \frac{St}{\pi} \sec\left(\frac{\pi}{2} r^2\right) \left\{ \left[i - \frac{\pi}{St} \cos\left(\frac{\pi}{2} r^2\right) \right] \bar{u}_z \right. \\ & \left. + \frac{1}{rSt} \sin\left(\frac{\pi}{2} r^2\right) \frac{\partial \bar{u}_z}{\partial r} + \varepsilon \left(\frac{\partial^2 \bar{u}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_z}{\partial r} \right) \right\} \end{aligned} \quad (13)$$

Equation (13) suggests using separation of variables in order to investigate a solution of the form

$$\bar{u}_z(r, z) = V(r)Z(z) \quad (14)$$

Substituting Eq. (14) into Eq. (13) and rearranging to show that terms which are purely functions of r or z can be isolated, Eq. (13) becomes

$$\begin{aligned} \frac{z}{Z} \frac{dZ}{dz} = & \frac{St}{\pi V} \sec\left(\frac{\pi}{2} r^2\right) \left\{ \left[i - \frac{\pi}{St} \cos\left(\frac{\pi}{2} r^2\right) \right] V \right. \\ & \left. + \frac{1}{rSt} \sin\left(\frac{\pi}{2} r^2\right) \frac{dV}{dr} + \frac{\varepsilon}{r} \frac{d}{dr} \left(r \frac{dV}{dr} \right) \right\} = \lambda_n \end{aligned} \quad (15)$$

where it can be easily shown that the separation constant λ_n must be strictly positive for a physically acceptable solution. For every value of the eigenvalue λ_n , a solution to the axially dependent and radially dependent eigenfunctions $Z_n(z)$ and $V_n(r)$ must be

obtained. Integration of the axially dependent equation is straightforward. The exact result is

$$\frac{dZ_n}{dz} - \frac{\lambda_n}{z} Z_n = 0 \Rightarrow Z_n(z) = c_n z^{\lambda_n} \quad (16)$$

where c_n is an integration constant corresponding to a given value of the separation eigenvalue. Since the governing equation is linear, any linear combination of two or more solutions, corresponding to two or more separation constants, is also a solution. So one can write, in general, for all possible eigenvalues λ_n

$$\bar{u}_z(r, z) = \sum_{\lambda_n} c_n z^{\lambda_n} V_n(r) \quad (17)$$

where λ_n is to be determined from no-slip boundary condition at the wall. This is achieved when, at the wall, the rotational and irrotational components of the time-dependent velocity cancel out:

$$\mathbf{u}^{(1)} = \mathbf{0} \text{ or } \tilde{\mathbf{u}}_z = -\hat{\mathbf{u}}_z \quad (18)$$

which translates into

$$\bar{u}_z \exp(-ik_m t) = -\frac{\varepsilon_w}{\gamma} i \sin(k_m z) \exp(-ik_m t) \quad (19)$$

$$\Rightarrow \bar{u}_z \Big|_{r=1} = -\frac{\varepsilon_w}{\gamma} i \sin(k_m z) \quad (20)$$

Inserting Eq. (20) into Eq. (17) and writing out the MacLaurin series expansion for the sine function, the following equality must be true

$$\sum_{\lambda_n} c_n z^{\lambda_n} V_n \Big|_{r=1} = -\frac{\varepsilon_w}{\gamma} i \sum_{n=0}^{\infty} \frac{(-1)^n (k_m z)^{2n+1}}{(2n+1)!} \quad (21)$$

Clearly, the left and right-hand sides of Eq. (21) will be identical when

$$\lambda_n = 2n+1, \quad n = 0, 1, 2, \dots \quad (22)$$

$$c_n = -\frac{\varepsilon_w}{\gamma} i \frac{(-1)^n (k_m)^{2n+1}}{(2n+1)!} \quad (23)$$

$$V_n(1) = 1 \quad (24)$$

Substituting back Eqs. (22-23), Eq. (17) becomes

$$\bar{u}_z(r, z) = -\frac{\varepsilon_w}{\gamma} i \sum_{n=0}^{\infty} \frac{(-1)^n (k_m z)^{2n+1}}{(2n+1)!} V_n(r) \quad (25)$$

The velocity eigenfunction V_n is left to be determined from the separable, radially dependent part of Eq. (15):

$$\varepsilon \left(\frac{d^2 V_n}{dr^2} + \frac{1}{r} \frac{dV_n}{dr} \right) + \frac{1}{rSt} \sin\left(\frac{\pi}{2} r^2\right) \frac{dV_n}{dr}$$

$$+ \left[i - \frac{\pi}{St} (1 + \lambda_n) \cos\left(\frac{\pi}{2} r^2\right) \right] V_n = 0 \quad (26)$$

which is subject to two boundary conditions:

$$V_n \Big|_{r=1} = 1 \text{ (no-slip at the wall)} \quad (27)$$

$$\frac{dV_n}{dr} \Big|_{r=0} = 0 \text{ (centerline symmetry)} \quad (28)$$

The Radially-Dependent Velocity Function

Integration of Eq. (26) is not straightforward. The inherent difficulty in analytically integrating this second order ODE stems from the existing variable coefficients which, unless linearized, do not appear to permit an exact solution. This impediment is behind the multiple formulations of one-dimensional,^{11,13,14} two-dimensional inviscid,¹⁷ and two-dimensional viscous,¹⁸ models, whose purpose was, in reality, to overcome this mathematical obstacle using different levels of approximations. As will be shown here, this difficulty can be circumvented by the multiple-scales approach, provided that the proper characteristic scales are known. The multiple-scales technique will be capable of integrating the radially dependent momentum equation to a relatively high degree of accuracy, theoretically, to the order of ε^2 , which is more than sufficiently accurate, since ε is always much smaller than M_b .

