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COMPACT THERMAL REPRESENTATIONS FOR SEVERAL FUNDAMENTAL SHAPES IN FORCED CONVECTION

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In the cooling of electronic packages, the task of simulating large arrays of heat sinks is often accomplished by the use of compact models. These simpler models attempt to capture the thermal and flow resistance characteristics of a representative heat sink while ignoring secondary detail. In the porous block model, an equivalent thermal conductivity is assigned to the fluid that enters the 'porous' space above the heat sink base that was once occupied by the fins. This artificially enhanced thermal conductivity enables the porous block of fluid to exhibit the same thermal resistance as that of the original heat sink. Due to the three-dimensional distribution of the thermal resistance in space, temperature maps associated with the resulting model provide better agreement with detailed numerical simulations than is possible with other models based on two-dimensional flat plate or thin sheet approximations. In this paper, we present closed-form expressions for the equivalent thermal conductivity associated with a large number of heat sink shapes in a forced convection environment.

1 INTRODUCTION

Electronic packages are constantly gravitating towards higher densities and increased power dissipation. The increased power dissipation is mitigating the use of cooling methods based solely on natural convection. In most high power applications today, not only is the use of forced airflow a necessity, but also the need for multiple and innovative heat sink designs becoming an absolute requirement.

In chip populated circuit boards that involve multiple heat sinks, the CPU time for numerically simulating the entire system including heat sink details can be unaffordable in view of critical turnaround time and prolonged delays between thermal analyses and manufacturing deadlines [1]. In order to meet the competitive demands to expedite the product 'out the door,' it is becoming customary to use compact models of heat sinks that can be resolved more rapidly than detailed prototypes [2-8]. The key in compact simulations is to incorporate into the coarser model the equivalent thermal and flow resistance characteristics of a representative heat sink.

In forced convection applications based on the porous block model, the thermal character of the detailed heat sink can be captured by calculating an equivalent thermal conductivity for the geometry at hand. The equivalent thermal conductivity is an artificial property that can be assigned to the fluid which, upon entering the porous space above the heat sink base, will exhibit a reduced thermal resistance that matches that associated with the detailed heat sink. Several compact models have been proposed in the past including one-, two-, and three-dimensional representations [6-8]. By assigning the thermal conductivity to the control volume above the heat sink base, the reduced thermal resistance of the resulting model will be distributed over the entire porous block. For this reason, previous investigations based on the porous block model have produced temperature maps that more closely resembled the realistic, 3-D isotherms obtained from laboratory measurements or complete CFD tests.

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Nomenclature

C_p	= constant pressure specific heat
D	= diameter of cylinder or sphere
g	= acceleration due to gravity
h	= effective heat transfer coefficient
k	= thermal conductivity
k_e	= effective thermal conductivity
L	= characteristic length
Pr	= Prandtl number, $\mu C_p / k$
Re_D	$= \rho VD / \mu$
T_b	= bulk mean coolant temperature, $(T_i + T_o)/2$
T_f	= film temperature, $(T_s + T_b)/2$
T_i	= inlet coolant temperature
T_j	= junction temperature
$T_{\max}$	= maximum surface temperature, T_j
T_o	= outlet coolant temperature
T_s	= surface temperature
T_∞	= ambient coolant temperature
U	= overall heat transfer coefficient, $\dot{Q}/(A\Delta T)$
ΔT	$= T_s - T_b$
μ, ν	= dynamic and kinematic viscosities
ρ	= density

In that regard, the reader may consult the work of Bar-Cohen and Kraus [9], Bar-Cohen [10], Linton and Agonafer [7], Narasimhan and Kusha [11], Narasimhan and Majdalani [12], and Brucker, Ressler and Majdalani [13].

