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All current solid propellant rocket instability calculations (*e.g.* Standard Stability Prediction Program, SSPP) account only for the evolution of acoustic energy with time. However, the acoustic component represents only part of the total unsteady system energy; additional kinetic energy resides in the shear waves that naturally accompany the acoustic oscillations. Since most solid rocket motor combustion chamber configurations support gas oscillations parallel to the propellant grain, an acoustic representation of the flow does not satisfy physically correct boundary conditions. It is necessary to incorporate corrections to the acoustic wave structure arising from generation of vorticity at the chamber boundaries. Modifications of the classical acoustic stability analysis have been proposed that partially correct this defect by incorporating energy source/sink terms arising from rotational flow effects. One of these is Culick's *flow-turning* stability integral; related terms appear that are not found in the acoustic stability algorithm. In this paper, a more complete representation of the linearized motor aeroacoustics is utilized to determine the growth or decay of the system energy with *all* rotational flow effects accounted for. Significant changes in the motor energy gain/loss balance result; these help to explain experimental findings that are not accounted for in the present acoustic stability assessment methodology. In particular, the origins of several types of vortex-driven instabilities observed in large solid propellant motors are illuminated.

Nomenclature

a_0	Mean speed of sound
e	Oscillatory energy density
$\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z$	Unit vectors in r, θ and z directions
E	Time-averaged oscillatory system energy
E_m^2	Normalization constant for mode m
k_m	Wave number for axial mode m
L	Chamber length
m	Mode number
M_b	Reference Mach number at burning surface
$\boldsymbol{n}$	Outward pointing unit normal vector
p	Oscillatory Pressure
P_0	Mean chamber pressure
r	Radial position
R	Chamber radius
S	Strouhal Number, k_m / M_b
t	Time
$\boldsymbol{u}$	Oscillatory velocity vector
U_r, U_z	Mean flow velocity component
y	Radial distance from the wall, $1-r$
z	Axial position
α	Growth rate (dimensional, $\sec^{-1}$)
γ	Ratio of specific heats
δ	Inverse square root of the acoustic Reynolds number, $\sqrt{\nu / a_0 R}$
ε	Wave amplitude
ν	Kinematic viscosity, μ / ρ
ρ	Density
$\phi(r)$	Function defined in Eq. (13)
$\psi(r)$	Exponential argument, Eq. (11)
ω	Unsteady vorticity amplitude
Ω	Mean vorticity amplitude

Subscripts

b	Combustion zone
m	Mode number

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Superscripts

*	Dimensional quantity
~	Vortical (rotational) part
^	Acoustic (irrotational) part
(r), (i)	Real and imaginary parts

I. Introduction

CULICK'S papers on combustion instability[1-5] published in the early 1970's are the foundation for all stability prediction methods now in use.[6,7] His method is based on three crucial assumptions:

- small amplitude pressure fluctuations superimposed on a low-speed mean flow,
- thin, chemically reacting surface layer with mass addition, and
- oscillatory flowfield represented by chamber acoustic modes.

The first assumption allows linearization of the governing equations both in the wave amplitude and the surface Mach number of the mean injected flow. The second causes all surface reaction effects, including combustion, to collapse to simple acoustic admittance boundary conditions imposed at the chamber surfaces. The last assumption oversimplifies the oscillatory gas dynamics by suppressing all unsteady rotational flow effects; the acoustic representation is strictly irrotational. Concern for this omission was addressed partially by Culick in a paper in which he introduced his well-known rotational mean flow model.[2] Stability calculations based on this improved mean flow representation produced no significant changes in the system stability characteristics. On this basis, it has since been generally assumed that all vorticity (rotational flow effects), including the unsteady part, have negligible influence on combustion instability growth rate calculations.

Considerable progress has been made in the last decade in understanding both the precise source of the vorticity and the resulting changes in the oscillatory flowfield. Approximate analytical,[8-16] numerical,[17-22] and experimental investigations[23-26] have demonstrated that rotational flow effects play an important role in the unsteady gas motions in solid rocket motors. Much effort has been directed to constructing the required corrections to the acoustic model. This has culminated in a comprehensive picture of the unsteady motions that agrees with experimental measurements,[8-10] as well as numerical simulations.[11]

These models were used in carrying out three-dimensional system stability calculations,[8,9] in a first attempt to account for rotational flow effects by correcting the acoustic instability algorithm. In this process one discovers the origin, and the three-dimensional form, of the classical *flow-turning* correction; related terms appear that are not accounted for in the SSPP algorithm. In particular, a rotational correction term was identified that cancels the flow-turning energy loss in a full-length cylindrical grain. However, all of these results must now be questioned because they are mistakenly founded on an incomplete representation of the system energy balance.

Culick's stability estimation procedure is based on calculating the exponential growth (or decay) of an irrotational acoustic wave; the results are equivalent to energy balance models used earlier by Cantrell and Hart.[27] In all of these calculations the system energy is represented by the classical Kirchoff (acoustic) energy density. Consequently, it does not represent the *full* unsteady field, including both acoustic and rotational flow effects. Kinetic energy carried by the vorticity waves is ignored. It will be demonstrated in this paper that the actual average unsteady energy contained in the system at a given time is about 25% larger than the acoustic energy alone. Furthermore, representation of the energy sources and sinks that determine the stability characteristics of the motor chamber must also be modified. Attempts to correct the acoustic growth rate model by retention of rotational flow source terms only,[8,9] preclude a full representation of the effects of vorticity generation and coupling.

In fact, there is a convincing body of evidence pointing to the existence of other aeroacoustic coupling mechanisms that are not incorporated in the current acoustic-stability theory. For example, the so-called parietal or surface vortex shedding (PVS) has been identified some years ago as a source of instability that eludes classic theory.[28] The corresponding phenomenon was first detected by numerical simulations of 1:5 subscale models of the French Ariane V P230 MPS booster[29-32] that was known to exhibit large amplitude oscillations.[33] This new type of instability was especially important in long, segmented rocket motors such as the Japanese H-II vehicle,[34] the Titan 34D SRM,[35] the Titan IV SRM/SRMU (upgrade),[36-39] the Shuttle Rocket Booster SRB,[40] and other elongated motors whose dimensionless lengths ranged from 15 to 25.

In order to compensate for the inability of classic theory[3,5,41-48] to explain the large pressure oscillations driven by so-called "crawling" vortices,[29] a number of dedicated studies have been carried out hoping to improve our basic understanding of the suspected mechanism.[28] Credit should be given, in that regard, to Vuillot, Avalon, Casalis, Griffond, Lupoglazoff, Traineau, Dupays, Pineau, Tissier, Ugurtas and co-workers who have tried all three experimental,[28,49-52] numerical,[53-59] and theoretical avenues[60-63] to elucidate the origin of PVS coupling. It should also be noted that Casalis, Avalon, Pineau, and Griffond have based their recent theoretical study on linear instability theory introduced in 1969 by Varapaev and Yagodkin.[64] Their efforts have provided an alternate source of instability whose omission in classic analyses has led them to associate some of the unforeseen experimental and numerical instabilities to the hydrodynamic evolution and inception of turbulence.[60,61] At the outset, their results have been limited in fully explaining the observed PVS-related mechanisms.

At the conclusion of these studies,[30-32,50-52] speculations that resonance-like pressure amplifications were caused only by vortex shedding at annular restrictors or inhibitor rings were laid to rest when similarly intense vorticity-generated oscillations were observed in unsegmented rocket motors. As noted by Ugurtas *et al.*,[65] two-dimensional compressible flow simulations of the Navier-Stokes equations by Lupoglazoff and Vuillot[29-31] have confirmed the measurements acquired from subscale firings; as such, the collection of all available data has pointed out to the existence of a "powerful" vorticity-driven coupling irrespective of whether inhibitor rings or other surface anomalies are present.[29-31]

The main objective of this paper is construction of a more complete stability model that accounts for *all* of the system energy and correctly portrays all energy sources and sinks. This task is most readily accomplished by application of the energy balance approach. As is usually the case with energy analyses, our method will promote an improved physical understanding of the results.

Considerable work is needed to implement these changes. The outcome is a stability algorithm that accounts fully for both the acoustic and vortical flow interactions. Significant differences between these new results and those presently utilized for rocket motor stability computations are demonstrated. The new model is tested by applying it to several solid propellant rocket motor designs. Despite significant changes in the mathematical formulation, it is not necessary to discard the current solid rocket motor stability estimation methodology; required modifications are readily accomplished by means of minor changes to the existing codes. A new energy source term is identified that suggests a possible origin of the unexplained instabilities in large solid booster motors. This energy source, arising from production of unsteady vorticity, is comparable in size to the key pressure coupling term itself. It may also be related to velocity coupling effects, which cannot be fully represented in the context of irrotational acoustic instability theory.

II. Unsteady Flow Analysis

The goal of this section is the construction of a comprehensive model of the unsteady flow- field in a rocket chamber that realistically accounts for vortical (rotational) as well as acoustical (irrotational) effects. In particular, close attention is paid to properly satisfying the correct boundary conditions on all chamber boundaries including both inert and reactive burning surfaces.