The Multiple-Scales Procedure

The key issue for this method to work is a prior knowledge of the location and form of the characteristic scales. To that end, the first step will be to determine the local coordinate transformation(s) that can be associated with particular phenomena which dominate in specific regions of interest.

Characteristic Scales

Mathematically, the characteristic scales are those that are capable of providing a balance in the governing equation between terms representing the interacting phenomena that are appreciable in a given region. Physically, they represent the scale at which order the momentum exchange between dominating phenomena, such as convection, diffusion, or inertia takes place in reality. As will be demonstrated below, four distinct regions with known characteristic scales are identified. Three of them correspond to the near-wall, centerline, and buffer regions. Both centerline and buffer regions are characterized by similar magnified scales, and occur in a zone that is far from the solid boundary. The near-wall region, which is highly rotational, is characterized by a compressed scale. Between the two zones, a transition zone exists where all existing physical

phenomena are in balance. This region is characterized by a base scale that is neither stretched nor compressed, corresponding to the unmodified spatial coordinate itself. In what follows, a systematic method that allows identifying each particular scale will be presented.

A Near-Wall Scale

Near the wall, unsteady inertia and mean flow convection dominate. Changes in the oscillatory velocity in this region are slow, with almost constant amplitude. Instead of stretching the spatial coordinate near the wall, a compression of the coordinate is required here since the oscillatory amplitude decays slowly near the wall where the effect of blowing is appreciable. In compressing the spatial dimension, a transformation of the independent variable is needed. Introducing the near-wall transformation variable, $r_1 = \varepsilon(1-r)$, a balance between unsteady inertia and convection is achieved in Eq. (26) since the actual velocity gradient near the wall is always large¹⁴:

$$\begin{aligned} \frac{\varepsilon}{St} U_r \frac{dV_n}{dr_1} + \left[i - \frac{\pi}{St} (1 + \lambda_n) \cos\left(\frac{\pi}{2} r^2\right) \right] V_n \\ = -\varepsilon^3 \left(\frac{d^2 V_n}{dr_1^2} - \frac{1}{\varepsilon - r_1} \frac{dV_n}{dr_1} \right) \end{aligned} \quad (29)$$

A Centerline Scale

Approaching the centerline, blowing effects become negligible since the steady radial velocity becomes very small. The unsteady velocity amplitude approaches the acoustic amplitude. Instead of compressing the radial coordinate, a stretching or magnification of the scale is required here, since the wavelength, which is controlled by the mean flow velocity, becomes vanishingly small. An expansion of the region is hence required to account for small, yet rapid spatial changes in the oscillatory amplitude. Introducing the scale, $r_1 = r^q \varepsilon^{-1}$, or its inverse $r_1 = \varepsilon r^{-q}$, a balance between unsteady inertia and diffusion can be achieved in Eq. (26):

$$\begin{aligned} q^2 \varepsilon^{1-2/q} \left(r_1^{2(1+1/q)} \frac{d^2 V_n}{dr_1^2} + r_1^{(1+2/q)} \frac{dV_n}{dr_1} \right) \\ + \left[i - \frac{\pi}{St} (1 + \lambda_n) \cos\left(\frac{\pi}{2} r^2\right) \right] V_n \\ = -\frac{q}{St} \varepsilon^{-1/q} r_1^{(1+1/q)} U_r \frac{dV_n}{dr_1} \end{aligned} \quad (30)$$

which shows that, when $\varepsilon^{1-2/q} = 1$, or $q = 2$, the first two terms representing viscous and inertial forces will be of the same order of magnitude where U_r vanishes.

A Transition Scale

The scaling analysis also shows that there exists a buffer region adjacent to the centerline region where viscous and mean flow convection forces are in balance. When the radial mean velocity component is no more negligible, a value of $q = 1$ will characterize this layer of undetermined size. In this buffer region that is sufficiently removed from the solid surface, the same scaling analysis shows that mean flow convection and viscous diffusion become of the same order. Blowing effects become as important as viscous diffusion. In order to expand the transition region, a less intense stretching of the scale is required by comparison to the centerline scale. Introducing a magnified scale of a form that is similar to the centerline scale, a balance between convection and viscous diffusion can be achieved in Eq. (26):

$$\begin{aligned} q^2 \varepsilon^{1-2/q} \left(r_1^{2(1+1/q)} \frac{d^2 V_n}{dr_1^2} + r_1^{(1+2/q)} \frac{dV_n}{dr_1} \right) \\ + \frac{q}{St} \varepsilon^{-1/q} r_1^{(1+1/q)} U_r \frac{dV_n}{dr_1} \\ = - \left[i - \frac{\pi}{St} (1 + \lambda_n) \cos\left(\frac{\pi}{2} r^2\right) \right] V_n \end{aligned} \quad (31)$$

where it can be easily seen that convective and viscous force terms will be of the same order of magnitude when $\varepsilon^{1-2/q} = \varepsilon^{-1/q}$, or $q = 1$.

An Unmodified Base Scale

In a region extending from the edge of the buffer region to the edge of the near-wall region, all three physical phenomena, namely, convection, diffusion, and inertia, need to be included in order to account for the resulting fluid motion. Mathematically, all the terms in Eq. (26) must be retained. The base scale at which level all three physical phenomena are in balance will correspond to the spatial coordinate itself. The viscous term is no more negligible because, in this region, velocity gradients vary rapidly in the radial direction due to a reduction in the spatial wavelength. Hence, for $r_0 = r$, Eq. (26) is unaltered.

Note that, from the above scaling analysis, viscous forces appear to play a significant role in a region that is “blown off” the wall by the strong radial mean flow component. Due to radial injection, the traditionally thin and highly viscous region, which usually occurs at the wall in no-injection flowfields, takes the form here of a free unlocalized shear layer separating the rotational from the irrotational flow. This result is very similar to that of the Blowhard problem and is well described by Cole and Aroesty¹⁹ who analyzed a steady flow with sidewall injection.

