AIAA 2002-2856

Pulsatory Channel Flow for an Arbitrary Volumetric Flowrate

Daniel E. Muntges and Joseph Majdalani
Marquette University
Milwaukee, WI 53233

32nd AIAA Fluid Dynamics Conference
24–26 June 2002
St. Louis, MO

Pulsatory Channel Flow for an Arbitrary Volumetric Flowrate

Daniel E. Muntges* and Joseph Majdalani†
Marquette University, Milwaukee, WI 53233

In this study we present an analytical solution of the Navier-Stokes equations for pulsatory laminar duct flow specified by an arbitrary flowrate. The novelty of this solution lies in the application. The specification of a variant volumetric flowrate rather than a pressure gradient can be advantageous in those physical settings where pressures are more difficult to measure than flowrates. Using the method of residues in conjunction with Laplace transforms, we apply the Das and Arakeri approach to the classic viscous flow in a rectangular duct for a general pulsatory piston motion. We then present the special solutions for oscillatory and trapezoidal piston motions. After identifying explicit relations that prescribe shear reversal at the wall, limiting process verifications are carried out to confirm the validity of the analytical results.

Nomenclature

a = core-to-wall distance/half height
A = $a/\sqrt{\nu}$
b = total width of the duct
k = kinetic Reynolds number, $\omega a^2/\nu$
$\mathfrak{L}$ = Laplace transform
P = total pressure
Q = volumetric flowrate
T = dimensionless time, ωt
u = total velocity in the $x-$direction
u_m = mean velocity
$u_{n,+}$ = time-dependent component of the total velocity derived from positive residues
$u_{n,-}$ = time-dependent component of the total velocity derived from negative residues
u_p = total piston velocity
u_{sn} = sine coefficient in the Fourier series that is used to define the piston velocity
u_{cn} = co-sinusoidal Fourier coefficient
Y = $y/\sqrt{\nu}$
β = kinetic ratio of the fluctuating and mean velocity amplitudes
η = dimensionless spatial coordinate, y/a
μ = dynamic viscosity, $\mu=\nu\rho$
ν = kinematic viscosity, $\nu=\mu/\rho$
ρ = fluid density

Subscripts
0 = time-independent component
$+,-$ = owing to positive or negative residues
c = cosine part of a complex variable
m = cross-sectional mean
p = piston
s = sine part of a complex variable
$-$ = variable that has been transformed from the $t-$domain to the $s-$domain

I. Introduction

THE assessment of pulsatory motions has a number of captivating applications involving wave propagation and control in chemical, biological, civil, and mechanical engineering.[1] Since pulsatory flow in these applications can be either advantageous, promoting mixing in thermal applications, or detrimental, causing cyclical damage to structures, detailed analytical models of the flow can be helpful in designing for optimal performance and endurance. This work adds to the

*Undergraduate Research Assistant.
†Assistant Professor, Department of Mechanical and Industrial Engineering. Member AIAA.

existing body of knowledge upon which such designs are based by providing the design engineer with an additional methodological tool. This tool is focused on a user-specified volumetric flowrate performance, rather than pressure. The technical relevance lies in the fact that it is practically easier to measure or calculate a time-dependent flowrate than it is to track a time-varying pressure gradient.

Our methodology is to solve the governing Navier-Stokes equation and supporting conditions by Laplace transformations paired with the method of residues. This methodology was recently proposed by Das and Arakeri[2] in their pipe flow application. The reader must be cautioned that the solution appears to be only possible insofar as the specified mean flow is Laplace transformable. In the current study, the generalized model will be developed and applied to a user-defined pulsatory motion arising in the viscous duct flow geometry. It will also be applied to the trapezoidal piston motion.

To start, we shall develop an analytical solution from the Navier-Stokes equation for the motion of a viscous fluid in a duct under the influence of a periodic volumetric flowrate. A generalization will be pursued for an arbitrary volumetric flowrate. After identifying the dynamic similarity parameters, their role will be examined in the limiting processes for which either a purely oscillatory motion or a fully-developed profile can be recovered. Furthermore, criteria for shear-reversal at the wall will be identified using asymptotic tools. Finally, the same method will be re-applied to the trapezoidal piston motion.

II. Problem Formulation

The problem concerns the two-dimensional flow within a rectangular channel of height $2a$ and width b. Here we use Cartesian coordinates, where x is the direction of flow and y is the direction of flow variation. The axial velocity reduces to $u(y,t)$ so long as we consider incompressible bi-directional flow in an infinitely long channel. The condition of incompressibility ensures that variations in the pressure gradient act throughout the channel. Due to the remission of transverse velocities, the condition of flow continuity translates into

$$\frac{\partial(\rho u)}{\partial x} = 0 \text{ or } u \neq u(x)\,. \tag{1}$$

Subsequently, the governing equation becomes

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

where P is the pressure. Equation (2) indicates that $\partial P / \partial x \neq f(x)$ since $u \neq u(x)$. One is left with $\partial P / \partial x = f(y,t)$. However, from the $y-$momentum equation, one deduces that