Due to the importance of accurately estimating the effective thermal conductivity k_e for a given heat sink shape, the purpose of this article is to determine general canonical forms that can be used to explicitly express k_e associated with a number of fundamental geometric shapes used in the literature [14-19]. The availability of analytic expressions for k_e will not only facilitate the development of compact models but will also help accelerate numerical algorithms that will no longer be required to iterate in finding estimates for k_e.

2 GENERAL FORCED CONVECTION

The simplest form of an empirical correlation, that can be used to predict forced convection heat transfer from bodies of various shapes is

$$\mathrm{Nu}_L = \begin{cases} C_l \,\mathrm{Re}_L^m \,\mathrm{Pr}^n, \text{ laminar} \\ C_t \,\mathrm{Re}_L^p \,\mathrm{Pr}^q, \text{ turbulent} \end{cases} \tag{1}$$

for which
$$\mathrm{Nu}_x = \begin{cases} mC_l \,\mathrm{Re}_x^m \,\mathrm{Pr}^n, \text{ laminar} \\ pC_t \,\mathrm{Re}_x^p \,\mathrm{Pr}^q, \text{ turbulent} \end{cases} \tag{2}$$

These correlations have been abbreviated by using generic forms that depend on the flow regime.

Equations (1) and (2), which define the average and local Nusselt numbers, can be integrated to determine the combined laminar and turbulent flow correlation if the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ is known, yielding

$$\mathrm{Nu}_L = [C_t \,\mathrm{Re}_L^p - (C_t \,\mathrm{Re}_{\mathrm{cr}}^{p-m} - C_l)\,\mathrm{Re}_{\mathrm{cr}}^m]\,\mathrm{Pr}^q \,;\; n = q\,. \tag{3}$$

The technical literature is abound with correlations of this form. As an example, consider a flow over an isothermal plate, in this case $C_l = 0.664$, $C_t = 0.037$, $m = \frac{1}{2}$, $n = q = \frac{1}{3}$, $p = \frac{4}{5}$, and $\mathrm{Re}_{\mathrm{cr}} = 5\times10^5$ to verify that [20]

$$\mathrm{Nu}_L = (0.037\,\mathrm{Re}_L^{4/5} - 871.3)\,\mathrm{Pr}^{1/3} \tag{4}$$

If the base plate is subjected to a sufficiently uniform heat flux, only leading coefficients in Eq. (2) is modified. By setting $C_l = 0.906$ and $C_t = 0.0385$, the corresponding correlation becomes

$$\mathrm{Nu}_L = (0.0385\,\mathrm{Re}_L^{4/5} - 754.6)\,\mathrm{Pr}^{1/3}\,. \tag{5}$$

In compact models, the assumption is usually made that, surfaces are isothermal, even in the presence of a uniform heat flux. This is justified by the small surface area and large conductivity of the base plate.

Regardless of the correlation used for forced convection, determination of the effective thermal conductivity is straightforward, in most cases. For equations that take the form of Eq. (1) this can be accomplished by setting $\mathrm{Nu}_L = UL/k_e$ and solving for k_e, yielding the following relations for the effective thermal conductivity

$$k_e = \begin{cases} \mathrm{Re}_L^m \,\mu Cp[UL/(C_l \,\mathrm{Re}_L^m \,\mu Cp)]^{1/(1-n)} \\ \quad \text{laminar} \\ \mathrm{Re}_L^p \,\mu Cp[UL/(C_t \,\mathrm{Re}_L^p \,\mu Cp)]^{1/(1-q)} \\ \quad \text{turbulent} \\ \left\{UL/[C_t \,\mathrm{Re}_L^p - (C_t \,\mathrm{Re}_{\mathrm{cr}}^{p-m} - C_l)\,\mathrm{Re}_{\mathrm{cr}}^m (\mu Cp)^q]\right\}^{1/(1-q)} \\ \quad n = q, \text{combined laminar and turbulent} \end{cases} \tag{6}$$