A Composite Scale

Since the problem involves events occurring at three different scales, other than the base scale, a strict multiple-scales technique will require utilizing four scales, and integrating three times up to the third term in order for a one-term solution to be feasible.¹⁵ It can be easily shown that, in this particular problem, for a complete analytical expression to be possible, the method of multiple-scales will have to be limited to two scales only. Hence, a solution which is uniformly valid in the entire range of interest is reachable if a composite scale can be devised to provide the same geometric description that is available from the near-wall, transition and centerline scales in their regions of applicability. Therefore, a composite scale that is valid in the entire region extending from the centerline to the wall must be constructed such that

$$s(r) = \frac{(1-r)}{r^q} \approx \begin{cases} 1-r & (\text{as } r \rightarrow 1) \\ \frac{1}{r} & (\text{in the buffer region}) \\ \frac{1}{r^2} & (\text{as } r \rightarrow 0) \end{cases} \quad (35)$$

If the stretching coefficient^{13,14} q is defined to be a spatial function which approaches 0 in the neighborhood of $r = 1$, 1 in the buffer region, and 2 at the centerline, then a controllable agreement with the exact solution to Eq. (26) can be achieved everywhere ($0 \leq r \leq 1$). It is found that, using a variable stretching coefficient that is dependent on the geometry involved, such as $q(r) = q(1-r)^g$, minimizes the error between the numeric and multiple-scales solutions. Fortunately, the parameters involved in the variable stretching coefficient $q(r)$ can be determined to any degree of desired precision. For instance, the error between the numerical and analytical predictions is found to be less than 1% when $q = g = 3/2$.

The Composite-Scale Solution

One major advantage of the present method is its accuracy; using a combination of two scales, the zeroth order term in this solution will have the same order as a two-term solution using regular perturbations.¹⁵ Another advantage is its compactness. Unlike regular perturbations where the solution to the first order is a linear combination of two terms, the first order solution here is only one term that contains information from both the zeroth order and first order perturbed equations as well. This so-called derivative-expansion method will consist of expanding the derivatives, as well as the dependent variables in powers of the small parameter ε , in order to attain uniformly valid expansions.¹⁵ In

what follows, a generalized solution will be derived that is applicable to any characteristic scale. From the generalized expression, solutions corresponding to any characteristic scale can be obtained.

A Generalized Two-Term Expansion

The two scales to be used in Eq. (26) are: 1) the base $r_0 \equiv r$, and 2) the modified scale $r_1 \equiv \varepsilon s(r) \equiv \varepsilon s(r_0)$, written in the most general form for any scale function corresponding to a transformation of the coordinate in the near-wall, transition, or centerline regions. The value of the modified scale can thus be any one of

$$s(r_0) = \begin{cases} 1-r_0 & (\text{near - wall}) \\ r_0^{-q} & (\text{near - center}) \\ (1-r_0)r_0^{-q(r_0)} & (\text{composite}) \end{cases} \quad (33)$$

Using the chain rule for differentiation, and carrying as many terms as will be needed to retain a final order of ε^2 , the derivatives can be expanded as follows

$$\frac{d}{dr} = \frac{\partial}{\partial r_0} + \varepsilon \frac{ds}{dr_0} \frac{\partial}{\partial r_1} + O(\varepsilon^2); \quad \frac{d^2}{dr^2} = \frac{\partial^2}{\partial r_0^2} + O(\varepsilon) \quad (34)$$

After substitution of Eq. (34) into Eq. (26), the latter is transformed into a partial differential equation that is function of two independent coordinates r_0 and r_1

$$\varepsilon \left(\frac{\partial^2 V_n}{\partial r_0^2} + \frac{1}{r_0} \frac{\partial V_n}{\partial r_0} \right) - \frac{U_r}{St} \frac{\partial V_n}{\partial r_0} - \varepsilon \frac{U_r}{St} \frac{ds}{dr_0} \frac{\partial V_n}{\partial r_1} + \left[i - \frac{\pi}{St} (1 + \lambda_n) \cos\left(\frac{\pi}{2} r_0^2\right) \right] V_n + O(\varepsilon^2) = 0 \quad (35)$$

Following the multiple-scales procedure, the velocity is now written using a two-term expansion which is function of the two independent scales:

$$V_n(r_0, r_1) = V_n^{(0)}(r_0, r_1) + \varepsilon V_n^{(1)}(r_0, r_1) + O(\varepsilon^2) \quad (36)$$

Substituting the perturbed two-term expansion given by Eq. (36) into Eq. (35), rearranging and collecting terms of order ε^0 and ε^1 , two first order coupled PDEs must be satisfied for any value of ε . These are

$$\begin{aligned} & -\frac{U_r}{St} \frac{\partial V_n^{(0)}}{\partial r_0} + \left[i - \frac{\pi}{St} (1 + \lambda_n) \cos\left(\frac{\pi}{2} r_0^2\right) \right] V_n^{(0)} = 0 \quad (37) \\ & -\frac{U_r}{St} \frac{\partial V_n^{(1)}}{\partial r_0} + \left[i - \frac{\pi}{St} (1 + \lambda_n) \cos\left(\frac{\pi}{2} r_0^2\right) \right] V_n^{(1)} \\ & = \frac{U_r}{St} \frac{ds}{dr_0} \frac{\partial V_n^{(0)}}{\partial r_1} - \frac{\partial^2 V_n^{(0)}}{\partial r_0^2} - \frac{1}{r_0} \frac{\partial V_n^{(0)}}{\partial r_0} \end{aligned} \quad (38)$$

The boundary conditions given by Eqs. (27-28) for the velocity V_n can be translated to the first perturbation term $V_n^{(0)}$ which must satisfy the same condition as V_n in the limit when ε approaches zero, as can be clearly seen from Eq. (36). The resulting boundary conditions that must be satisfied by $V_n^{(0)}$ are

$$V_n^{(0)}(r_0 = 1) = 1 \quad (\text{no-slip}) \quad (39)$$

$$\frac{\partial V_n^{(0)}}{\partial r_0}(r_0 = 0) = 0 \quad (\text{axial symmetry}) \quad (40)$$

Nonsecular Integration

The next step in the multiple-scales technique is to integrate Eq. (37), which is solely a function of the first scale, for the zeroth term solution. The integration constant that must be a function of the second scale will have to be determined after substitution into Eq. (38) and examining the possibility of removing secular terms which, if retained, will cause the solution to become nonuniformly valid. Complete closure is later accomplished by applying the boundary conditions.