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} \text{ or } P \neq P(y) \tag{3}$$

This prohibits the possibility of a spatially dependent pressure gradient. Rather, one must have

$$\frac{\partial P}{\partial x} = \frac{\partial P(t)}{\partial x}\,. \tag{4}$$

The boundary conditions for the flow profile are determined by boundary layer theory and flow symmetry; these require that

$$u(a,t) = u(-a,t) = 0 \tag{5}$$

$$\frac{\partial u(0,t)}{\partial y} = 0\,. \tag{6}$$

A further initial condition is determined from the known volumetric flowrate in the absence of knowledge of the pressure gradient. This can be specified from

$$Q(t) = 2b\int_0^a u(y,t)\mathrm{d}y = 2abu_p(t) \tag{7}$$

where $u_p(t)$ represents the velocity of a hypothetical piston that is generating the periodic volumetric flowrate Q. Using this third condition, it is possible to solve Eq. (2) by the Laplace technique. In fact, careful application of the Laplace transforms to Eqs. (2), (5), (6), and (7) yields the coupled set

$$\frac{\partial^2 \overline{u}(y,s)}{\partial y^2} - \frac{s}{\nu}\overline{u}(y,s) = \frac{1}{\mu}\frac{\partial \overline{P}(x,s)}{\partial x} - \frac{1}{\nu}u(y,0) \tag{8}$$

$$\overline{u}(a,s) = \overline{u}(-a,s) = 0 \tag{9}$$

$$\frac{\partial \overline{u}(0,s)}{\partial y} = 0 \tag{10}$$

$$2b\int_0^a \overline{u}(y,s)dy = 2ab\overline{u}_p(s) \tag{11}$$

Equation (8) is a second-order, non-homogeneous differential equation whose family of solutions exhibits the form

$$\overline{u}(y,s) = \overline{u}_h(y,s) + \phi_p$$

$$= C_1 \exp(-y\sqrt{s/\nu}) + C_2 \exp(y\sqrt{s/\nu}) + \phi_p \quad (12)$$

Here ϕ_p is the particular solution. The unknown coefficients are determined by fulfilling the boundary conditions at $y = a$ and $y = 0$. At the outset, one finds

$$C_1 = C_2 = \frac{-\phi_p}{\exp(pa) + \exp(-pa)} \quad (13)$$

where

$$p = \sqrt{s/\nu}\,. \quad (14)$$

These constants give

$$\overline{u}_h(y,s) = \phi_p\left[1 - \cosh(py)/\cosh(pa)\right]. \quad (15)$$

Using the boundary condition of the known flowrate, the particular solution can now be determined by substitution of Eq. (15) into Eq. (7). One puts

$$2b\int_0^a \phi_p\left[1 - \frac{\cosh(py)}{\cosh(pa)}\right]\mathrm{d}y = 2ab\overline{u}_p(s) \quad (16)$$

to obtain

$$\phi_p = -\frac{\overline{u}_p(s)pa}{\tanh(pa)}\,. \quad (17)$$

At this juncture, the total solution of the velocity profile is expressible in the $s-$domain as

$$\overline{u}(y,s) = \frac{-\overline{u}_p(s)a\sqrt{s/\nu}}{\tanh(a\sqrt{s/\nu})}\left[1 - \frac{\cosh(y\sqrt{s/\nu})}{\cosh(a\sqrt{s/\nu})}\right] \quad (18)$$

Transforming this solution to the time domain requires a Laplace inversion according to contour integration based on the Bromwich integral,

$$u(y,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{u}(y,s)e^{st}\mathrm{d}s\,. \quad (19)$$

The solution of this integral requires that the form of the piston velocity be specified. Being a pulsatory motion, it can be described most generally as a mean component superposed over sinusoidal components. The latter are best represented by a Fourier series in the time domain. In this vein, one can express an arbitrary piston motion in the form

$$u_p(t) = u_0 + \sum_{n=1}^{\infty} u_{cn}\cos(\omega n t) + \sum_{n=1}^{\infty} u_{sn}\sin(\omega n t) \quad (20)$$

which can be normalized via

$$u_p^*(T) = 1 + \sum_{n=1}^{\infty}\left[\frac{u_{cn}}{u_0}\cos(nT) + \frac{u_{sn}}{u_0}\sin(nT)\right] \quad (21)$$

In the $s-$domain, Eq. (20) converts to

$$\overline{u}_p(s) = \mathcal{L}\left[u_p(t)\right] = \frac{u_0}{s} + \sum_{n=1}^{\infty}\frac{u_{cn}s}{s^2 + n^2\omega^2} + \sum_{n=1}^{\infty}\frac{u_{sn}\sqrt{n^2\omega^2}}{s^2 + n^2\omega^2} \quad (22)$$