The validity of the stability calculations depends critically on a sufficiently detailed and physically correct representation of the unsteady velocity field. The assumption of an irrotational unsteady flow as used in most stability calculations must be discarded. The analytical methods and notation employed here closely follow earlier papers.[8-10] Dimensionless variables are the same as those used in the classical combustion instability analyses: velocities are normalized by dividing by the average sound speed; pressure is normalized by dividing by the product of the mean chamber pressure and the specific heat ratio, γ. Other thermodynamic properties are made dimensionless by dividing them by their mean chamber reference values. Lengths are referenced to the chamber radius, R. Figure 1 illustrates the geometry and coordinate systems to be employed in the analysis.

The analysis starts by assuming small amplitude unsteady perturbations on a mean flow described by the vector, $M_b\boldsymbol{U}$. In this paper, Culick's mean flow model[2] for a cylindrical burning port,

$$\boldsymbol{U}=U_r\boldsymbol{e}_r+U_z\boldsymbol{e}_z=-r^{-1}\sin(x)\boldsymbol{e}_r+\pi z\cos(x)\boldsymbol{e}_z \quad (1)$$

will be used. Note that the combination

$$x\equiv\tfrac{1}{2}\pi r^2 \quad (2)$$

appears frequently in the analysis.

Isentropic conditions are assumed but viscous forces, both shear and dilatational, are retained. The linearized continuity and momentum equations then become

$$\frac{\partial\rho^{(1)}}{\partial t}+\nabla\cdot\boldsymbol{u}^{(1)}=-M_b\boldsymbol{U}\cdot\nabla\rho^{(1)} \quad (3)$$

$$\frac{\partial\boldsymbol{u}^{(1)}}{\partial t}+\nabla p^{(1)}=M_b\left\{\begin{aligned}&-\nabla\left[\boldsymbol{U}\cdot\boldsymbol{u}^{(1)}\right]+\boldsymbol{u}^{(1)}\times\nabla\times\boldsymbol{U}\\&+\boldsymbol{U}\times\nabla\times\boldsymbol{u}^{(1)}\end{aligned}\right\}$$
$$+\delta^2\left\{(2+\lambda/\mu)\nabla\left[\nabla\cdot\boldsymbol{u}^{(1)}\right]-\nabla\times\nabla\times\boldsymbol{u}^{(1)}\right\} \quad (4)$$

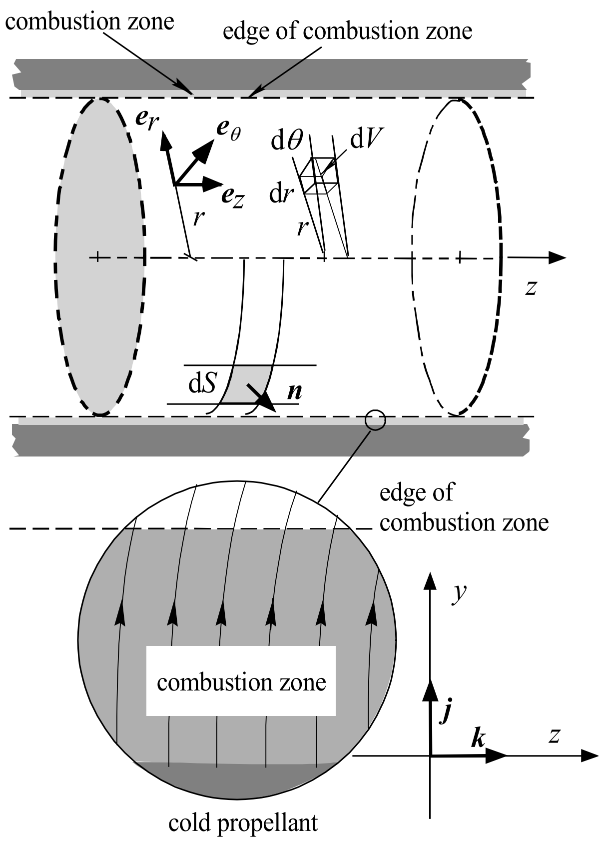

Fig. 1 Motor geometry and coordinate system used.

This dimensionless set of equations is the starting point for all previous combustion instability models. The superscript (1) denotes the first-order terms in a perturbation parameter, ε, proportional to the amplitude of the unsteady pressure fluctuation. A second small parameter, the mean flow Mach number at the burning surface, M_b, is also employed in the linearization process; only terms to the first order in M_b are retained as in the classical combustion instability analyses. The relative size of the viscous forces is set by the parameter δ^2, which is the inverse of the acoustic Reynolds number based on the chamber radius R. Viscous terms are usually dropped, but are retained here for completeness by assuming that $\delta^2 = O(M_b)$.

The main variables will be represented as combinations of irrotational and rotational components:

$$\begin{cases} p^{(1)} = \hat{p}^{(1)} + \tilde{p}^{(1)} \\ \rho^{(1)} = \hat{p}^{(1)} \\ \boldsymbol{u}^{(1)} = \hat{\boldsymbol{u}}^{(1)} + \tilde{\boldsymbol{u}}^{(1)} \end{cases} \tag{5}$$

where the caret (^) denotes the irrotational part of the solution and the tilde (~) indicates the rotational part. The acoustic field is irrotational ($\nabla \times \hat{\boldsymbol{u}}^{(1)} = 0$) and the superimposed vortical component is incompressible ($\nabla \cdot \tilde{\boldsymbol{u}}^{(1)} = 0$). Because of the assumption of constant entropy, the density and the thermodynamic (acoustic) pressure are interchangeable as indicated in Eq. (5). The density in the continuity equation is routinely replaced in the classical analysis by the pressure variable. The reader is cautioned that care must be taken in interpreting the unsteady pressure, $p^{(1)}$. If it appears in the continuity equation, this parameter will then represent a *density* fluctuation. This distinction is not important if the flow is irrotational, but it is vitally important when rotational effects are to be incorporated.

If irrotational flow is assumed, Eqs. (3) and (4) yield the usual acoustic wave solutions. For axial motions in a rocket chamber, the solution is the standing plane wave,

$$\hat{p}^{(1)} = e^{-ikt} \cos\left(k_m z\right) + O\left(M_b\right) \tag{6}$$

$$\hat{\boldsymbol{u}}^{(1)} = ie^{-ikt} \sin\left(k_m z\right) \boldsymbol{e}_z + O\left(M_b\right) \tag{7}$$

where k_m is the dimensionless wave number. For axial modes, satisfaction of closed-end boundary conditions requires that

$$k_m = m\pi R / L \tag{8}$$

where m is the mode integer and R/L is the chamber radius-to-length ratio. As Eqs. (6) and (7) indicate, there exist corrections to the acoustic field of the order of the mean Mach number. These were described in Ref. 8, but are not required in the stability calculations. Only the first-order velocity and pressure (in M_b) are needed in evaluating the stability integrals as in the classical analyses.

It is immediately apparent that Eq. (7) cannot satisfy physically correct boundary conditions either at the edge of the combustion zone or on inert sidewalls. The axial velocity component must approach zero at the wall to satisfy the no-slip condition. Thus the production of axial acoustic oscillations gives rise to corrections which must come from the rotational flow effects. Flandro solved Eqs. (3) and (4) for the vortical (rotational, incompressible) field by two different methods. The inviscid case was treated in Ref. 8; a more complete analysis was presented in Ref. 9, in which viscous shear wave damping was included. Another improvement in the latter paper was the introduction of an equivalent closed form solution in place of the infinite series used in Ref. 8. Additional improvements were made in Ref. 10, in which turbulent mean flow corrections were introduced. For clarity, we will display here only the laminar results for the simplest motor geometry and acoustic mode structure, namely, a cylindrical port with axial oscillations; other geometries and transverse modes of oscillation can be handled similarly. Other investigators such as Majdalani and Roh[16] have verified the basic results to be used here by quite different analytical approaches and by direct CFD computations.[11-19]

One finds for rotational pressure and velocity corrections that must accompany the assumed acoustic solution of Eqs. (6) and (7),

$$\tilde{p}^{(1)} = iM_b e^{-ikt}\sin(2x)e^{(\phi+i\psi)}\left(\tfrac{1}{2}\pi z\right)\sin\left[\sin(x)k_m z\right] + O\left(M_b^2\right) \tag{9}$$

$$\tilde{\boldsymbol{u}}^{(1)} = iCe^{-ikt}rU_r e^{(\phi+i\psi)}\sin\left[\sin(x)k_m z\right]\boldsymbol{e}_z + O(M_b) \tag{10}$$

Flandro[9] derived these expressions directly from Eqs. (3) and (4) without recourse to the *splitting* theorem applied in earlier papers. Critics have not understood that splitting Eqs. (3) and (4) into acoustical and vortical parts (following the method of Chu and Kovásznay[66,67]) is a simplifying, but not a necessary step in arriving at the solutions displayed.[68] Another misunderstanding centers on the pressure correction, $\tilde{p}^{(1)}$, sometimes called the *pseudopressure*.[68] It is important to understand that this was not simply set to zero by assumption in the earlier works. Careful solution of the momentum equation, Eq. (4), shows that it is an additional correction of the order of the mean Mach number; the result is shown in Eq. (9). Corrections of similar size appear in the irrotational (acoustic) pressure solution, Eq. (6). These are not required in the stability calculations for reasons already given. It is also necessary to point out that the pseudopressure must not be inserted in the continuity equation, Eq. (3), as part of the density variable as suggested by Brown.[68]

The complex exponential factor $\exp(\phi + i\psi)$ in the solution exhibited in Eq. (10) suggests that the vortical motion can be interpreted as a damped traveling shear wave. Both ϕ and ψ are functions only of radial position. The imaginary part of the exponential argument, namely,

$$\psi(r) = -\frac{k_m}{\pi M_b}\ln\tan\left(\tfrac{1}{2}x\right) \tag{11}$$

sets the wavelength and spatial frequency of the shear wave. For later use note that the derivative of ψ with respect to the radius is simply