First Integration

Equation (37) is a homogeneous first order PDE with a variable coefficient that is solely a function of r_0 .

Using $\theta_0 = \pi r_0^2 / 2$, direct integration yields

$$V_n^{(0)} = C_1(r_1) \exp\left[\left(1 + \lambda_n\right) \ln \sin \theta_0 - i \frac{St}{\pi} \ln \tan \frac{\theta_0}{2}\right] \quad (41)$$

Since integration is carried out with respect to r_0 only, the constant of integration C_1 can, in general, be a function of r_1 as well.

Removing Secular Terms

First and second partial derivatives of $V_n^{(0)}$ are

$$\begin{aligned} \frac{\partial V_n^{(0)}}{\partial r_0} &= \left[-iSt r_0 \csc \theta_0 + \pi(1 + \lambda_n)r_0 \cot \theta_0\right] V_n^{(0)} \\ &= C_1(r_1) \left[-iSt r_0 \csc \theta_0 + \pi(1 + \lambda_n)r_0 \cot \theta_0\right] \\ &\quad \times \exp\left[\left(1 + \lambda_n\right) \ln \sin \theta_0 - i \frac{St}{\pi} \ln \tan \frac{\theta_0}{2}\right] \end{aligned} \quad (42)$$

$$\frac{\partial V_n^{(0)}}{\partial r_1} = \frac{dC_1}{dr_1} \exp\left[\left(1 + \lambda_n\right) \ln \sin \theta_0 - i \frac{St}{\pi} \ln \tan \frac{\theta_0}{2}\right] \quad (43)$$

$$\begin{aligned} \frac{\partial^2 V_n^{(0)}}{\partial r_0^2} &= \left\{-St^2 r_0^2 \csc^2 \theta_0 + \pi(1 + \lambda_n) \cot \theta_0\right. \\ &\quad \left.- iSt \csc \theta_0 \left[1 + \pi(1 + 2\lambda_n)r_0^2 \cot \theta_0\right]\right\} \end{aligned}$$

$$+ \pi^2(1 + \lambda_n)r_0^2(\lambda_n \cot^2 \theta_0 - 1)\} V_n^{(0)} \quad (44)$$

Now by substituting Eqs. (42-44) into Eq. (38), the right-hand-side becomes a source of secular terms. To remove secular terms which cause the solution to degenerate, a mathematical requirement that satisfies the symmetry condition, given by Eq. (40), is that the right-hand-side be zero; this implies that

$$\begin{aligned} &-\frac{U_r}{St} \frac{\partial V_n^{(1)}}{\partial r_0} + \left[i - \frac{\pi}{St}(1 + \lambda_n) \cos \theta_0\right] V_n^{(1)} \\ &= \left[\frac{U_r}{St} \frac{ds}{dr_0} \frac{dC_1}{dr_1} - C_1 \left\{-St^2 r_0^2 \csc^2 \theta_0\right. \right. \\ &\quad \left. \left.+ 2\pi(1 + \lambda_n) \cot \theta_0 + \pi^2(1 + \lambda_n)r_0^2(\lambda_n \cot^2 \theta_0 - 1)\right. \right. \\ &\quad \left. \left.- iSt \csc \theta_0 \left[2 + \pi(1 + 2\lambda_n)r_0^2 \cot \theta_0\right]\right\}\right] \\ &\quad \times \exp\left[\left(1 + \lambda_n\right) \ln \sin \theta_0 - i \frac{St}{\pi} \ln \tan \frac{\theta_0}{2}\right] \equiv 0 \end{aligned} \quad (45)$$

yielding a first order ODE that can be solved for C_1 :

$$\begin{aligned} &\frac{dC_1(r_1)}{dr_1} + C_1(r_1) St \left\{-St^2 r_0^2 \csc^2 \theta_0\right. \\ &\quad \left.+ \pi^2(1 + \lambda_n)r_0^2(\lambda_n \cot^2 \theta_0 - 1)\right. \\ &\quad \left.- iSt \csc \theta_0 \left[2 + \pi(1 + 2\lambda_n)r_0^2 \cot \theta_0\right]\right. \\ &\quad \left.+ 2\pi(1 + \lambda_n) \cot \theta_0\right\} r \csc \theta_0 \left(\frac{ds}{dr_0}\right)^{-1} = 0 \end{aligned} \quad (46)$$

Second Integration

Integration of Eq. (46), subject to satisfaction of the no-slip condition given by Eq. (39), allows the complete determination of C_1 . Using $\theta = \pi r^2 / 2$, the result is

$$\begin{aligned} C_1(r) &= \exp\left(\xi \left[r^3 \csc^3 \theta \eta(r) - \eta(1)\right] - \frac{\xi}{St^2} \pi(1 + \lambda_n)\right) \\ &\quad \times \left\{r \csc \theta \eta(r) \left[2 \cot \theta + \pi r^2(\lambda_n \cot^2 \theta - 1)\right] + \pi \eta(1)\right\} \\ &\quad + \frac{i\pi\xi}{St} \left\{r \csc^2 \theta \eta(r) \left[2 + \pi r^2 \cot \theta(1 + 2\lambda_n)\right] - 2\eta(1)\right\} \end{aligned} \quad (47)$$