Equation (6) can be used to determine k_e for many geometries used in forced convection studies. These are summarized in Table 1. Also given in Table 1, are the Jacob correlations [21] which pertain to planar cross-sections whose correlations collapse into the simple expression .

$$\mathrm{Nu}_L = C\,\mathrm{Re}_L^m \,\mathrm{Pr}^{1/3}\,;\; k_e = \left\{UL/\left[C\,\mathrm{Re}_L^m \left(\mu C_P\right)^{1/3}\right]\right\}^{3/2} \tag{7}$$

It may be instructive to note that, for flow over a flat plate, the special equation used in a recent study by Narasimhan and Majdalani [12, 22] can be restored from Eq. (6). In these studies, results based on Eq.. (6) were shown to fall within $\pm7\%$ of both computational and experimental measurements.

Table 1 Constants for forced convection direct solutions

Case	C	C_0	m	p	Eq.	L, T or L&T*	Notes, Range
Flat Plate [23] (isothermal surface)	0.664	0	¼	⅓	(6)	L	$\mathrm{Re}_L \le 5\times10^5$, $\mathrm{Pr} \ge 0.6$
Flat Plate [23] (isothermal surface)	0.037	-871	⅘	⅓	(6)	L&T	Form of $C(\mathrm{Re}_L^{4/5}+C_0)$, $5\times10^5 \le \mathrm{Re}_L \le 10^8$, $0.6 \le \mathrm{Pr} \le 60$
Flat Plate [20] ($\dot{Q} = \mathrm{const}$)	0.906	0	½	⅓	(6)	L	$\mathrm{Re}_L \le 5\times10^5$, $\mathrm{Pr} \ge 0.6$
Flat Plate [20] ($\dot{Q} = \mathrm{const}$)	0.0385	755	⅘	⅓	(6)	L&T	Form of $C(\mathrm{Re}_L^{4/5}+C_0)$, $5\times10^5 \le \mathrm{Re}_L \le 10^8$, $0.6 \le \mathrm{Pr} \le 60$
Vertical Flat Plate [24]	0.2	0	⅔	0	(6)	L&T	$1 \le \mathrm{Re}_L \le 4\times10^5$, $\mathrm{Pr} = 0.7$
Square Plate [25]	0.93	0	½	⅓	(6)	L&T	$2\times10^4 \le \mathrm{Re}_L \le 10^5$, air
Disk [26]	1.05	0	½	0.36	(6)	L&T	$5\times10^3 \le \mathrm{Re}_D \le 5\times10^4$, $0.7 \le \mathrm{Pr} \le 380$
Disk [27]	0.591	0	0.564	⅓	(6)	L&T	$9\times10^2 \le \mathrm{Re}_D \le 3\times10^4$, $0.7 \le \mathrm{Pr} \le 380$
Tubes – Jakob's Correlations [21]							
Cylinder	0.989		0.330	⅓	(7)	L	$0.4 < \mathrm{Re}_D < 4$
Cylinder	0.911		0.385	⅓	(7)	L	$4 < \mathrm{Re}_D < 40$
Cylinder	0.683		0.466	⅓	(7)	L	$40 < \mathrm{Re}_D < 4\times10^4$
Cylinder	0.193		0.618	⅓	(7)	L	$4\times10^4 < \mathrm{Re}_D < 4\times10^5$
Cylinder	0.0266		0.805	⅓	(7)	L	$4\times10^5 < \mathrm{Re}_D < 4\times10^6$
Square	0.104		0.675	⅓	(7)	L	$5\times10^4 < \mathrm{Re}_D < 1\times10^6$
Square	0.180		0.699	⅓	(7)	L	$2.5\times10^4 < \mathrm{Re}_D < 8\times10^4$
Inclined Square	0.251		0.588	⅓	(7)	L	$5\times10^4 < \mathrm{Re}_D < 1\times10^6$
Inclined Square	0.293		0.624	⅓	(7)	L	$2.5\times10^4 < \mathrm{Re}_D < 7.5\times10^4$
Ellipse	0.252		0.612	⅓	(7)	L	$2.5\times10^4 < \mathrm{Re}_D < 1.5\times10^5$
Ellipse	0.096		0.804	⅓	(7)	L	$3\times10^4 < \mathrm{Re}_D < 1.5\times10^5$
Flat Plate	0.232		0.731	⅓	(7)	L	$4\times10^4 < \mathrm{Re}_D < 1.5\times10^5$
Hexagon	0.156		0.638	⅓	(7)	L	$5\times10^4 < \mathrm{Re}_D < 1\times10^6$
Hexagon	0.163		0.638	⅓	(7)	L	$5\times10^4 < \mathrm{Re}_D < 1\times10^6$
Hexagon	0.039		0.782	⅓	(7)	L	$1.95\times10^5 < \mathrm{Re}_D < 1\times10^6$