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{k_m}{M_b U_r} \tag{12}$$

This is proportional to the reciprocal of the radial mean flow velocity. The real part of the argument, $\phi(r)$, is given to good approximation by

$$\phi(r) = \frac{\xi}{\pi^2}\left[1 - \frac{1}{\sin(x)} - x\frac{\cos(x)}{\sin^2(x)}\right] \tag{13}$$

where S and ξ are dimensionless scaling factors

$$S \equiv \frac{k_m}{M_b},\quad \xi \equiv \frac{k_m^2\delta^2}{M_b^3} = \frac{S^2\delta^2}{M_b} \tag{14}$$

where S is the Strouhal number and ξ is a parameter of $O(1)$ that reflects the relative importance of the viscous damping; its physical description and verification is detailed by Majdalani and Roh.[16] Note that the damping of the shear wave in the radial direction increases quadratically as the frequency of the acoustic oscillation increases. Similarly, the wavelength of the shear wave decreases as frequency increases. For the fundamental acoustic mode and the lowest order harmonics, the shear wave may fill the entire chamber; for high-order acoustic modes, the shear oscillations become confined to a thin acoustic boundary layer, which may lie entirely within the combustion zone.[13]

An undetermined complex constant C is shown in Eq. (10). In the earlier work, this constant was evaluated by assuming that, for a thin combustion layer, the physical boundary at the solid surface and the edge of the combustion zone are nearly coincident.[8-10] Then if one applies the no-slip condition to the composite unsteady axial velocity ($\boldsymbol{u}^{(1)} = \hat{\boldsymbol{u}}^{(1)} + \tilde{\boldsymbol{u}}^{(1)}$) at the edge of the combustion zone, one finds that $C = 1$. This result has been questioned, since the actual solid surface lies within a region of nonuniformity, the combustion zone.[68] It is a straightforward process to show that the original value is correct; the calculation is too lengthy to display here. This is done by constructing a two-dimensional model for the combustion zone including all gas phase effects. The axial momentum equation in the flame zone is solved numerically to account for the gradients in temperature and density due to combustion. The resulting *inner solution* is then matched asymptotically to the *outer solution* expressed by Eq. (10). Standard methods of singular perturbation theory apply. The simplest approach is to determine solutions in an intermediate region in which both the outer and inner representations are valid. Then the matching is readily accomplished. The outcome confirms that the original result, $C = 1$, is correct unless the oscillation frequency is very high. In that case, the rotational flow effects become entirely buried within the combustion zone, and the outer solution collapses to the simple plane wave acoustic model.

The composite axial velocity solution found by superimposing Eqs. (7) and (10) has been shown[8-10,12] to agree quite closely with the experimental findings of Brown and co-workers,[12,25,26] and Barron, Van Moorhem, and Majdalani.[69] Figures 2 and 3 show comparisons of the oscillation amplitude and phase angle from the theory just discussed superimposed on some of Brown's earlier cold flow work.[24,70] It is important to understand that no "curve fitting" has been used here; the differences between the experimental and theoretical results arise both from unavoidable experimental errors (orientation and positioning of the hot film sensor, data reduction, etc.) and from the assumption set used in the analysis. Nevertheless, very good agreement is shown. Since nitrogen was the working fluid at measured temperature and pressure, all parameters characterizing the flowfield are well known.

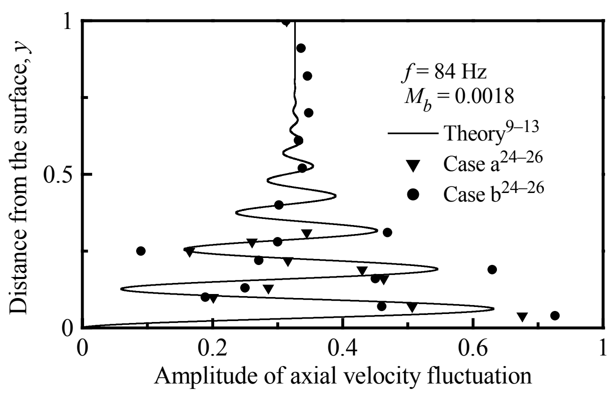

Fig. 2 Comparison of theoretical and experimental amplitudes of unsteady axial velocities. Here S = 51.3m, z^*/L = 0.106, M_b = 0.0018, L/R = 34, k_m = 0.0924m, ξ = 0.539m^2, and m = 1. Triangles and circles correspond to experimental data acquired with a pressure wave amplitude of a) 0.0005 and b) 0.0039.

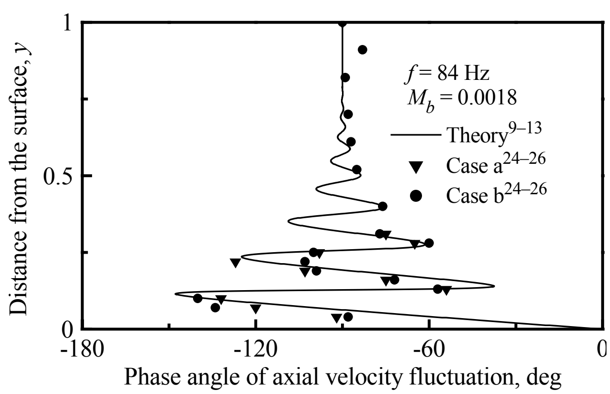

Fig. 3 Comparison of theoretical and experimental phase angles (degrees).

Numerical solutions based on the full Navier-Stokes representation of the unsteady field also indicate excellent agreement with the theory reviewed here. Furthermore, other analytical methods based on WKB and multiple-scale theories have been applied by Majdalani and co-workers.[71] Theirs have demonstrated, based on the principle of least singularity, the existence of nonlinear scales underlying the physics of the problem.[72] In addition to providing a rigorous mathematical treatment that obviates the need for guesswork, their approach has led to the exact form of the characteristic length scales as function of Culick's mean flow solution. More importantly, their solutions have yielded results indistinguishable from those displayed here.[11-16]

In previous analyses of the rocket motor rotational flowfield, special attention was given to the radial component of velocity.[8,9,11] This was done since, in classical instability theory, the main source of acoustic energy is the radial velocity fluctuation produced by interaction of the pressure wave with the combustion processes and heat transfer in the burning zone. It was shown in Refs. 8 and 9 that, in addition to the radial acoustic velocity, there appears a rotational correction that does not vanish at the edge of the combustion zone. The presence of this additional unsteady radial velocity component implies that, by analogy to pressure coupling, there is an additional source of unsteady mass flux at the combustion zone interface. For want of better terminology, this was called the *rotational flow correction* as first displayed in Eq. (89) of Ref. 8. The new term, which does not appear in any previous analysis, has given rise to considerable controversy.

By way of providing a precise description of the source of the *rotational flow correction*, a complete physical interpretation, and a full resolution of the controversy appear in the next section of this paper. The answers are found by means of a careful derivation of the system stability energy balance. It is not necessary to calculate the vortical radial velocity correction in determining the system stability. The surface acoustic admittance function will be utilized in the standard fashion to represent the pressure coupling of the oscillatory flow with the combustion processes. Then experimental measurements (usually secured with a T-burner or similar device) of the surface response can be utilized as in the accepted methodology for accounting for this important source of oscillatory energy.

Earlier attempts to incorporate vortical flow corrections in the stability calculations were based on the idea that one must restore rotational source terms that were dropped in the classical analyses. For example, Brown, *et al.* had suggested that retention of the rotational convective acceleration term $\boldsymbol{U}\times\nabla\times\boldsymbol{u}^{(1)}$

on the right side of Eq. (4) might have important stability implications.[24] This term was first evaluated by Flandro,[8,9] who proved that this term yields a damping effect that is indistinguishable from the classical flow-turning loss. Additional terms appear in this analysis that cannot be found in the standard stability computations. One of these, the *rotational flow correction,* has led to much debate, since it exactly cancels the flow turning effect in some chamber configurations. Brown's assertion[68] that the *rotational flow correction* is just another way to represent flow-turning can be quite readily refuted. In fact, all misunderstandings are resolved when a consistent energy method is used to deduce motor stability characteristics. The flaw in the earlier approach is that it represented the system by a model that accounts for growth or decay of the acoustic energy alone. Simply retaining the rotational convective source terms[24] in the perturbed wave equation, following Culick's method, does not address this important defect.