The effective scale function η that appears in the solution is the ratio of the scale and its first derivative

$$\eta(r) = s(r) \left(\frac{ds}{dr}\right)^{-1} \quad (48)$$

From Eq. (36), V_n can be written to the order of ε^2 , for any effective scale η , and any eigenvalue λ_n . This is typically the case because information from both first

and second perturbation levels are included in the first term of the multiple-scales expansion for V_n :

$$\begin{aligned}
V_n(r) &= (\sin\theta)^{(1+\lambda_n)} \exp\left(\xi\left[r^3 \csc^3 \theta \eta(r) - \eta(1)\right]\right) \\
&+ i\pi \frac{\xi}{St} \left\{ 2r \csc^2 \theta \eta(r) \left[1 + \theta \cot \theta (1 + 2\lambda_n) \right] - 2\eta(1) \right\} \\
&\frac{-\xi}{St^2} \left\{ 2r \csc \theta \eta(r) \left[\cot \theta + \theta (\lambda_n \cot^2 \theta - 1) \right] + \pi \eta(1) \right\} \\
&\times \pi (1 + \lambda_n) - i \frac{St}{\pi} \ln \tan \frac{\theta}{2} \quad (49)
\end{aligned}$$

Uniformly Valid Solution

In this case, for $y = 1 - r$, $s(r) = yr^{-qy^g}$, we get

$$\begin{aligned}
\eta(r) &= -y \left[1 + qy^g (yr^{-1} - g \ln r) \right]^{-1}; \quad q = g = 3/2 \quad (50) \\
V_n(r) &= \sin^{1+\lambda_n} \theta \exp\left\{ \xi r^3 \csc^3 \theta \eta(r) - \frac{iSt}{\pi} \ln \tan \frac{\theta}{2} \right. \\
&\frac{-2\pi\xi}{St^2} (1 + \lambda_n) r \csc \theta \eta(r) \left[\cot \theta + \theta (\lambda_n \cot^2 \theta - 1) \right] \\
&\left. + 2i\pi \frac{\xi}{St} r \csc^2 \theta \eta(r) \left[1 + \theta \cot \theta (1 + 2\lambda_n) \right] \right\} \quad (51)
\end{aligned}$$

Description of the Radially Dependent Velocity

The radially dependent velocity function corresponding to the multiple-scales solution given by Eq. (51) is virtually indistinguishable from the numeric solution to Eq. (26). The function $V_n(r)$ is best described as an upward-traveling harmonic wave with an amplitude which suffers an exponential damping with increasing distance from the wall. This damping is found to depend on the penetration number, S_p or ξ^{-1} .

The Rotational Flowfield

Substituting Eq. (51) into Eq. (25), the result is

$$\begin{aligned}
\bar{u}_z(r, z) &= -\frac{\varepsilon_w}{\gamma} i \sum_{n=0}^{\infty} \frac{(-1)^n (k_m z)^{2n+1}}{(2n+1)!} (\sin\theta)^{(2n+2)} \\
&\times \exp\left\{ \xi r^3 \csc^3 \theta \eta(r) - i \frac{St}{\pi} \ln \tan \frac{\theta}{2} \right. \\
&\frac{-2\pi\xi}{St^2} (1 + \lambda_n) r \csc \theta \eta(r) \left[\cot \theta + \theta (\lambda_n \cot^2 \theta - 1) \right] \\
&\left. + 2i\pi \frac{\xi}{St} r \csc^2 \theta \eta(r) \left[1 + \theta \cot \theta (1 + 2\lambda_n) \right] \right\} \quad (52)
\end{aligned}$$

Infinite Series Solution

The rotational component \bar{u}_z can now be written in terms of an infinite series, including all the terms that originate from the full momentum equation (10).

Regrouping and inserting elementary trigonometric identities, the solution can be written in a form that clearly displays the dominating terms and the smaller terms of order St^{-2} :

$$\begin{aligned}
\bar{u}_z(r, z, t) &= -\frac{\varepsilon_w}{\gamma} i \sin\theta \sum_{n=0}^{\infty} \frac{(-1)^n (k_m z \sin\theta)^{2n+1}}{(2n+1)!} \\
&\times \exp\left(-ik_m t - i \frac{St}{\pi} \ln \tan \frac{\theta}{2} + \xi \eta r^3 \csc^3 \theta \left\{ 1 - \frac{\pi^2}{St^2} \right. \right. \\
&\times (1 + \lambda_n) \left[\cos 2\theta + \frac{\sin 2\theta}{2\theta} + (\lambda_n - 1) \cos^2 \theta \right] \left. \left. \right\} \right) \\
&+ 2i\pi \frac{\xi}{St} \eta r \csc^2 \theta \left[1 + \theta \cot \theta (1 + 2\lambda_n) \right] \quad (53)
\end{aligned}$$

Fortunately, this is a rapidly converging series that requires only two terms to yield results which compare extremely well with the solution corresponding to a very large number of terms.