* L: laminar; T: turbulent; L&T: laminar and turbulent.

3 APPROXIMATE k_e FOR LAMINAR AND TURBULENT FORCED CONVECTION ACROSS A CYLINDER

Churchill and Bernstein [14] have proposed three equations to describe the heat transfer from a cylinder in forced convection. These are, first,

$$\mathrm{Nu}_D = 0.3 + \frac{0.62\mathrm{Re}_D^{1/4}\,\mathrm{Pr}^{1/3}}{\left(1+0.543\,\mathrm{Pr}^{-2/3}\right)^{1/4}}; \tag{8}$$

for

$$\begin{cases} 10^4 < \mathrm{Re}_D \le 4\times10^5 \\ \mathrm{Pr} > 0.5 \end{cases} \tag{9}$$

second,

$$\mathrm{Nu}_D = 0.3 + \frac{0.62\mathrm{Re}_D^{1/4}\,\mathrm{Pr}^{1/3}\left(1+\frac{1}{531}\mathrm{Re}_D^{1/2}\right)}{\left(1+0.543\,\mathrm{Pr}^{-2/3}\right)^{1/4}} \tag{10}$$

for

$$\begin{cases} 4\times10^5 < \mathrm{Re}_D \le 5\times10^6 \\ \mathrm{Pr} > 0.5 \end{cases} \tag{11}$$

and third,

$$\mathrm{Nu}_D = 0.3 + \frac{0.62\mathrm{Re}_D^{1/4}\,\mathrm{Pr}^{1/3}\left(1+\frac{1}{2550}\mathrm{Re}_D^{5/8}\right)^{0.8}}{\left(1+0.543\,\mathrm{Pr}^{-2/3}\right)^{1/4}} \quad (12)$$

for

$$\begin{cases} 3.5 \le \mathrm{Re}_D \le 7.6\times10^4 \\ 0.7 \le \mathrm{Pr} \le 380 \end{cases}. \quad (13)$$

Equations (8), (10), and (12) can be used in a compact model, with the stipulation that the solution will only be valid for the less popular cylindrical fins, rather than the circular fins. It is also noted that they have nearly the same form as the ones presented by Curchill and Chu [28] for natural convection heat transfer from many geometric shapes. A solution for the most general form of this equation is given by Brucker, Ressler and Majdalani [13].

$$UL/k_e = a_0 + a_1\left(A_0/k_e\right)^m\left\{1+[a_2k_e/(\mu C_P)]^n\right\}^p \quad (14)$$

here the constant A_0 is given by

$$A_0 \equiv \mu C_p\,\mathrm{Re}_D^{1/4} \text{ for Eq. (8)} \quad (15)$$

$$A_0 \equiv \mu C_p\,\mathrm{Re}_D^{1/4}\left(1+\tfrac{1}{531}\mathrm{Re}_D^{1/2}\right) \text{ for Eq. (10)} \quad (16)$$

$$A_0 \equiv \mu C_p\,\mathrm{Re}_D^{1/4}\left(1+\tfrac{1}{2550}\mathrm{Re}_D^{5/8}\right)^{0.8} \text{ for Eq. (12)} \quad (17)$$

and the constants a_0, a_1, and a_2 are specified by the geometry, and posted with m, n, and p in Table 3. U is the overall heat transfer coefficient from the base of the heat sink.