The total Bromwich integral can then be represented by

$$u(y,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}a\sqrt{s/\nu}}{a\sqrt{s/\nu} - \tanh(a\sqrt{s/\nu})} \times\left[1 - \frac{\cosh(y\sqrt{s/\nu})}{\cosh(a\sqrt{s/\nu})}\right] \times\left(\frac{u_0}{s} + \frac{u_{cn}s}{s^2 + n^2\omega^2} + \frac{u_{sn}\sqrt{n^2\omega^2}}{s^2 + n^2\omega^2}\right)\mathrm{d}s \quad (23)$$

This integral can be split into two distinct integrals for the purpose of evaluating the mean and sinusoidal contributions separately. This is accomplished by writing

$$u(y,t) = u_1(y,t) + u_2(y,t) \quad (24)$$

where

$$u_1(y,t) = \frac{u_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{A[\cosh(A\sqrt{s}) - \cosh(Y\sqrt{s})] \times e^{st}/\left[As\cosh(A\sqrt{s}) - \sqrt{s}\sinh(A\sqrt{s})\right]\right\}\mathrm{d}s \quad (25)$$

and

$$u_2(y,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(u_{cn}s + \sqrt{n^2\omega^2}u_{sn})}{(s^2 + n^2\omega^2)}e^{st} \times\frac{A\sqrt{s}[\cosh(A\sqrt{s}) - \cosh(Y\sqrt{s})]}{[A\sqrt{s}\cosh(A\sqrt{s}) - \sinh(A\sqrt{s})]}\mathrm{d}s \quad (26)$$

The hyperbolic function arguments correspond to the normalized quantities

$$A = a/\sqrt{\nu}\text{ , and } Y = y/\sqrt{\nu}\,. \quad (27)$$

III. Evaluation

To determine the flow profile, the Bromwich integrals are evaluated by summing the residues at the poles of the integrands. First, a Laurent series expansion of the first integral reveals that $s = 0$ is a simple pole with the residue

$$res_{1,0} = \tfrac{3}{2}u_0(1 - y^2/a^2) \quad (28)$$

Additionally we must consider other singular points, including the zeros of the integrand denominator. Laurent series expansions about these points reveal no poles, thus leading to a residue of zero. It is noted that these zero points are identical for the hyperbolic contributions of integrand denominators. We thus conclude that

$$u_1(y,t) = \tfrac{3}{2} u_0 (1 - y^2 / a^2) \tag{29}$$

Moving on to the second integral, a series expansion of the entire integral reveals no poles. Thus, we must evaluate the singular points, or zeros of the denominator. The zero of the hyperbolic contribution, $A\sqrt{s} - \tanh(A\sqrt{s}) = 0$, has no residue. One evaluates the residues at the zeros of the term $s^2 = \omega^2 n^2$, which, for $n = 1, 2, \ldots, \infty$ yield

$$s_n = \pm i n \omega\,. \tag{30}$$

The corresponding residues are found to be

$$res_n = \sum_{n=1}^{\infty} 2A\, e^{s_n t} s_n^{3/2} (u_{cn} s_n + \sqrt{n^2\omega^2}\, u_{sn}) \times [\cosh(A\sqrt{s_n}) - \cosh(Y\sqrt{s_n})] / \delta \tag{31}$$

where

$$\delta = (-3s_n^2 + A^2 s_n^3 + n^2\omega^2 + A^2 n^2 \omega^2 s_n)\sinh(A\sqrt{s_n}) + A\sqrt{s_n}(3s_n^2 - n^2\omega^2)\cosh(A\sqrt{s_n}) \tag{32}$$

The total solution is thus the sum of $res_{1,0}$ and res_n. Further evaluation of Eq. (31) at the zeros given by Eq. (30) reveals the following form of the second residue: for $s_n = +in\omega$,

$$res_{n,+} = \sum_{n=1}^{\infty} \left[\cosh(A\sqrt{i\omega n}) - \cosh(Y\sqrt{i\omega n})\right] \times \frac{\tfrac{1}{2}\exp(i\omega n t)\omega n A\left(u_{cn} i + u_{sn}\right)}{\left[i\omega n A\cosh(A\sqrt{i\omega n}) - \sqrt{i\omega n}\sinh(A\sqrt{i\omega n})\right]} \tag{33}$$

and, for $s_n = -in\omega$,

$$res_{n,-} = \sum_{n=1}^{\infty} \left[\cosh(A\sqrt{-i\omega n}) - \cosh(Y\sqrt{-i\omega n})\right] \times \frac{\tfrac{1}{2}\exp(-i\omega n t)\omega n A\left(u_{cn} + i u_{sn}\right)}{\left[\omega n A\cosh(A\sqrt{-i\omega n}) + \sqrt{i\omega n}\sinh(A\sqrt{-i\omega n})\right]} \tag{34}$$