Closed Form Solution

Analyzing carefully the infinite expression above shows that it can be written in a closed form by ignoring terms that are much smaller than the order of precision associated with the infinite series itself. The result is an alternative closed form expression which can be written in a compact form in terms of a spatial damping function ζ and a spatial phase angle Φ . A practically equivalent expression to Eq. (53) is hence

$$\bar{u}_z = -\frac{\varepsilon_w}{\gamma} i \sin\theta \sin(k_m z \sin\theta) \exp[\zeta - i(k_m t + \Phi)] \quad (54)$$

where zeroth and first order terms in ζ and Φ are

$$\zeta = \zeta_0 + \zeta_1, \quad \Phi = \Phi_0 + \Phi_1 \quad (55)$$

$$\zeta_0(r) = \xi \eta(r) r^3 \csc^3 \theta, \quad \Phi_0(r) = \frac{St}{\pi} \ln \tan \frac{\theta}{2} \quad (56)$$

$$\zeta_1(r) = -\frac{2\pi^2}{St^2} \xi \eta(r) r^3 \csc^3 \theta \left(\cos 2\theta + \frac{\sin 2\theta}{2\theta} \right) \quad (57)$$

$$\Phi_1(r) = -2\pi \frac{\xi}{St} \eta(r) r \csc^2 \theta (1 + 3\theta \cot \theta) \quad (58)$$

Simplified Solution

The closed form expression can be further simplified by neglecting terms of order St^{-2} and higher. As one would expect, since St^{-2} is smaller than M_b , no loss in accuracy is incurred by so doing. The resulting simple expression agrees with both numerical and infinite series results with a margin of error that is smaller than M_b . Since there is no need to be more accurate than the governing momentum equation itself

which neglects \tilde{u}_r , of order M_b , a short practical expression can now be offered by retaining the zeroth order terms and dropping smaller terms given by Eqs. (57-58). The result is a simple yet accurate form:

$$\tilde{u}_z = -\frac{\varepsilon_w}{\gamma} i \sin \theta \sin(k_m z \sin \theta) e^{\frac{\xi \eta r^3}{\sin^3 \theta} - i \frac{St}{\pi} \ln \tan \frac{\theta}{2} - ik_m t} \quad (59)$$

Other than its simplicity and remarkable agreement with the infinite series solution [Eq. (53)], the advantage of Eq. (59) is that it only contains the important terms which affect the solution. It can be easily shown that: 1) retaining the axial convection of unsteady vorticity leads to the axially dependent sinusoidal function; 2) retaining radial convection leads to the term involving the Strouhal number which controls the spatial wavelength; and 3) retaining viscous diffusion leads to the rotational wave damping term controlled by the penetration number.

The Radial Velocity Component

The radial component of the rotational velocity can be derived analytically from the continuity equation which suggests first assuming an ersatz of the form¹⁸

$$\tilde{u}_r = \frac{\varepsilon_w}{\gamma} \frac{F(r)}{r} \cos(k_m z \sin \theta) \exp[\zeta - i(k_m t + \Phi)] \quad (60)$$

Substituting Eq. (60) into the continuity equation and balancing the highest order terms, one can deduce the spatial function $F(r)$

$$\frac{1}{r} \frac{\partial(r\tilde{u}_r)}{\partial r} \Big|_{\text{largest}} = -\frac{\partial \tilde{u}_z}{\partial z} \Rightarrow F(r) = -M_b \sin^3 \theta \quad (61)$$

Consequently, the radial component is determined:

$$\tilde{u}_r = -\frac{\varepsilon_w}{\gamma} \frac{M_b}{r} \sin^3 \theta \cos(k_m z \sin \theta) e^{\zeta - i(k_m t + \Phi)} \quad (62)$$

This reassuring result justifies the assumption of $\tilde{u}_r = \mathcal{O}(M_b)$ in Eq. (9). From Eq. (2), $u_r^{(1)} = \tilde{u}_r$ also.

The Total Time-Dependent Velocity

The total unsteady axial velocity can now be constructed by juxtaposition of the rotational [Eq. (54)] and irrotational [Eq. (46)] parts in Eq. (2). The result is

$$u_z^{(1)} = \frac{\varepsilon_w}{\gamma} \left[\sin \bar{z} \sin \bar{t} - \sin \theta \sin(\bar{z} \sin \theta) e^{\zeta} \sin(\bar{t} + \Phi) \right] \quad (63)$$

where $\bar{z} = k_m z$ and $\bar{t} = k_m t$.

This result has been tested for the entire range of possible parameters¹⁴ and has been found to match very closely both the numerical solution to the governing momentum equation (using a fourth order method with

a truncation error of 10^{-20}), and Flandro's latest result.¹⁸ The results are so close that, for comparison purposes, when curves are overlaid that correspond to Eq. (63), the numerical solution, and Flandro's solution,¹⁸ only one curve is visually discerned. The ability of this multiple-scales solution, and of Flandro's regular perturbation solution, to match the numerical solution, is quite remarkable. For illustration purposes, Eq. (63) is evaluated for three cardinal cases spanning the range of rocket motors (given in Reference 17) for typical tactical rocket, small research rocket, and large shuttle rocket booster. The error between analytical and computational predictions appears only in the third or fourth significant digit, causing a relative error of less than 1%, which is difficult to discern visually. This agreement is found to get even better at higher modes. A graphical representation of Culick's steady flowfield along with the total unsteady velocity derived here is given in Figs. 1-2, at several discrete axial locations and at times of maximum amplitude for the three cardinal cases at the fundamental oscillation mode. Indiscernible from the regular perturbation results obtained by Flandro¹⁸ using a totally different formulation, or from the numerical solution to the momentum equation, these profiles illustrate an asymmetric behavior with respect to the chamber midpoint and a Richardson's velocity overshoot²⁰ near the wall that is not accounted for in the

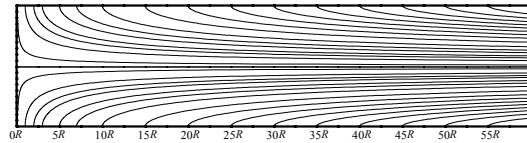


Fig. 1 Culick's steady flow streamlines.