As noted in Brucker, Ressler and Majdalani [13], the power-law embedment of k_e in the universal Prandtl number function $\left[1+\left(a_2/\mathrm{Pr}\right)^n\right]^p$ eliminates the possibility of obtaining an exact expression for k_e. An asymptotic approximation of the form $k_e \approx k_0 + k_1$ will have to be settled for. To proceed, the Prandtl number function has to be expanded first. Letting $\kappa = a_2/\mu C_P$, it follows that two cases must be considered separately, due to the convergence criteria of the Taylor series used in the expansion.

3.1 Type 1: Small κk_e case

For small $\left(\kappa k_e\right)^n$, one can expand the universal Prandtl number function using a Taylor series. One obtains

$$\left[1+\left(\kappa k_e\right)^n\right]^p = 1 + p\left(\kappa k_e\right)^n + \tfrac{p(p-1)}{2!}\left(\kappa k_e\right)^{2n} + \tfrac{p(p-1)(p-2)}{3!}\left(\kappa k_e\right)^{3n} + \tfrac{p(p-1)(p-2)(p-3)}{4!}\left(\kappa k_e\right)^{4n} + \ldots \quad (18)$$

By substituting Eq. (18) into Eq. (14), one gathers the multi-order polynomial

$$UL = \underset{1}{a_0k_e} + \underset{2}{B_0k_e^{1-m}} + \underset{(-)}{C_0k_e^{1-m+n}} + \underset{3}{D_0k_e^{1-m+2n}} + \underset{(-)}{E_0k_e^{1-m+3n}} + \underset{4}{F_0k_e^{1-m+4n}} \quad (19)$$

where,

$$\begin{cases} B_0 \equiv a_1A_0^m; \quad C_0 \equiv pB_0\kappa^n \\ D_0 \equiv \frac{p(p-1)}{2!}B_0\kappa^{2n};\ E_0 \equiv \frac{p(p-1)(p-2)}{3!}B_0\kappa^{3n} \\ F_0 \equiv \frac{p(p-1)(p-2)(p-3)}{4!}B_0\kappa^{4n} \end{cases} \quad (20)$$

The leading order solution can be shown to be

$$k_0 = (UL/B_0)^{1/(1-m)} \quad (21)$$

Having identified Eq. (21) as the leading order solution, the next step is to put

$$k_e^{\mathrm{I}} = \left(UL/B_0\right)^{1/(1-m)} + k_1 \quad (22)$$

where the superscript is used to denote a type-I solution. To solve for the first-order correction k_1, the notion of successive approximations is used. The first order correction is found to be

$$k_1 = -\left[UL\left(1-m\right)\right]^{-1}\left(a_0k_0{}^2 + C_0k_0{}^{2-m+n} + D_0k_0^{2-m+2n} + E_0k_0{}^{2-m+3n} + F_0k_0{}^{2-m+4n}\right). \quad (23)$$

Equation (22) is valid as long as $0 \le k_e^{\mathrm{I}} \le k_{//}$ [13]. This physical limitation is due to the divergence of the Taylor series at larger values of $\left(\kappa k_e\right)^n$.