Combining all residues, one arrives at

$$u_2(y,t) = res_{n,+} + res_{n,-} \tag{35}$$

At this point, it may be useful to recall that the mean and time-dependent velocities are simply the residues of integration. By recognizing that the mean part stems from $u_m = u_1 = res_{1,0}$, and that the time dependent velocities are the sum of $u_{n,+} = res_{n,+}$, and $u_{n,-} = res_{n,-}$, one can put

$$u(\eta,t) = u_m(\eta) + u_{n,+}(\eta,t) + u_{n,-}(\eta,t) \tag{36}$$

where $\eta = y/a$ is the dimensionless transverse coordinate. The full solution is reproducible from

$$u_m = \tfrac{3}{2} u_0 (1 - \eta^2) \tag{37}$$

$$u_{n,+} = \sum_{n=1}^{\infty} \frac{kn\left[\cosh(\sqrt{ikn}) - \cosh(\eta\sqrt{ikn})\right]}{\left[ikn\cosh(\sqrt{ikn}) - \sqrt{ikn}\sinh(\sqrt{ikn})\right]} \times \tfrac{1}{2}\exp(i\omega n t)\left(u_{cn} i + u_{sn}\right) \tag{38}$$

and,

$$u_{n,-} = \sum_{n=1}^{\infty} \frac{kn\left[\cosh(\sqrt{-ikn}) - \cosh(\eta\sqrt{-ikn})\right]}{\left[kn\cosh(\sqrt{-ikn}) + \sqrt{ikn}\sinh(\sqrt{-ikn})\right]} \times \tfrac{1}{2}\exp(-i\omega n t)\left(u_{cn} + i u_{sn}\right) \tag{39}$$

where $k = \omega a^2 / \nu$ is the known kinetic Reynolds number.[3,4] It may be useful to recall that k is the square of the Womersley number ($\alpha = a\sqrt{\omega/\nu}$) and has been referred to at times by the frequency parameter[5-8] or the oscillation Reynolds number.[9] In arterial flows, the kinetic Reynolds number is in the order of $k \approx 11$.[10,11] It should also be noted that, despite the complex form of $u(y,t)$, it can be shown through the aid of a symbolic program that its imaginary part vanishes upon expansion.

IV. Discussion

To illustrate the results of the method, two sets of plots have been generated. But first, an essential flow attribute has been introduced in the form of the kinetic ratio β. This parameter is the ratio of the fluctuating and mean velocity components,

$$\beta = u_{sn} / u_m = u_{cn} / u_m \tag{40}$$

For this simple pulsatory piston motion, simplifying assumptions can be realized and these can be exploited in the generation of the plots. These include the truncation of the residues to one term, for $n = 1$, and the equalization of the single sine and cosine terms. The resulting piston motion is illustrated in Fig. 1.

The corresponding viscous flow is illustrated in Fig. 2 where the mean and fluctuating components are equalized by setting $\beta = 1$. This equalization leads to a nearly even superposition of oscillatory and mean amplitudes in prescribing the total axial profile. The plain superposition is particularly seen at very low frequency in Fig. 2a where the core velocity is augmented by the amplitude of the oscillatory motion.

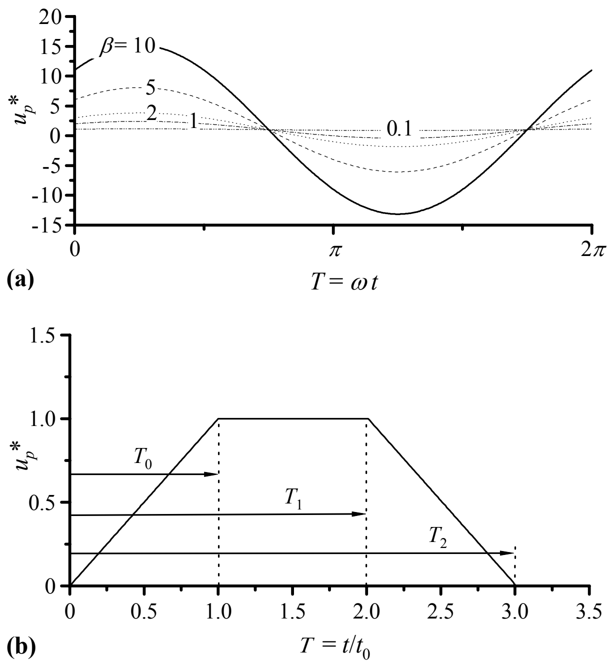

Fig. 1 The normalized piston velocity shown for a) pulsatory and b) trapezoidal motions. The trapezoidal motion is for $t_1 = 2t_0$ and $t_2 = 3t_0$.

In this quasi-steady superposition of amplitudes, the parabolic shape associated with a fully-developed velocity can be discerned. During a pulsatory cycle, the entire flow appears to reciprocate, forward and backward, in phase with the piston motion. With successive increases in the frequency parameter, however, this behavior is no longer the case. Figures 2b–d describe the evolution of the pulsatory profile into a reciprocating flow in which the motion reverses at some point above the wall during part of a cycle. The flow reversal splits the profile into a core region in the streamwise direction, and a shear region near the wall that seems to be induced by sudden piston retraction during a cycle. This rapid withdrawal is certainly more pronounced at higher frequency parameters.