To correctly describe the evolution of the oscillatory flow in a rocket motor, *all* unsteady energy must be accounted for. The method used here follows the known energy balance approach described by Kirchoff[73] and used extensively in rocket stability calculations by Cantrell and Hart.[27,41-43] It is now necessary to account for the entire kinetic energy fluctuation. To accomplish this, multiply the continuity equation, Eq. (3), by the acoustic pressure (or density) and add this to the momentum equation, Eq. (4), multiplied by the composite unsteady velocity vector,

$$\boldsymbol{u}^{(1)} = \hat{\boldsymbol{u}}^{(1)} + \tilde{\boldsymbol{u}}^{(1)} \tag{15}$$

This operation constitutes the main departure from the previous approach. Isolating the terms differentiated with respect to time and retaining only those terms that are linear in the Mach number, M_b, one finds

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) &= -\left(\hat{p}\nabla\cdot\hat{\boldsymbol{u}} + \boldsymbol{u}\cdot\nabla\hat{p}\right)\\ &-M_b\left[\tfrac{1}{2}\boldsymbol{U}\cdot\nabla\hat{p}^2 + \boldsymbol{u}\cdot\nabla(\boldsymbol{U}\cdot\boldsymbol{u}) - \boldsymbol{u}\cdot\left(\boldsymbol{u}\times\nabla\times\boldsymbol{U} + \boldsymbol{U}\times\nabla\times\tilde{\boldsymbol{u}}\right)\right]\\ &+\delta^2\boldsymbol{u}\cdot\left[\left(2+\lambda/\mu\right)\nabla\left(\nabla\cdot\hat{\boldsymbol{u}}\right) - \nabla\times\nabla\times\tilde{\boldsymbol{u}}\right]\end{aligned} \tag{16}$$

Note that the superscript (1) has now been discarded, since it is no longer necessary to emphasize that only first-order terms are being retained. If the full unsteady velocity vector, $\boldsymbol{u}$, were to be replaced at this stage by the acoustic part, $\hat{\boldsymbol{u}}$, then the final result of the stability calculation would be exactly that described in earlier papers.[8,9] If all rotational terms on the right of Eq. (16) were also to be removed by assuming that $\tilde{\boldsymbol{u}} = 0$ then the classical three-dimensional stability result would be reproduced. Assumptions such as those just described are motivated mainly by a desire to keep the analysis simple. However, as we shall now demonstrate, much physical substance is lost in such simplifications. We choose not to follow the traditional approach here –*all* rotational effects are retained as Eq. (16) shows; the kinetic energy per unit mass on the left side reflects all laminar unsteady motions in the chamber. Turbulent corrections will require attention in future refinement of the computations; discussions of turbulent flow effects are found in the numerical studies by Yang,[10,18,19,74] Beddini and Roberts,[75,76] and Vuillot, *et al*.[57,58] In order that the analysis is not overly complicated, effects of solid particles in the flow are not displayed; required modifications are readily incorporated later as needed.

Following the standard approach,[73] one defines the oscillatory energy density

$$e = \tfrac{1}{2}\left(\hat{p}^2 + \boldsymbol{u}\cdot\boldsymbol{u}\right) \tag{17}$$

and the time-averaged oscillatory energy residing in the chamber at any instant as

$$E = \iiint_V \langle e\rangle \,\mathrm{d}V = \tfrac{1}{2}\iiint_V \left\langle \hat{p}^2 + \boldsymbol{u}\cdot\boldsymbol{u}\right\rangle \mathrm{d}V \tag{18}$$

where triangular brackets denote the time-average of the enclosed function. Then from Eq. (16), it is clear that the evolution of the system energy is controlled by

$$\begin{aligned}\frac{\partial E}{\partial t} = \iiint_V \Big\langle &\overbrace{-\nabla\cdot\left(\hat{p}\hat{\boldsymbol{u}}\right) - \tfrac{1}{2}M_b\left(\boldsymbol{U}\cdot\nabla\hat{p}^2\right) - M_b\left[\hat{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\hat{\boldsymbol{u}}\right)\right]}^{\text{irrotational}}\\ &+\delta_d^2\hat{\boldsymbol{u}}\cdot\nabla\left(\nabla\cdot\hat{\boldsymbol{u}}\right) + M_b\left[\hat{\boldsymbol{u}}\cdot\left(\hat{\boldsymbol{u}}\times\boldsymbol{\Omega}\right) + \hat{\boldsymbol{u}}\cdot\left(\boldsymbol{U}\times\boldsymbol{\omega}\right)\right]\\ &\overbrace{-\tilde{\boldsymbol{u}}\cdot\nabla\hat{p} - M_b\begin{bmatrix}\tilde{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\hat{\boldsymbol{u}}\right) + \hat{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}\right) + \tilde{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}\right)\\ -\tilde{\boldsymbol{u}}\cdot\left(\boldsymbol{U}\times\boldsymbol{\omega}\right) - \tilde{\boldsymbol{u}}\cdot\left(\tilde{\boldsymbol{u}}\times\boldsymbol{\Omega}\right)\end{bmatrix}}^{\text{rotational}}\\ &+\delta_d^2\tilde{\boldsymbol{u}}\cdot\nabla\left(\nabla\cdot\hat{\boldsymbol{u}}\right) - \delta^2\left[\hat{\boldsymbol{u}}\cdot\left(\nabla\times\boldsymbol{\omega}\right) + \tilde{\boldsymbol{u}}\cdot\left(\nabla\times\boldsymbol{\omega}\right)\right]\Big\rangle \mathrm{d}V\end{aligned} \tag{19}$$

where the irrotational and rotational contributions to the energy rate-of-change have been partitioned for clarity. The last two "irrotational" terms are in reality due to rotational effects. They are placed with the irrotational terms to conform to Culick's methodology.[2,47] The mean and unsteady vorticity vectors are represented in Eq. (19) by $\boldsymbol{\Omega} = \nabla\times\boldsymbol{U}$ and $\boldsymbol{\omega} = \nabla\times\tilde{\boldsymbol{u}}$, respectively.

The dilatational viscous term is simplified by defining

$$\delta_d^2 \equiv \delta^2\left(2+\lambda/\mu\right) \tag{20}$$

where λ is the second coefficient of viscosity. Several terms cancel. For example,

$$-\hat{\boldsymbol{u}}\cdot\left(\tilde{\boldsymbol{u}}\times\boldsymbol{\Omega}\right) - \tilde{\boldsymbol{u}}\cdot\left(\hat{\boldsymbol{u}}\times\boldsymbol{\Omega}\right) = 0 \tag{21}$$

Further simplifications can be made. For example, several terms can be expressed as integrals over the surface bounding the control volume. It is first necessary to review the protocol needed in evaluating the stability integrals.

The terms on the right side of Eq. (19) control the rate at which the system energy changes. From this information, one can estimate the growth or decay rate for a given motor configuration.

The complex wave number (or frequency) k in the assumed exponential time dependence is written as

$$k = k_m + (\omega_m + i\alpha_m) + O\left(M_b^2\right) \tag{22}$$

where k_m is the wave number for the unperturbed acoustic mode m. In the multidimensional case, m may consist of three mode integers. We will restrict the evaluation to axial modes for clarity; then a single integer, m, identifies the mode being investigated. More complex modes can be handled in an entirely similar way. The result for a simple plane wave axial mode is shown in Eq. (8). The dimensionless corrections, $\omega_m + i\alpha_m$, are of the order of the mean flow Mach number. In the conventional energy-balance procedure, the frequency correction, ω_m, is not evaluated. For simplicity, we follow this approach. The frequency correction is implicitly accounted for in the wave number k_m.

Since quadratic combinations of the variables appear, it is first necessary to take the real parts of the pressure and velocities to be inserted in the equations. Thus one writes

$$\hat{p} = \hat{p}_m \exp(\alpha_m t)\cos(k_m t) \tag{23}$$

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_m \exp(\alpha_m t)\sin(k_m t) \tag{24}$$

$$\tilde{\boldsymbol{u}} = \exp(\alpha_m t)\left[\tilde{\boldsymbol{u}}_m^{(r)}\cos(k_m t) + \tilde{\boldsymbol{u}}_m^{(i)}\sin(k_m t)\right] \tag{25}$$

where superscripts (r) and (i) refer to the real and imaginary parts. This notation is necessary since the rotational velocity vector contains the spatial exponential term $\exp(\phi + i\psi)$. Inserting these expressions into Eq. (17) yields the time-averaged energy density

$$\langle e \rangle = \tfrac{1}{4}\exp(2\alpha_m t)\Big[\overbrace{p_m^2 + \hat{\boldsymbol{u}}_m\cdot\hat{\boldsymbol{u}}_m}^{\text{irrotational}}$$

$$\overbrace{+2\hat{\boldsymbol{u}}_m\cdot\tilde{\boldsymbol{u}}_m^{(i)} + \tilde{\boldsymbol{u}}_m^{(r)}\cdot\tilde{\boldsymbol{u}}_m^{(r)} + \tilde{\boldsymbol{u}}_m^{(i)}\cdot\tilde{\boldsymbol{u}}_m^{(i)}}^{\text{rotational}}\Big] \tag{26}$$

which has been partitioned to emphasize the new terms. This result should be compared to the classical expression for the energy density, which consists of only the first two terms in Eq. (26). The additional three terms represent the kinetic energy residing in the unsteady vorticity. Inserting these expressions into Eq. (18), carrying out the time averaging, and differentiating with respect to time, one finds

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \alpha_m \exp(2\alpha_m t)E_m^2 \tag{27}$$

where the energy normalization function

$$E_m^2 = \tfrac{1}{2}\iiint_V \Big[(p'_m)^2 + \hat{\boldsymbol{u}}_m\cdot\hat{\boldsymbol{u}}_m + 2\hat{\boldsymbol{u}}_m\cdot\tilde{\boldsymbol{u}}_m^{(i)}$$

$$+\tilde{\boldsymbol{u}}_m^{(r)}\cdot\tilde{\boldsymbol{u}}_m^{(r)} + \tilde{\boldsymbol{u}}_m^{(i)}\cdot\tilde{\boldsymbol{u}}_m^{(i)}\Big]\mathrm{d}V \tag{28}$$

is expressed here by the same symbol used in earlier stability analyses. It is written as a squared quantity to indicate that it is positive-definite.