3.2 Type 2: Large κk_e case

The series expansion in Eq. (18) can become divergent as $\left(\kappa k_e\right)^n$ increases. When $\left(\kappa k_e\right)^n$ is no longer small, one needs to re-expand Eq. (18) in the reciprocal of $\left(\kappa k_e\right)^n$. One obtains

$$\left(\kappa k_e\right)^{np}\left[1+1/\left(\kappa k_e\right)^n\right]^p = \left(\kappa k_e\right)^{np}\left[1 + p\left(\kappa k_e\right)^{-n} + \tfrac{p(p-1)}{2!}\left(\kappa k_e\right)^{-2n} + \tfrac{p(p-1)(p-2)}{3!}\left(\kappa k_e\right)^{-3n} + \ldots\right] \quad (24)$$

When Eq. (24) is substituted back into Eq. (15), a number of terms arise. Some are so small that they can be ignored. The remaining terms are found to be

$$UL = \underset{1}{a_0k_e} + \underset{2}{B_1k_e^{1+np-m}} + \underset{(-)}{C_1k_e^{1-n+np-m}} + \underset{3}{D_1k_e^{1-2n+np-m}} + \underset{(-)}{E_1k_e^{1-3n+np-m}} \quad (25)$$

where

$$\begin{cases} B_1 \equiv B_0\kappa^{np}; \quad C_1 \equiv B_0p\kappa^{np-n} \\ D_1 \equiv \frac{p(p-1)}{2!}B_0\kappa^{np-2n} \\ E_1 \equiv \frac{p(p-1)(p-2)}{3!}B_0\kappa^{np-3n} \end{cases} \quad (26)$$

Here the leading order solution must be extracted from

$$-UL + a_0K_0 + B_1K_0{}^{1+np-m} = 0 \quad (27)$$

As it can be seen in Eqs. (8)-(12), $1+np-m=\frac{1}{2}$ (see Eq. (14) and Table 3). This fractional power leads to a quadratic equation. Based on the meaningful quadratic root, the two-term expansion for k_e^{II} becomes

$$k_e^{\mathrm{II}} = \left[UL + \tfrac{1}{2}B_1\left(B_1 - \sqrt{B_1^2 + 4a_0UL}\right)/a_0\right]/a_0 + K_1 \qquad (28)$$

where the superscript is used to denote a type-II solution. Using successive approximations [13], K_1 is determined to be

$$K_1 = 16k_0\left(E_1 + D_1K_0^{9/16} + C_1K_0^{9/8}\right)/ \left(19E_1 + 10D_1K_0^{9/16} + C_1K_0^{9/8} - 8B_1K_0^{27/16} - 16a_0K_0^{35/16}\right) \qquad (29)$$

Equation (28) insofar as $k_e^{\mathrm{II}} > k_{//}$. Again, the range of applicability is limited by the divergence of the Taylor series expansion involved in the solution.

For the case where $1+np-m \neq \frac{1}{2}$ the reader is referred to Brucker, Ressler and Majdalani [15] for a discussion on a more general solution technique. Nonetheless, the authors are not aware of a forced convection correlation not corresponding to $1+np-m=\frac{1}{2}$.

3.3 Cut-off Value $k_{//}$

The cut-off value that delimits the small and large k_e solutions can be obtained following Ref. [13]. Letting

$$x = s_0 + s_1\bar{k} + s_2\bar{k}^2 + s_3\bar{k}^3 + s_4\bar{k}^4 ; \quad \bar{k} = 2UL/(k_e^- + k_e^+) \qquad (30)$$

the $s_i = (s_0, s_1, s_2, s_3)$ coefficients are provided in Table 3. Using Eq. (30), the maximum asymptotic error at the delineation point where both approximations are optimally patched is contained within 2% when k_e is at least 5% away from $k_{//}$. In practice, once the cut-off $k_{//}$ is calculated for a given application, one may safely use the appropriate correlation depending on the operating range. In summary, one can use

$$k_e = \begin{cases} k_e^{\mathrm{I}}; & 0 < k_e \le k_{//} \\ k_e^{\mathrm{II}}; & k_e > k_{//} \end{cases} \qquad (31)$$

where both approximations are equivalent at the cut-off point. The degree of precision associated with Eqs. (22) and (28) is illustrated in Fig. 1 where analytical and numerical predictions for k_e are compared over a wide range of U and three flow configurations. All fluid and geometric properties used in the preparation of Fig. 1 are given in Table 2.