The presence of flow reversal above the wall appears to be a special feature of a pulsatory volumetric flowrate. No such separation occurs in the analogous problem involving a pulsatory pressure gradient.[1] This characteristic behavior is only observed at sufficiently high kinetic Reynolds numbers for a fixed kinetic ratio.

In the second set of plots, Fig. 3, the kinetic Reynolds number is held constant at $k = 10$,

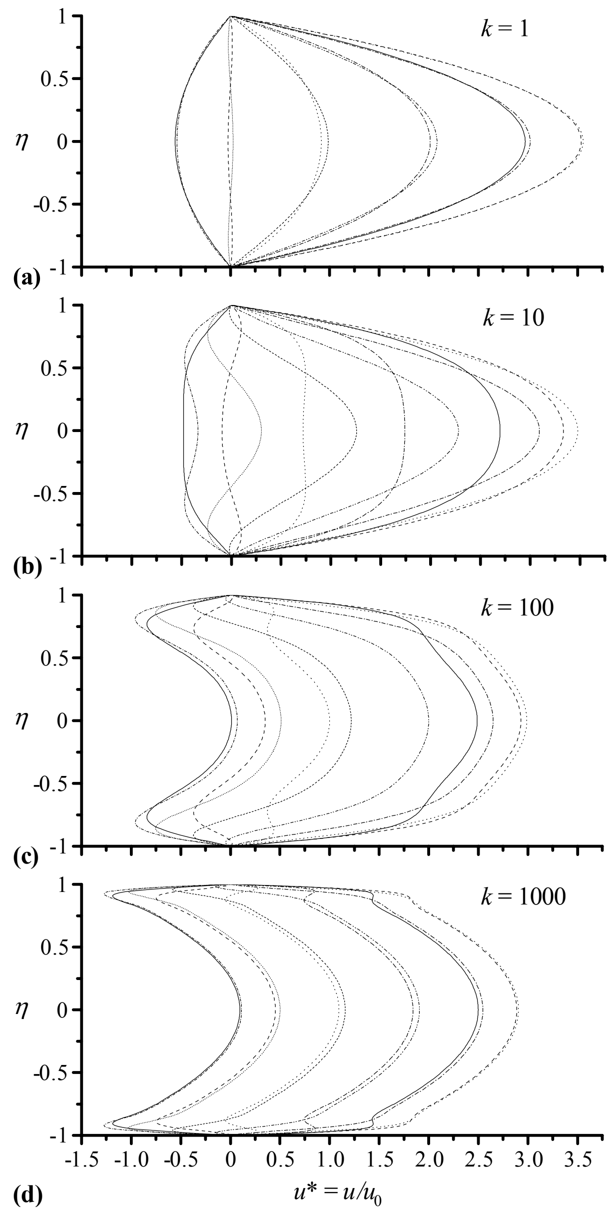

Fig. 2 Non-dimensional velocity profiles for $\beta = 1$ and a kinetic Reynolds number of a) 1, b) 10, c) 100, and d) 1000. Timelines are shown for $T = \omega t$ [deg] =
—— 0 - - - - 30 ······ 60 –·–·– 90 –··–·· 120 ------- 150
·········· 180 –·–·–·– 210 —— 240 - - - - 270 ······ 300 –·–·– 330.

while the kinetic ratio β is varied. In Fig. 3a, the large unsteady amplitude derived from $\beta = 100$ leads to a predominantly oscillatory flow with no mean component. The classic Stokes solution with sinusoidal piston displacement at infinity is recovered. Conversely, when β is reduced to 1% of the mean velocity in Fig. 3c, the parabolic Poiseuille flow is restored. The central plot, Fig.

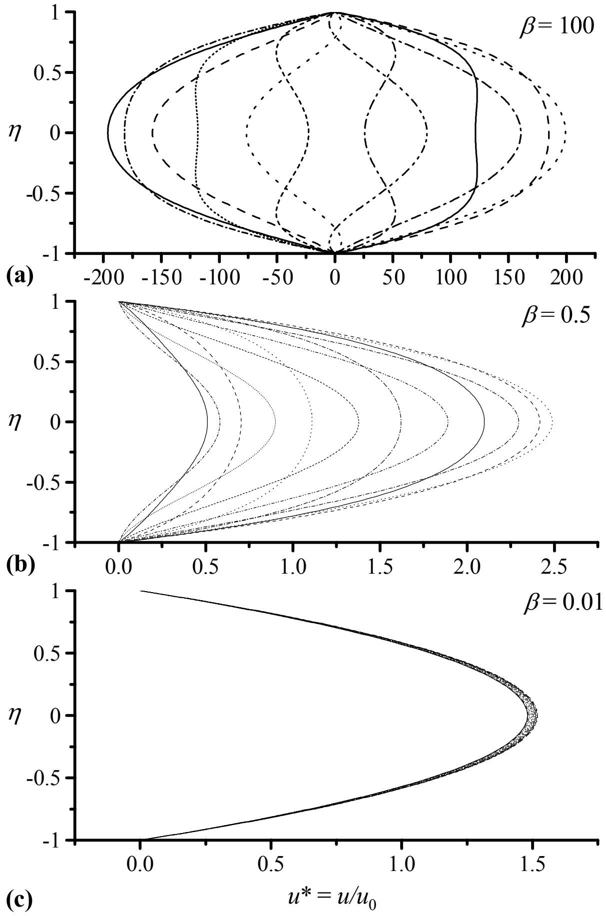

Fig. 3 Non-dimensional velocity profiles for $k=10$ and a displacement parameter of a) 100, b) 0.5, and c) 0.01. Timelines are shown using a constant increment of 30 degrees as indicated in Fig. 2.