It is interesting to compare values from Eq. (28) to those found in earlier computations. The mode shapes are needed for this purpose. Equations (6), (7) and (10) give the required information for the assumed axial oscillations to zeroth-order in M_b:

$$\hat{p}_m = \cos(k_m z) \tag{29}$$

$$\hat{\boldsymbol{u}}_m = \sin(k_m z)\boldsymbol{e}_z \tag{30}$$

$$\tilde{\boldsymbol{u}}_m^{(r)} = \sin(x)\exp(\phi)\sin(\psi)\sin\left[\sin(x)k_m z\right]\boldsymbol{e}_z \tag{31}$$

$$\tilde{\boldsymbol{u}}_m^{(i)} = -\sin(x)\exp(\phi)\cos(\psi)\sin\left[\sin(x)k_m z\right]\boldsymbol{e}_z \tag{32}$$

The acoustic pressure and velocity mode shapes are related by

$$\hat{\boldsymbol{u}}_m = -\nabla\hat{p}_m / k_m \tag{33}$$

for later use. Figure 4 shows the results of integration of the kinetic energy terms in Eq. (28) across the chamber from the centerline to the wall at a given axial location. Note that the term involving the cross-product of the acoustic and vortical parts does not contribute to the integrated result because it oscillates in the radial direction, and is zero at the upper limit. However, the rotational correction shows a net value, which is half of

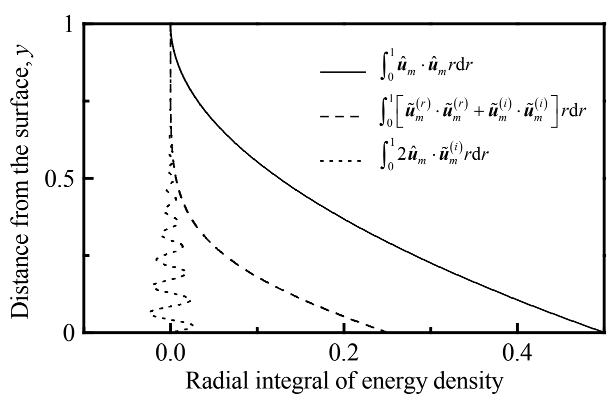

Fig. 4 Radial integration of the kinetic energy density components.

the integrated acoustic energy density. Clearly, the rotational effects represent a significant change in the system energy.

When the entire volume integral of Eq. (28) is carried out for the axial mode case, using Eqs. (29)–(32) one finds that

$$E_m^2 = \tfrac{5}{8}\pi L / R \tag{34}$$

which is 25% larger than the conventional acoustic value for longitudinal oscillations. However, the most important changes in the growth rate, α_m, appear in the integral terms on the right side of Eq. (19). We must now attend to this important part of the stability analysis.

A. Growth Rate Calculations

In estimating the system growth rate for a given mode of oscillation, it can be seen that it consists of the linear superposition of fifteen volume integrals as shown in Eq. (19). The simplest procedure is to consider each of these (or selected groups) to be evaluated individually. One can then represent the net growth rate as a linear sum of gains and losses as

$$\alpha_m = \alpha_1 + \alpha_2 + \alpha_3 + \cdots = \sum_{i=1}^{N} \alpha_i \tag{35}$$

in the usual fashion. The dimensionless values of the several terms are given in the following paragraphs. The reader is reminded that, in order to compute the corresponding dimensional values, it is necessary to multiply by a_0 / R, the inverse of the characteristic time used in the formulation. Then the growth rates are obtained in the familiar units of rad/sec ($\sec^{-1}$).

B. Irrotational Growth Rate Contributions

To illustrate the computational procedure, consider the first three irrotational terms:

$$\alpha_1 = \frac{1}{\exp(2\alpha_m t) E_m^2} \iiint_V \Big\langle -\nabla \cdot \left[\hat{p}\hat{\boldsymbol{u}} + \tfrac{1}{2} M_b \boldsymbol{U} (\hat{p})^2 \right] - M_b \left[\hat{\boldsymbol{u}} \cdot \nabla (\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \right] \Big\rangle \mathrm{d}V \tag{36}$$

where the first two terms have been combined by using the vector identity for the divergence of the product of a scalar and a vector and by noting that $\nabla \cdot \boldsymbol{U} = 0$ for continuity of the incompressible mean flow. Thus, the first term can be converted to a surface integral using Gauss's divergence theorem. The second can be written in terms of the pressure by means of Eq. (33); after some algebra it also reduces to a surface integral. Assuming a short, quasi-steady nozzle, one finds

$$\alpha_1 = \frac{M_b}{2E_m^2} \left[\iint_{S_b} (A_b + 1) \hat{p}_m^2 \, \mathrm{d}S - \iint_{S_N} \left(A_N + \tfrac{1}{2} U_z \right) \hat{p}_m^2 \, \mathrm{d}S \right]$$
$$\simeq \frac{\pi M_b}{2E_m^2} \frac{L}{R} \left[(A_b + 1) - (\gamma + 1) \right] \tag{37}$$

The first term in Eq. (37) is, of course, the classical *pressure coupling effect* except that it is now somewhat reduced in size because of the larger energy normalization integral. The entire surface integral has not been represented in Eq. (37). Non-burning chamber surfaces can be accommodated in the surface integrals if required. The several admittance functions are taken to be quantities of $O(1)$; the mean flow Mach number has been factored out as in most traditional computations. The admittance for non-burning surfaces, A_s, is usually neglected, although rough chamber walls and insulation layers may display substantial (usually negative) admittance.

The fourth term, the volume (dilatational) acoustic viscous energy loss, is usually ignored in standard combustion stability computations; it is discussed extensively in the acoustic literature since it represents a main source of decay of acoustic waves in a variety of applications. It is readily evaluated for the present case with the result that

$$\alpha_2 = \frac{1}{E_m^2 \exp(2\alpha_m t)} \iiint_V \left\langle \delta_d^2 \hat{\boldsymbol{u}} \cdot \nabla (\nabla \cdot \hat{\boldsymbol{u}}) \right\rangle \mathrm{d}V$$
$$= -\frac{\delta_d^2 k_m^2}{E_m^2} \iiint_V \hat{p}_m^2 \, \mathrm{d}V \simeq -\frac{\delta_d^2 \pi k_m^2}{2E_m^2} \frac{L}{R} \tag{38}$$

where Eq. (33) has again been used. Note that this damping effect may not be negligible when turbulence is present; then the transport properties are modified, and the effective viscosity coefficient may be much larger than the laminar values over a substantial volume of the chamber.[10] Other irrotational growth rate terms are not displayed here. These include effects of aluminum particulates and residual combustion.[77-79] Since the system is linear, they may be superposed later as required.

Two of the rotational terms are traditionally included with the strictly irrotational growth rate contributions just evaluated. These are the effects of the mean flow rotationality and the flow-turning effect. To account for the rotational mean flow, one writes

$$\alpha_3 = \frac{1}{E_m^2 \exp(2\alpha_m t)} \iiint_V \left\langle M_b \left\{ \hat{\boldsymbol{u}} \cdot (\hat{\boldsymbol{u}} \times \boldsymbol{\Omega}) \right\} \right\rangle \mathrm{d}V = O\left(M_b^2\right) \tag{39}$$

as evaluated by Culick[2] in his rotational mean flow paper, and shown by him to represent a negligible correction.

Before proceeding to the next term in Eq. (19), the reader is reminded that Culick first identified the *flow-turning* loss in his one-dimensional acoustic stability analysis.[4,47,80] As shown by Flandro,[8,9] this damping effect appears because, by requiring the unsteady flow to enter the duct in a direction perpendicular to the burning surface, Culick effectively invoked the no-slip condition. Although this term cannot arise in the multidimensional irrotational stability analysis, Culick insisted that it be "patched" onto the acoustic growth results. If we follow this dictum, we must add

$$\alpha_4 = \alpha_{\text{flow turning}} = \frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V \left\langle M_b \hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\right\rangle \,\mathrm{d}V$$

$$= -\frac{M_b}{k_m E_m^2 \exp(2\alpha_m t)}\iiint_V \left\langle \nabla\hat{p}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\right\rangle \,\mathrm{d}V \qquad (40)$$

to the other irrotational terms. This is not justified in reality, since the unsteady vorticity must be used in the evaluation of Eq. (40). Inclusion of this single rotational term in the energy balance to the exclusion of all the others is a not a mathematically legitimate step. Nevertheless, we place it in Eq. (19) with the irrotational terms in accordance with accepted practice. In evaluating Eq. (40), it is necessary to remember that the unsteady vorticity is

$$\boldsymbol{\omega} = \nabla\times\tilde{\boldsymbol{u}} = \left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right)\boldsymbol{e}_\theta = -\frac{\partial\tilde{u}_z}{\partial r}\boldsymbol{e}_\theta + O(M_b) \quad (41)$$

The derivative of the radial velocity component with respect to z is of the order of the mean Mach number, and can therefore be dropped as indicated. Using Eq. (25) for the axial rotational velocity, the amplitude of the vorticity vector becomes

$$\omega = -\exp(\alpha_m t)\left[\frac{\partial\tilde{u}_z^{(r)}}{\partial r}\cos(kt) + \frac{\partial\tilde{u}_z^{(i)}}{\partial r}\sin(kt)\right] \qquad (42)$$

Taking the cross product with the mean flow vector yields

$$\boldsymbol{U}\times\boldsymbol{\omega} = (-U_z\omega)\boldsymbol{e}_r + (U_r\omega)\boldsymbol{e}_z \qquad (43)$$

For longitudinal modes, the pressure gradient is in the z-direction, so the flow-turning integral reduces to

$$\alpha_4 = -\frac{M_b}{2E_m^2}\iiint_V U_r \frac{\partial\tilde{u}_z^{(i)}}{\partial r}\sin(k_m z)\,\mathrm{d}V \qquad (44)$$

Flandro[8] showed that the volume integral of Eq. (44) can be reduced to a surface integral that is identical to Culick's original flow-turning expression (see sections V.B and V.C of Ref. 8). Since the flow-turning integral is considered to be a key damping effect, it is appropriate to review its evaluation in detail for the axial mode case.