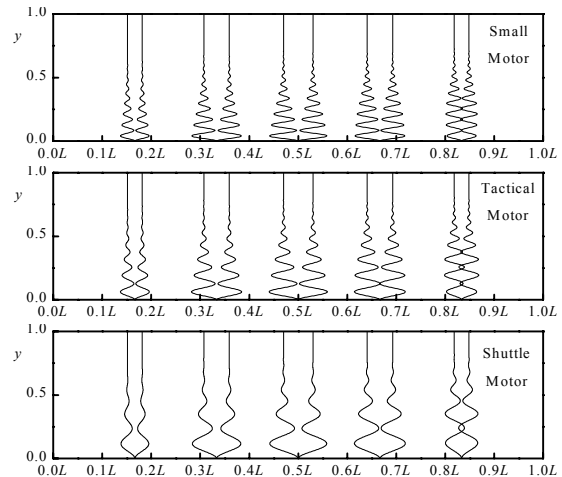


Fig. 2 Time-dependent velocity patterns.

plain one-dimensional acoustic model. More flow details are given in Figs. 3-8 corresponding to the forward and aft halves of each particular motor. Note that the rotational velocity amplitude is larger in the aft half by comparison to its corresponding value at a point that is equidistant from the chamber midpoint in the forward half. This is to be expected due to the added rotational effects that accompany the downstream convection of unsteady vorticity by Culick's axial mean flow component.¹² Contrary to one-dimensional prediction, the unsteady velocity amplitude is not zero everywhere at $z^* = L$. Also, note that for higher oscil-

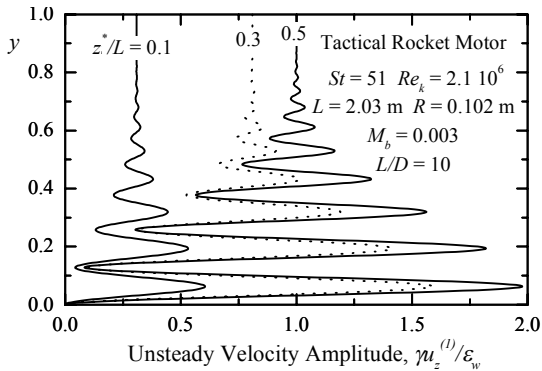


Fig. 3 Unsteady velocity amplitude (fore-half).

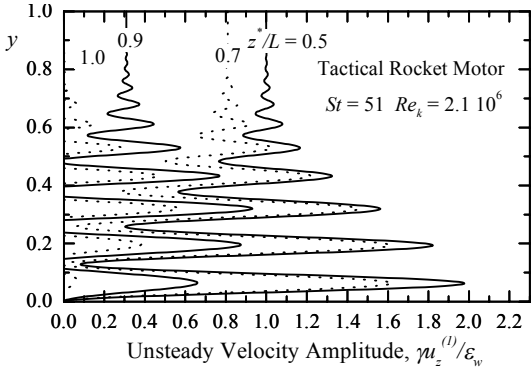


Fig. 4 Unsteady velocity amplitude (aft-half).

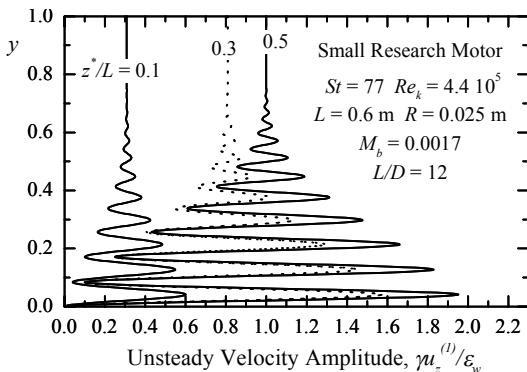


Fig. 5 Unsteady velocity amplitude (fore-half).

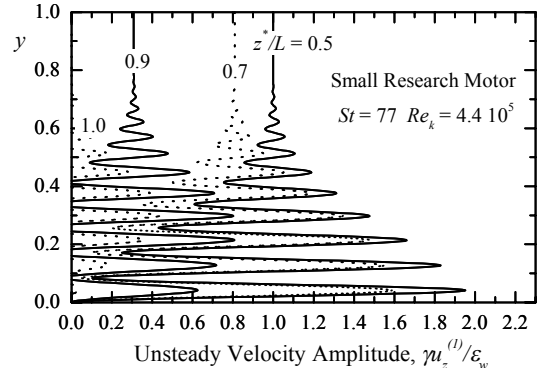


Fig. 6 Unsteady velocity amplitude (aft-half).

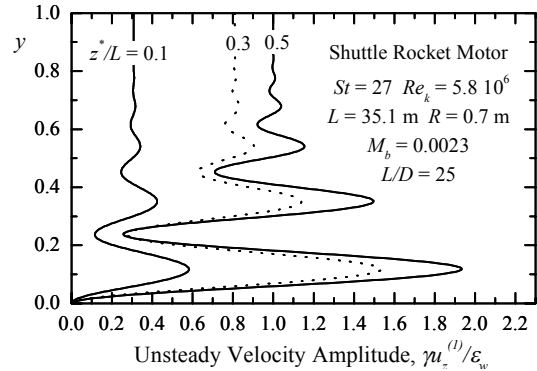


Fig. 7 Unsteady velocity amplitude (fore-half).

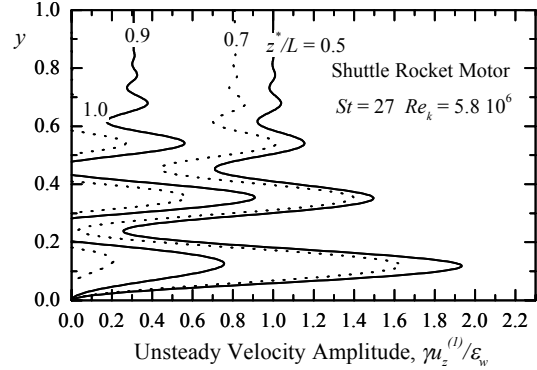


Fig. 8 Unsteady velocity amplitude (aft-end).

lation frequencies corresponding to shorter rocket motors, the rotational wavelength is decreased, being an inverse function of the Strouhal number.