3b, illustrates a transitional case for which both fluctuating and mean components are of the same order.

For a given frequency parameter, it is possible to calculate the minimum kinetic ratio that can lead to the total flow reversing direction during a cycle. When the flow reverses, the direction of shear at the wall switches as well. Based on Eq. (36), the relevant criterion translates into

$$\partial u(1,t)\,/\,\partial \eta = 0 \tag{41}$$

hence

$$\sum_{n=1}^{\infty}\left\{\frac{e^{-i\omega nt}\left(u_{cn}+iu_{sn}\right)n\sqrt{-ikn}\,\sinh(\eta\sqrt{-ikn})}{\left[kn\cosh(\sqrt{-ikn})+\sqrt{ikn}\,\sinh(\sqrt{-ikn})\right]} + \frac{e^{i\omega nt}\left(u_{cn}i+u_{sn}\right)n\sqrt{ikn}\,\sinh(\eta\sqrt{ikn})}{\left[ikn\cosh(\sqrt{ikn})-\sqrt{ikn}\,\sinh(\sqrt{ikn})\right]}\right\}+\frac{6}{k}=0 \tag{42}$$

This condition necessitates a sufficiently large kinetic ratio. For a pulsatory flowrate variation prescribed after Eq. (40), one must have

$$\beta \ge \left\{3\left[(k-1)\cos\sqrt{2k}+(k+1)\cosh\sqrt{2k} -\sqrt{2k}(\sin\sqrt{2k}+\sinh\sqrt{2k})\right]\right\} \Big/ \left\{\left[\cosh\sqrt{2k}(\cos T-\sin T)+\cos\sqrt{2k}(\sin T-\cos T) +\sqrt{2k}\left(\sin\sqrt{2k}\sin T-\sinh\sqrt{2k}\cos T\right)\right]k\right\} \tag{43}$$

The minimum value $\beta_{\min}$ required to trigger flow reversal can be calculated at the time $T_{\min}$ that causes the right-hand-side of the inequality to pass through a minimum. Setting $\partial\beta\,/\,\partial T=0$, one can extract

$$T_{\min}=\pi+\cos^{-1}\Big(\tfrac{1}{2}\Big\{[(1+k)\cosh\sqrt{2ik}-1] \times \operatorname{csch}\sqrt{ik}\,\sin\sqrt{ik}-2ik\cos\sqrt{ik}\,\cosh\sqrt{ik} -k\csc\sqrt{ik}\,\sinh\sqrt{ik}-2i\sqrt{2k}\,\sinh\sqrt{2k}\Big\}^{1/2} \times(\sqrt{ik}\cos\sqrt{ik}-\sinh\sqrt{ik})^{-1/2} \times(\sqrt{ik}\cosh\sqrt{ik}-\sinh\sqrt{ik})^{-1/2} \tag{44}$$

At this value of $T_{\min}$ one obtains the smallest possible wall shear stress that can be realized in a given cycle. Substitution of Eq. (44) into Eq. (43) provides the exact value for $\beta_{\min}$. Since the expression is too long to display, we opt to provide an analytical equivalent, albeit approximate. For small and large k, explicit approximations can be carefully derived over the entire range of frequency parameters. Noting that

$$T_{\min}\sim\begin{cases}\frac{5}{4}\pi-\frac{1}{15}k; & k\to 0\\ \pi; & k\to\infty\end{cases} \tag{45}$$

one finds, after some effort,

$$\beta_{\min}=\begin{cases}\frac{1}{\sqrt{2}}(1+\frac{13}{3150}k^2); & k\le 7.5117656\\ \dfrac{-3(1-\sqrt{2k}+k)}{k(1-\sqrt{2k})}; & k>7.5117656\end{cases} \tag{46}$$

These two approximations are practically equivalent to the exact solution by virtue of their maximum relative errors of 8.54% across the entire range of k. These errors occur at the patching point $k^*=7.5118$ where both approximations exhibit the same relative error. The asymptotic error in both approximations quickly improves everywhere else. These exact and approximate solutions are plotted in Fig. 4 where the region of reverse flow is clearly delineated. Note that, for $k=10$, shear reversal will occur when $\beta_{\min}=0.54265$; this explains why no flow reversal can be seen in Figs. 3b–c (as, in both instances, $\beta \le \beta_{\min}$). When $k=1$, however, $\beta_{\min}$ drops to $0.704212<1$; this value is not sufficient to mitigate reversal in all four cases of Fig. 2.