First, consider the derivative of the axial rotational velocity (the vorticity) appearing in Eq. (44),

$$\frac{\partial\tilde{u}_z^{(i)}}{\partial r} = -\sin x \exp\phi \frac{\partial(\cos\psi)}{\partial r}\sin(k_m z\sin x) + \ldots \quad (45)$$

Only the leading term in this derivative is shown, since it is several orders of magnitude larger than the other terms resulting from chain-rule differentiation of factors such as $\sin x$ or $\exp\phi$. The reason for this expansion becomes obvious when it is remembered that the derivative of ψ with respect to r is proportional to the inverse of the mean flow Mach number as shown in Eq. (12). Then

$$\frac{\partial\tilde{u}_z^{(i)}}{\partial r} = \frac{k_m}{M_b U_r}\sin x\exp\phi\sin\psi\sin(k_m z\sin x) + O(1) \quad (46)$$

and the volume integral of Eq. (44) can be evaluated for the assumed cylindrical chamber as

$$\alpha_4 = -\tfrac{1}{2}k_m E_m^{-2}\int_0^{2\pi} d\theta \int_0^{L/R}\int_0^1 r\sin x\exp\phi\sin\psi$$

$$\times\sin(k_m z\sin x)\sin(k_m z)\,\mathrm{d}r\mathrm{d}z \qquad (47)$$

To find the exact value, it is necessary to use numerical integration due to the complicated radial dependence. Following the observation made in Ref. 8 (see section V.B) that the integrand oscillates around zero from the chamber axis to the surface, the value of the integral depends only on its behavior near the upper limit. Then to very good approximation

$$\alpha_4 = -\frac{\pi k_m}{E_m^2}\int_0^{L/R}\sin^2(k_m z)\,\mathrm{d}z\int_0^1 r\sin x\exp\phi\sin\psi\,\mathrm{d}r$$

$$= -\frac{\pi k_m}{2E_m^2}\frac{L}{R}\int_0^1 r\sin x\exp\phi\sin\psi\,\mathrm{d}r \simeq -\frac{\pi M_b}{2E_m^2}\frac{L}{R} \qquad (48)$$

The final numerical value of this damping effect is the same as in Culick's original calculation if the acoustic form of the normalization parameter ($E_m^2 = \tfrac{1}{2}\pi L/R$) is inserted. Then $\alpha_4 = -M_b$, in dimensionless form, and the dimensional value becomes

$$\alpha_4{}^* = -M_b\frac{a_0}{R}\ \ \sec^{-1} \qquad (49)$$

C. The Rotational Flow Correction

Consider now the first rotational flow stability integral in Eq. (19),

$$\alpha_5 = \frac{1}{E_m^2\exp(2\alpha_m t)}\iiint_V \left\langle -\tilde{\boldsymbol{u}}\cdot\nabla\hat{p}\right\rangle \,\mathrm{d}V \qquad (50)$$

A companion term, the product of the acoustic velocity with the pressure gradient term in the momentum equation, has already been evaluated and shown to give rise to the pressure coupling effect of central importance. The physical interpretation of Eq. (50) is apparent. It represents the rate-of-work done on the rotational part of the unsteady flow by the oscillatory pressure forces. In this respect, it is analogous to the pressure coupling, which as already shown, collapses to a surface integral and is then interpreted as the $p\mathrm{d}V$ work done on the incoming flow.[1] Obviously, this term vanishes in a strictly irrotational environment; hence, it does not appear in the usual acoustic stability computations. Using Eqs. (23) and (25), and carrying out the time averaging, Eq. (50) reduces to

$$\alpha_5 = -\frac{1}{2E_m^2}\iiint_V \left[\tilde{\boldsymbol{u}}_m^{(r)} \cdot \nabla \hat{p}_m\right] \mathrm{d}V \qquad (51)$$

Notice that, for axial modes, it is only necessary to know the axial component of the rotational velocity field, since the pressure gradient is in the z-direction. No information regarding the rotational radial velocity is required in the evaluation of α_5. Equation (51) is readily evaluated for axial modes in a cylindrical chamber by the same method described for the flow-turning integral; one finds for low-order modes,

$$\alpha_5 \simeq \frac{\pi M_b}{2E_m^2}\frac{L}{R} \qquad (52)$$

which is equal to the flow-turning result <u>but opposite in sign</u>.

The growth rate contribution, α_5, has been the source of much dispute[68] in the rocket combustion instability community. It must therefore be scrutinized here in full detail. It appears in Eq. (51) in its natural form as an integral over the chamber volume. In its original apparition,[8,9] α_5 arose as a surface integral because Culick's perturbed wave equation method was used for computing the growth rate. To display α_5 in its original format, convert Eq. (51) to surface integral form by application of Gauss's divergence theorem. Remembering that the rotational field is solenoidal, $\nabla \cdot \tilde{\boldsymbol{u}} = 0$, and using the vector identity for the divergence of a scalar times a vector, one may use

$$\nabla \cdot \left(\tilde{\boldsymbol{u}}\hat{p}\right) = \tilde{\boldsymbol{u}} \cdot \nabla \hat{p} + \hat{p}\nabla \cdot \tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}} \cdot \nabla \hat{p} \qquad (53)$$

The conversion to a surface integral follows immediately, one finds

$$\iiint_V \left(\tilde{\boldsymbol{u}} \cdot \nabla \hat{p}\right) \mathrm{d}V = \iiint_V \nabla \cdot \left(\tilde{\boldsymbol{u}}\hat{p}\right) \mathrm{d}V = \iint_S \left(\boldsymbol{n} \cdot \tilde{\boldsymbol{u}}\hat{p}\right) \mathrm{d}S \qquad (54)$$

The growth rate contribution in surface integral form then becomes

$$\alpha_5 = -\frac{1}{2E_m^2}\iint_S \left[\boldsymbol{n} \cdot \tilde{\boldsymbol{u}}_m^{(r)} \hat{p}_m\right] \mathrm{d}S \qquad (55)$$

This is precisely the term described in Eq. (89) of Ref. 8 as the *rotational flow correction*, for want of a better name. The difficulty with the surface integral form is that its evaluation implies detailed knowledge of the normal component of the rotational velocity at the surface. This approach was followed in Refs. 8 and 9, and the radial component, $\boldsymbol{n} \cdot \tilde{\boldsymbol{u}} = \tilde{u}_r$, was deduced by integrating the continuity equation.[11] It was found that the radial component of the rotational velocity, $\tilde{u}_r$, is proportional to the mean flow Mach number, does not vanish at the edge of the combustion zone, and is proportional to the unsteady pressure mode shape.[8,9,11] These findings lead to an unfortunate interpretation, reported in Ref. 8, that α_5 can be treated as a correction to the surface coupling effect deduced in the irrotational part of the stability calculation. This forces one to search for new sources of mass flux within the flame zone. Critics[68] misinterpreted this observation to imply that changes in the surface response to pressure fluctuations are implied. The actual source of the new radial mass flux can be readily identified in the parallel gas motions within the combustion layer. Since there exists a momentum defect at the surface due to the no-slip requirement, additional oscillatory mass flux in the radial direction is generated. The simple control volume shown in Fig. 5 describes the two-dimensional field in the combustion zone and illustrates these ideas.

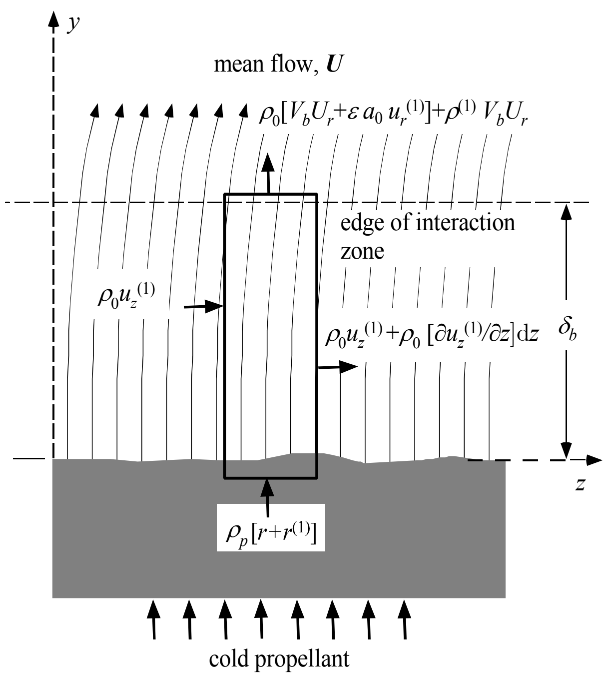

Fig. 5 Mass balance across the combustion zone.