Concluding Remarks

Flandro's latest model¹⁸ results from a careful application of perturbation theory. The methodology that is employed is analogous to Culick's approach¹² to the steady flow. The results are equally accurate for the time-dependent flowfield. They match both numerical and analytical predictions presented here using totally different procedures.

The multiple-scales formulation presented here is a successful attempt to obtain a uniformly valid solution to a singular perturbation problem using an innovative technique. In this approach, all the terms in the momentum equation are retained without neglecting any of the smaller terms in the process of integration except for the radial velocity component which is of order M_b , and which has been totally ignored in all previous analytical models. No approximation is applied to the boundary condition such as found necessary in Flandro's approach.¹⁸ The rotational velocity is derived by direct integration of the momentum equation. The analytical expression that is derived is theoretically accurate to the order of Re_k^{-2} . However, due to the need to match three different scales that arise in the problem with a single composite scale, the final outcome is a solution that is as accurate as the matching procedure. Fortunately, the matching process can be done very precisely to the point that the composite scale solution and the full numerical solution are indistinguishable in the entire range of the physical parameters. The main advantage of this technique over previous models is that it incorporates all of the effects that characterize the problem in a compact expression where the key elements that control the solution are clearly seen. The dependence of the solution on the penetration number, the mode number, and the axial location is clearly depicted from inspection of the solution. Another advantage of this method is that the inviscid formulation can be readily produced from the resulting solution by setting the viscous parameter ξ to be zero. In the previous model,¹⁸ the complete viscous solution relied on the inviscid formulation which had to be derived beforehand.¹⁷

Acknowledgments

The authors wish to thank Dr. Gary Flandro for his valuable comments and suggestions.

References

¹Culick, F.E.C., and Yang, V., "Stability Predictions in Rockets," *Nonsteady Burning and Combustion Stability of Solid Propellants, Progress in Astronautics and Aeronautics*, Vol. 143, 1992, pp. 719-799.

²Shaeffer, C.W., and Brown, R.S., "Oscillatory Internal Flow Studies," United Technologies, Chemical Systems Division Report No. 2060 FR, August 1992.

³Vuillot, F., and Avalon, G., "Acoustic Boundary Layer in Large Solid Propellant Rocket Motors Using Navier-Stokes Equations," *Journal of Propulsion and Power*, Vol. 7, No. 2, 1991, pp. 231-239.

⁴Smith, T.M., Roach, R.L., and Flandro, G.A., "Numerical Study of the Unsteady Flow in a Simulated

Rocket Motor," AIAA Paper 93-0112, AIAA 31st Aerospace Sciences Meeting, Reno, NV, Jan. 1993.

⁵Flandro G.A., and Roach, R.L., "Effects of Vorticity Production on Acoustic Waves in a Solid Propellant Rocket," AFOSR Final Report, AFOSR-90-0159, 1992.

⁶Sabnis, J.S., Giebeling, H.J., and McDonald, H., "Navier-Stokes Analysis of Solid Propellant Rocket Motor Internal Flows," *Journal of Propulsion and Power*, Vol. 5, No. 6, 1989, pp. 657-664.

⁷Zhao, Q., and Kassoy, D.R., "The Generation and Evolution of Unsteady Vorticity in a Model of a Solid Rocket Engine Chamber," AIAA Paper 94-0779, AIAA 32nd Aerospace Sciences Meeting, Reno, NV, 1994.

⁸Kirkkopru, K., Kassoy, D.R., and Zhao, Q., "Unsteady Vorticity Generation and Evolution in a Model of a Solid Rocket Motor: Sidewall Mass Addition Transients," AIAA Paper 95-0603, AIAA 33rd Aerospace Meeting and Exhibit, Reno, NV, Jan. 1995.

⁹Nickerson, G.R., Coates, D.E., and Hermesen, R.W., "A Computer Program for the Prediction of Solid Propellant Rocket Motor Performance," Interim Technical Report, AFRPL-TR-80-34, April 1981.

¹⁰Flandro, G.A., "Solid Propellant Acoustic Admittance Corrections," *Journal of Sound and Vibrations*, Vol. 36, No. 3, March 1974, pp. 297-312.

¹¹Flandro, G.A., "Effects of Vorticity Transport on Axial Acoustic Waves in a Solid Propellant Rocket Chamber," 1989 ASME Annual Meeting, San Francisco, CA, December 1989.

¹²Culick, F.E.C., "Rotational Axisymmetric Mean Flow and Damping of Acoustic Waves in a Solid Propellant Rocket," *AIAA Journal*, Vol. 4, No. 8, August 1966, pp. 1462-1464.

¹³Majdalani, J., and Van Moorhem, W.K., "The Unsteady Boundary Layer in Solid Rocket Motors," Paper No. AIAA-95-2731, July 10-12, 1995.

¹⁴Majdalani, J., and Van Moorhem, W.K., "A Multiple-Scales Solution to the Acoustic Boundary Layer in Solid Rocket Motors," *Journal of Propulsion and Power*, Vol. 13, No. 2, March 1997, pp. 186-193.

¹⁵Nayfeh, A., *Perturbation Methods*, John Wiley & Sons, Inc., 1973.

¹⁶Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, The Parabolic Press, Stanford, CA 1975.

¹⁷Flandro, G.A., "Effects of Vorticity on Rocket Combustion Stability," *Journal of Propulsion and Power*, Vol. 1, No. 4, 1995, pp. 607-625.

¹⁸Flandro, G.A., "On Flow Turning," AIAA Paper 95-2730, July 1995.

¹⁹Cole, J.D., and Aroesty, J., "The Blowhard Problem-Inviscid Flows with Surface Injection," *Int. J. Heat Mass Transfer*, Vol. 11, 1968, pp. 1167-1183.

²⁰Richardson, E.G., and Tyler, E., *Proc. phys. Soc. Lond.*, Vol. 42, 1929, p. 1.