V. Oscillatory Piston Motion

A purely harmonic piston motion can be obtained by setting $u_0=0$, $u_{cn}=0$, and

$$u_{sn}=\begin{cases}U_p, & n=1\\ 0, & n>1\end{cases} \tag{47}$$

The corresponding piston motion reduces to

$$u_p(t)=U_p\sin(\omega t) \tag{48}$$

and so

$$\overline{u}_p(s)=U_p\omega/(s^2+\omega^2) \tag{49}$$

The Bromwich integral becomes

$$u(y,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\omega U_p}{(s^2+\omega^2)}e^{st} \times\frac{A\sqrt{s}[\cosh(A\sqrt{s})-\cosh(Y\sqrt{s})]}{[A\sqrt{s}\cosh(A\sqrt{s})-\sinh(A\sqrt{s})]}\mathrm{d}s \tag{50}$$

The meaningful zeros of Eq. (50) correspond to $s=\pm i\omega$ whose residues proceed from

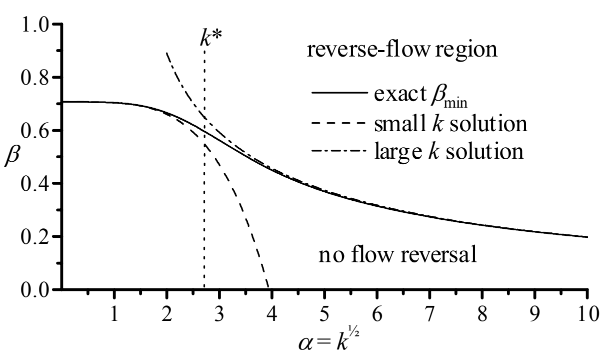

Fig. 4 Flow reversal chart relating β and $\alpha=k^{1/2}$ (the Womersley number). The solid line delineates the region of flow reversal according to the exact solution; the broken and chained lines correspond to the small and large k approximations given by Eq. (46). Results are shown for a pulsatory volumetric flowrate in a rectangular duct.

$$\frac{2res_+}{iU_pe^{i\omega t}}=\frac{A\sqrt{i\omega}\left[\cosh(Y\sqrt{i\omega})-\cosh(A\sqrt{i\omega})\right]}{\left[A\sqrt{i\omega}\cosh(A\sqrt{i\omega})-\sinh(A\sqrt{i\omega})\right]} \tag{51}$$

$$\frac{2res_-}{iU_pe^{-i\omega t}}=\frac{A\sqrt{-i\omega}\left[\cosh(A\sqrt{-i\omega})-\cosh(Y\sqrt{-i\omega})\right]}{\left[A\sqrt{-i\omega}\cosh(A\sqrt{-i\omega})-\sinh(A\sqrt{-i\omega})\right]} \tag{52}$$

Based on the sum of these residues, one collects

$$\frac{u(\eta,t)}{U_p}=\frac{ie^{i\omega t}\sqrt{ik}\left[\cosh(\eta\sqrt{ik})-\cosh(\sqrt{ik})\right]}{2\left[\sqrt{ik}\cosh(\sqrt{ik})-\sinh(\sqrt{ik})\right]} -\frac{ie^{-i\omega t}\sqrt{-ik}\left[\cosh(\eta\sqrt{-ik})-\cosh(\sqrt{-ik})\right]}{2\left[\sqrt{-ik}\cosh(\sqrt{-ik})-\sinh(\sqrt{-ik})\right]} \tag{53}$$

This result can be directly recovered from the pulsatory solution by setting in Eq. (36) $u_0=0$, $u_{cn}=0$, and $u_{sn}=U_p$. Despite its suggested form, Eq. (53) is real only. For $k>5$, one may use $\cosh x\approx\sinh x\approx\frac{1}{2}e^x$ to obtain

$$\frac{u(\eta,t)}{U_p}\approx\frac{ie^{i\omega t}\sqrt{ik}\left[1-2e^{-\sqrt{ik}}\cosh(\eta\sqrt{ik})\right]}{2(\sqrt{ik}-1)} -\frac{ie^{-i\omega t}\sqrt{ik}\left[1-2e^{-\sqrt{-ik}}\cosh(\eta\sqrt{-ik})\right]}{2(\sqrt{-ik}-1)} \tag{54}$$

VI. Trapezoidal Piston Motion

For purposes of further verification, a solution is sought for the governing equations corresponding to the piecewise stages of trapezoidal piston motion. As shown in Fig. 1b, this motion is prescribed by a constant positive acceleration from rest, constant velocity, constant negative acceleration, and finally, rest. The governing equation for the velocity profile remains the same as for the pulsatory case – given by Eq. (19). The exception lies in the piston velocity which, for a trapezoidal motion, can be expressed by

$$u_p^*=\frac{u_p(t)}{U_p}=\begin{cases}\dfrac{t}{t_0}; & 0\le t\le t_0\\ 1; & t_0\le t\le t_1\\ \dfrac{(t_2-t)}{(t_2-t_1)}; & t_1\le t\le t_2\\ 0; & t_2\le t<\infty\end{cases} \tag{55}$$