It is clear that, due to the gradient in axial velocity fluctuations parallel to the surface, there is a net flux of mass into the vertical sides of the control volume. This is reflected in the net unsteady mass flux at the edge of the combustion zone. It is this additional radial mass flux that is accounted for in Eq. (52); no modification of the oscillatory mass flux created by pressure coupling at the propellant side (*i.e.*, solid/gas interface) of the control volume is implied.

The formal approach just described, carried out in detail, fully justifies the results reported in the earlier papers.[8,9] Despite these findings, the combustion instability community is understandably wary of any suggestion indicating that a mechanism other than the time-honored pressure coupling arises in the combustion zone. This point of confusion is resolved by using the volume integral form, Eq. (51), in place of the surface integral, Eq. (52). Both forms represent the same destabilizing influence on system stability, but the volume integral is more easily evaluated, and is therefore the preferred form.

It is also important to understand that the rotational flow correction, Eq. (51), and the flow-turning integral, Eq. (40), are <u>separate and distinct parts</u> of the interaction of the acoustic field with vorticity production. Despite claims to the contrary, Eq. (51) is <u>not</u> an alternative way to represent flow-turning. Brown has attempted to prove that the sign on α_5 given in Refs. 8 and 9 should be reversed.[68] If he is correct in this assertion then both contributions to the growth rate have the same sign. Since, as shown in Ref. 8 they have nearly equal magnitude, Brown then assumes that they must represent, in reality, the same physical effect. He claims that Eq. (51) duplicates what is already accounted for by Eq. (40).[68] Since this paradoxical claim can have important consequences in the combustion stability research community, it is necessary to resolve the matter here.

Consider the combination of the flow turning and the rotational flow correction as represented by Eqs. (44) and (51). Their sum is

$$\alpha_4+\alpha_5=\frac{1}{2E_m^2}\iiint_V\left[-M_bU_r\frac{\partial\tilde{u}_z^{(i)}}{\partial r}+k_m\tilde{u}_z^{(r)}\right]\sin\left(k_mz\right)\mathrm{d}V \tag{56}$$

If the value for the derivative of the imaginary part from Eq. (46) is used and Eq. (31) is substituted for the real part of the rotational axial velocity, then the integrand becomes

$$-M_bU_r\frac{\partial\tilde{u}_z^{(i)}}{\partial r}+k_m\tilde{u}_z^{(r)}=-M_bU_r\frac{k_m}{M_bU_r}\sin x\exp\phi\sin\psi$$

$$\times\sin\left(k_mz\sin x\right)+k_m\sin x\exp\phi\sin\psi\sin\left(k_mz\sin x\right)=0 \tag{57}$$

This proves that the flow-turning is *exactly cancelled* by the rotational flow correction as asserted in Refs. 8 and 9; that is

$$\alpha_4+\alpha_5=0 \tag{58}$$

This result may seem contrary to some experimental results, which apparently require that flow-turning be included for acceptable agreement with the stability theory.[81] Contrary opinions have been expressed by many other experimentalists who find that removal of the flow-turning leads to better agreement with motor growth rate data. However, several other rotational flow corrections remain to be evaluated. Let us reserve judgment until all the pieces have been finally assembled.

D. More Rotational Growth Rate Contributions

It is now necessary to tackle the remaining terms in Eq. (19); a seemingly daunting task. None of these new terms has been accounted for in previous studies. Fortunately, some of them do not contribute significantly to the system energy balance. Again, it is instructive to examine these stability integrals individually.

The second term on the third line in Eq. (19) can be converted immediately to a surface integral by using standard vector identities and the fact that the rotational velocity field is solenoidal. One finds that

$$\iiint_V\left\langle -M_b\hat{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\hat{\boldsymbol{u}}\right)\right\rangle\mathrm{d}V$$

$$=\iint_S\left\langle -M_b\boldsymbol{n}\cdot\hat{\boldsymbol{u}}\left(\boldsymbol{U}\cdot\hat{\boldsymbol{u}}\right)\right\rangle\mathrm{d}S=O\left(M_b^2\right) \tag{59}$$

Since, as already shown, the normal velocity fluctuation at the surface is of order M_b, this term is negligible.

The third and fourth terms are most easily handled together. The volume integral can be converted to the sum of a surface integral and a simpler volume integral. The result is

$$\iiint_V\left\langle -M_b\left(\hat{\boldsymbol{u}}+\tilde{\boldsymbol{u}}\right)\cdot\nabla\left(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}\right)\right\rangle\mathrm{d}V$$

$$=-M_b\iint_S\left\langle\boldsymbol{n}\cdot\boldsymbol{u}\left(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}\right)\right\rangle\mathrm{d}S+M_b\iiint_V\left\langle\left(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}\right)\nabla\cdot\hat{\boldsymbol{u}}\right\rangle\mathrm{d}V \tag{60}$$

where the surface integral is again second-order in the Mach number because the normal velocity fluctuation is of the order of M_b. Then the remaining part must be evaluated. Using the continuity equation with Eq. (23)

$$\nabla \cdot \hat{\boldsymbol{u}} = -\frac{\partial \hat{p}}{\partial t} = k_m \hat{p}_m \exp(\alpha_m t)\sin(k_m t) + O(M_b) \quad (61)$$

and inserting Eq. (25), one finds the growth rate to be

$$\frac{k_m M_b}{2E_m^2}\iiint_V \left[\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}_m^{(i)}\right]\hat{p}_m \,\mathrm{d}V = 2\pi^2\int_0^{L/R} z\sin(k_m z)$$

$$\times\cos(k_m z)\,\mathrm{d}z\int_0^1 r\cos x\sin x\exp\phi\cos\psi\,\mathrm{d}r \quad (62)$$

The radial integral yields a value of the order of the mean Mach number, so this term represents a negligible growth rate contribution.

The fifth term is highly interesting since it is the companion of the original flow-turning effect represented by Eq. (40). Now, we must evaluate

$$\alpha_6 = \frac{1}{E_m^2\exp(2\alpha_m t)}\iiint_V \left\langle M_b\tilde{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\right\rangle dV \quad (63)$$

where the rotational velocity appears in the place of the acoustic velocity as in the flow-turning integral. Using Eq. (42) for the unsteady vorticity, and dropping terms of the second-order, one finds

$$\alpha_6 = -\frac{M_b}{4E_m^2}\iiint_V U_r\frac{\partial}{\partial r}\left[\tilde{\boldsymbol{u}}^{(r)}\cdot\tilde{\boldsymbol{u}}^{(r)} + \tilde{\boldsymbol{u}}^{(i)}\cdot\tilde{\boldsymbol{u}}^{(i)}\right]\mathrm{d}V \quad (64)$$

Inserting the expressions for the rotational velocity, and integrating over the volume, one finds, for a cylindrical chamber,

$$\alpha_6 = \frac{\pi M_b}{4E_m^2}\frac{L}{R}\int_0^1 \sin x\frac{\partial}{\partial r}\sin^2 x\exp^2\phi\,\mathrm{d}r \simeq \frac{3\pi M_b}{8E_m^2}\frac{L}{R} \quad (65)$$

This is an energy source rather than a sink as was the case for the flow turning. In a sense, it represents an unexpected finding. Conventional analyses indicate that the only energy source in the motor chamber is that produced by the pressure coupled combustion response; all other stability integrals are thought to be sinks of energy. This new source term is clearly related to the creation of unsteady vorticity at the boundaries. There are connections to the well-known vortex shedding energy source that we will examine carefully in the next section of the paper.

The sixth remaining term, involves the volume integral

$$\iiint_V -M_b\left\langle -\tilde{\boldsymbol{u}}\cdot(\hat{\boldsymbol{u}}\times\boldsymbol{\Omega})\right\rangle \mathrm{d}V \quad (66)$$

This growth rate contribution is negligible, since the cross product yields an axial vector component proportional to the radial acoustic velocity. Therefore, only terms of second-order in the mean flow Mach number are generated.

The last two terms are viscous damping expressions. In the classical (irrotational) combustion instability calculations, viscous effects are ignored completely on the basis that there are no strong velocity gradients at the surface to give rise to significant shearing stresses. Acoustic boundary layer corrections of the usual sort do not properly account for the shearing stresses when there is strong convection through the surface layer. A correction to the dilatational (volume damping) effect is represented in the seventh rotational term. Using the same methods used to evaluate the other terms, this one can be transformed into a surface integral, viz.

$$\iiint_V \left\langle \delta_d^2\tilde{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\hat{\boldsymbol{u}})\right\rangle \mathrm{d}V = -\delta_d^2\iint_S \left\langle \boldsymbol{n}\cdot\tilde{\boldsymbol{u}}\partial p^{(1)}/\partial t\right\rangle \mathrm{d}S \quad (67)$$

which, for realistic values of the parameters, must be negligible because both the dimensionless viscosity coefficient δ_d^2 and the normal velocity at the bounding surfaces are of the order of the mean flow Mach number.