Upon Laplace transformation of the piston motion, the velocity profile is determined using the method of residues described earlier. For the first case, the profile is determined simply from the residue at $s=0$; this yields a pole of order two, namely,

$$u_1(y,t)=\frac{U_p}{t_0}\left(1-\frac{y^2}{a^2}\right)\left[5\left(12t+\frac{y^2}{\nu}\right)-\frac{a^2}{\nu}\right]. \quad (56)$$

Similarly, for the second case, the profile is determined from the residue at $s=0$; the latter exhibits a simple pole corresponding to

$$u_2(y,t)=\tfrac{3}{2}U_p\left(1-y^2/a^2\right). \quad (57)$$

As one would expect, the constant piston displacement phase corresponds directly to the Poiseuille flow in a channel.

VII. Conclusions

In this study, the Das and Arakeri approach is used to derive the viscous channel flow solution for a user-prescribed volumetric flowrate. Three case studies are considered to illustrate the resulting formulation given in terms of Fourier coefficients. These cases employ either a pulsatory, oscillatory, or a trapezoidal piston motion. For pulsatory displacements, the kinetic Reynolds number and kinetic ratio of unsteady and mean velocity components are identified as dynamic similarity parameters. For a given kinetic Reynolds number (or frequency), there exists a sufficiently large kinetic ratio leading to the reversal in shear stress at the wall. Unlike flows prescribed by purely sinusoidal pressure oscillations, the observed flow reversal can, in some instances, occur above the wall. Criteria for shear reversal at the wall are provided in general, and explicit forms are furnished using asymptotic perturbation tools. For pulsatory piston motions, two approximations for the minimum kinetic ratio leading to shear reversal are obtained for small and large kinetic Reynolds numbers, respectively. These approximations remain suitable over the entire range of operating parameters and exhibit a maximum relative error of 8.54% at a kinetic Reynolds number of 7.51. For a sufficiently large kinetic ratio, the pulsatory motion reduces to the solution for a purely oscillatory flow. Conversely, for a sufficiently small kinetic ratio, the classic Poiseuille flow is recovered. Trapezoidal displacements also lead to the Poiseuille profile during their constant velocity segment.

Acknowledgments

The authors wish to thank NASA and the Wisconsin Space Grant Consortium whose support for this project is most gratefully acknowledged.

References

[1]Majdalani, J., and Chibli, H. A., "Pulsatory Channel Flows with Arbitrary Pressure Gradients," AIAA Paper 2002-2981, June 2002.

[2]Das, D., and Arakeri, J. H., "Unsteady Laminar Duct Flow with a Given Volume Flow Rate Variation," *Transactions of the American Society of Mechanical Engineers: Journal of Applied Mechanics, Series E,* Vol. 67, 2000, pp. 274-281.

[3]White, F. M., *Viscous Fluid Flow*, McGraw-Hill Book Company Inc., New York, 1991, pp. 135-136.

[4]Zhao, T. S., and Cheng, P., "The Friction Coefficient of a Fully Developed Laminar Reciprocating Flow in a Circular Pipe," *International Journal of Heat and Fluid Flow,* Vol. 17, 1996, pp. 167-172.

[5]Rott, N., "Theory of Time-Dependent Laminar Flows," *High Speed Aerodynamics and Jet Propulsion –Theory of Laminar Flows,* Section D, Vol. IV, edited by F. K. Moore, Princeton University Press, Princeton, New Jersey, 1964, pp. 395-438.

[6]Hall, P., "Unsteady Viscous Flow in a Pipe of Slowly Varying Cross-Section," *Journal of Fluid Mechanics,* Vol. 64, No. 2, 1974, pp. 209-226.

[7]Bhatnagar, R. K., "Fluctuating Flow of a Viscoelastic Fluid in a Porous Channel," *Transactions of the American Society of Mechanical Engineers: Journal of Applied Mechanics,* Vol. 46, 1979, pp. 21-25.

[8]Wang, C. Y., "Pulsatile Flow in a Porous Channel," *Transactions of the American Society of Mechanical Engineers: Journal of Applied Mechanics,* Vol. 38, 1971, pp. 553-555.

[9]Chow, J. C. F., and Soda, K., "Laminar Flow in Tubes with Constriction," *The Physics of Fluids,* Vol. 15, No. 10, 1972, pp. 1700-1706.

[10]Womersley, J. R., "Method for the Calculation of Velocity, Rate of Flow and Viscous Drag in Arteries When the Pressure Gradient Is Known," *Journal of Physiology,* Vol. 127, 1955, pp. 553-563.

[11]Womersley, J. R., "Oscillatory Motion of a Viscous Liquid in a Thin-Walled Elastic Tube –I. The Linear Approximation of Long Waves," *Philosophical Magazine, Series Ser. 7,* Vol. 46, No. 373, 1955, pp. 199-221.