Finally, the last term represents the viscous damping

$$\alpha_7 = \frac{1}{E_m^2\exp(2\alpha_m t)}\iiint_V \left\langle -\delta^2(\hat{\boldsymbol{u}}+\tilde{\boldsymbol{u}})\cdot(\nabla\times\boldsymbol{\omega})\right\rangle \mathrm{d}V \quad (68)$$

where the composite unsteady velocity appears instead of just the acoustic part as in previous works (cf. Eq. (95) in Ref. 8). After carrying out the indicated calculations and inserting the various components of the velocity vectors, the viscous growth rate reduces to

$$\alpha_7 = \frac{\delta^2}{2E_m^2}\iiint_V\left[\tilde{u}_z^{(r)}\frac{\partial^2\tilde{u}_z^{(r)}}{\partial r^2} + \tilde{u}_z^{(i)}\frac{\partial^2\tilde{u}_z^{(i)}}{\partial r^2}\right]\mathrm{d}V$$

$$= -\frac{\delta^2}{2E_m^2}\left(\frac{k_m}{M_b}\right)^2\iiint_V r^2\exp^2\phi\sin^2(k_m z)\,\mathrm{d}V \quad (69)$$

where smaller terms have been dropped. This expression is easily evaluated for a full-length cylindrical grain. The result is

$$\alpha_7 \simeq -\frac{\pi\delta^2}{8E_m^2}\frac{L}{R}\left(\frac{k_m}{M_b}\right)^2 = -\frac{\pi M_b\xi}{8E_m^2}\frac{L}{R} \quad (70)$$

to good approximation. Notice that the importance of viscous damping increases rapidly with frequency. Since the square of the Mach number appears in the denominator, this term may be as important as any of the others retained in the analysis. Contrary to previous assessments, we find that viscous damping must not be discarded, especially in the case of turbulent mean flows. Then the transport properties are modified, and δ^2 may be much larger than for the laminar case. In order to properly evaluate Eq. (69) in the turbulent case, a comprehensive numerical algorithm will be needed. Work of this sort has already begun.[10,18,19]

III. Discussion

The system stability is determined by superposition of a set of stability integrals that includes both the original set found from the classical irrotational analysis and several new ones that represent the effects of the rotational unsteady flow. We must now assess the impact of the proposed changes on stability assessment methodology.

It is useful to compare the theoretical irrotational and rotational calculations for typical motor parameters. Please note that account is not taken of other gain loss effects such as particulate drag and residual combustion that could be important in real rocket motors. Then for the classical model, the growth rate is

$$\alpha_{\text{standard}} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \tag{71}$$

which includes the flow-turning correction that is usually assumed, and where evaluations are to be made using the irrotational energy normalization value

$$\left(E_m^2\right)_{\text{irrotational}} = \tfrac{1}{2}\pi L / R \tag{72}$$

for the case of axial modes in a cylindrical combustion chamber.

For the full stability calculation, including all rotational corrections, one must use

$$\alpha_{\text{composite}} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 \tag{73}$$

where the cancellation of the *flow-turning* effect and the *rotational flow correction* is accounted for; furthermore, evaluations must be made using the corrected energy normalization value

$$\left(E_m^2\right)_{\text{composite}} = \tfrac{5}{8}\pi L / R \tag{74}$$

written, again, for axial modes in a full-length cylinder.

No attempt is made here to compare numerical values to experimental results published recently, which come from nonlinear decay data.[81,82] It has been claimed that some measurements demonstrate that the standard acoustic stability results give acceptable prediction of the observed behavior. It must be remarked that none of the experimental points illustrated are taken from sections of the data where there was rapid growth in the waves; only pulse decay data was utilized. The standard stability calculations did not predict linear growth for any of the motors tested. Careful study of the data reduction process suggests that the reason data points displayed show unusually high damping is that it was necessary to filter the data to eliminate the strong harmonic content; the presence of higher modes with quite appreciable amplitude accompanying the filtered first mode data indicates nonlinear interactions that cannot be ignored. Linear theory does not apply in the situation described.[81,82]

A few simple calculations will demonstrate some features of the results found here. It is quite easy to apply the formulas for a cylindrical chamber. Table 1 from Ref. 9, showing values for the key parameters for representative motors is reproduced here for convenience. Let us focus on large motors of the type described as Space Shuttle Solid Rocket Booster (SRB) in the table. As described in the Introduction, there have been many observations of longitudinal mode instability in large motors such as the SRB. These oscillations have most often been attributed to vortex shedding phenomena. In that model, the natural instability of the mean flow, especially when flow separation is present, suggests that large scale vortex structures are generated, which may add energy to the unsteady field in the manner of a wind musical instrument. However, as shown in the papers by French investigators,[29-32] there is overwhelming evidence that such instabilities appear even in the absence of protruding restrictors or inter-segment gaps that were thought to be sites of vortex shedding in earlier studies.

Let us test the result of the analysis given here by applying it to a simplified SRB geometry in which the grain is assumed to be a long, straight, and unsegmented cylinder. No measurements of the propellant admittance function need to be attempted for the SRB propellant because the natural frequencies in this long motor are very low (15 Hz first mode). Hence, one may assume a typical (dimensionless) value of $A_b \simeq 1$ for this situation. Standard short nozzle

Table 1 Physical parameters for typical motor systems

Motor	L (m)	R (m)	M_b	δ	k_m	S	ξ
Small Motor (Yang and Culick)	0.60	0.025	1.7^{-3}	5.49^{-4}	1.33^{-1}	76.87	1.0309
Tactical rocket (Typical geometry)	2.03	0.102	3.1^{-3}	2.74^{-4}	1.58^{-1}	50.84	0.0624
Cold Flow Experiment (Shaeffer and Brown)[25-26]	1.73	0.051	3.3^{-3}	6.07^{-4}	9.24^{-2}	28.3	0.0909
Space Shuttle SRB	35.1	0.700	2.3^{-3}	1.04^{-4}	6.27^{-2}	27.24	0.0035

Table 2 Comparison of stability estimates

Motor	A_b	f (Hz)	α_{standard} (sec^{-1})	$\alpha_{\text{composite}}$ (sec^{-1})
Small Motor (Yang and Culick)	2.50	1227	32.1	145.4
Tactical rocket (Typical geometry)	1.20	360	-43.7	26.9
Space Shuttle SRB	1.00	19.5	-5.70	2.2

damping is also assumed. No attempt is made to account for particle damping, since this effect is negligible for such a low frequency of oscillation. Application of the standard stability code yields, for this case,

$$\alpha_{\text{standard}} \approx -5.7 \text{ sec}^{-1} \tag{75}$$

Being comparable to values that were computed during the development of this motor, standard predictions lead to the impression that the SRB would be very stable. For such large motors, growth or decay rates are always found to be small. The result is misleading since, in practice, significant thrust oscillations (~ 20 klbf. peak-to-peak) are observed in static tests of all versions of the SRBs. Oscillations are also detected in flight with amplitudes sometimes exceeding 2 psi (peak-to-peak). If the new analysis is used instead, one finds

$$\alpha_{\text{composite}} \approx 2.2 \text{ sec}^{-1} \tag{76}$$

The new result reconciles with actual observations indicating that strong vorticity waves should be expected in this motor. It is interesting that the growth rate predictions for the next two or three longitudinal modes are of the same order of magnitude as the first mode results shown. In the SRB case, three or four modes were always readily discernible in the waterfall data.

Table 2 shows comparisons of stability computations (first axial mode) for three of the configurations defined in Table 1. Typical admittance values have been used, and no attempt has been made to include the effects of particle damping. In all cases, the new model predicts a less stable system. In comparing the standard approach to the composite method, in which all rotational effects are included, it is clear the numerical values are all of the same order of magnitude. When comparing the results to experimental data, it is noted that they exhibit the correct order of magnitude. This does not constitute proof that one or the other method is a correct representation of the system. However, at least for one situation, namely for the large SRB type motor, there is no question that the new method yields a closer prediction of the experimental observations.

IV. Conclusions

The new stability estimation method described in this paper displays a new energy source term not found in the acoustic instability methodology. The new results appear to explain the growth of longitudinal oscillations in large solid propellant motors that have appeared in many development programs. Considerable work lies ahead in fully utilizing what we have found as the basis for a predictive algorithm for use in motor design and in data reduction and interpretation. Although the general formulation is not geometry dependent, we have relied heavily on the assumption of a simple cylindrical geometry with longitudinal plane wave oscillations to enable the evaluations shown.

Clearly much work will be needed to determine the stability characteristics of waves in more realistic motor configurations. Extension of the result to partial grains requires further study. Complex geometrical features of the burning port such as slots and fins represent difficulties that may require a full numerical treatment of both the steady and the unsteady flow by application of computational fluid dynamics techniques. Inclusion of regions of separated flow, for example at segment interfaces in large motors, introduce additional complications. Full Navier-Stokes solutions may eventually provide the necessary information. It is possible that new numerical computations can be validated by use of the analytical models that have been described herein.

The new results introduced in this paper can readily be tested using an experimental apparatus already in place. For instance, the cold flow models used by French investigators in their Ariane work could be used to test the new theoretical framework. Since only growth rates are predicted in the linearized theory, it would be necessary to modify the test procedure in such a way that transient behavior could be assessed.

In the latter regard, new emphasis on modeling of the nonlinear aspects of unsteady flow with rotational flow corrections is clearly justified. There are no methods available for predicting important features of real motor operation such as limit-cycle amplitude or triggering. Certainly, rotational flow effects will play a central role in resolving these difficulties.

Finally, new stability integrals discovered in the present work, in particular, the "parietal" growth rate, α_6, may have a bearing on the unresolved problem of *velocity coupling*. This new growth rate term certainly represents one way in which velocity fluctuations affect the system stability. In fact, this term can be recast in the form of a surface integral and it is then obvious that a "velocity coupling" response function can be connected to it. These matters require further study.

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