4 APPROXIMATE k_e FOR LAMINAR AND TURBULENT FORCED CONVECTION ACROSS A SPHERE

Some correlations exist for which a standalone diffusive constant is added to the term containing the Prandtl number. An illustrative case arises in the context of a laminar or turbulent flow across a sphere of diameter D. This form of the equation applies to the natural convection correlation proposed by Yuge [19] for laminar heat transfer from a sphere. The pertinent Nusselt number relation has been developed by Whitaker [18] over the range $3.5 \le \mathrm{Re}_D \le 7.6\times10^4$ and $0.7 \le \mathrm{Pr} \le 380$. It exhibits the form

$$\mathrm{Nu}_D = C_0 + C_1\,\mathrm{Pr}^n ; \; n \equiv 2/5 ; \qquad (32)$$

$$C_0 \equiv 2 , \quad C_1 \equiv (0.4\,\mathrm{Re}_D^{1/2} + 0.06\,\mathrm{Re}_D^{2/3})(\mu/\mu_s)^{1/4} \qquad (33)$$

where both C_0 and C_1 are independent of k_e. Equation (32) can be rearranged into

$$k_e + Ak_e^m - B = 0 ; \; m \equiv 1-n , \qquad (34)$$

$$A \equiv (\mu C_p)^n (C_1/C_0) , \; B \equiv UD/C_0 \qquad (35)$$

Equation (34) is applicable to compact heat sink modeling, in the event that the heat sink base and fins are hemispherical or spherical in shape.

Table 2 Sample test cases

Case	L	ρ	μ	C_p	T_s	T_∞	Re_L
	m	kgm^{-3}	kgms^{-1}	Jkg^{-1}K^{-1}	K	K	
1*	0.03048	1.091	1.965$_{\mathrm{E}}$-05	1007	354.0	293.15	2.05$_{\mathrm{E}}$5
2*	0.07620	1.103	1.949$_{\mathrm{E}}$-05	1007	346.6	293.15	2.05$_{\mathrm{E}}$5
3	0.10	1.103	1.949$_{\mathrm{E}}$-05	1007	400	293.15	10^5

*Commercial heat sinks used by Narasimhan and Majdalani [12, 22].

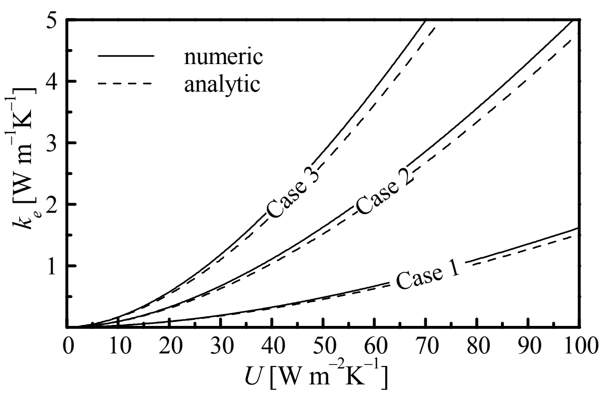


Figure 1. Three sample comparison between analytical and numerical predictions of the effective thermal conductivity. The three cases shown are described in Table 2.

then solved asymptotically. Since all physical problems exhibit $0<n<1$, a reasonably accurate two-term expansion can be obtained from

$$\begin{cases} Ak_0^m - B = 0,\ k_0 \text{ small} \\ K_0 - B = 0,\ K_0 \text{ large} \end{cases} \tag{36}$$

These yield

$$k_e = \begin{cases} \dfrac{m}{1/B + m(A/B)^{1/m}},\ k_e \le k_{//} \\ B\left[1 - 1/(m + B^{1-m}/A)\right],\ k_e > k_{//} \end{cases} \tag{37}$$

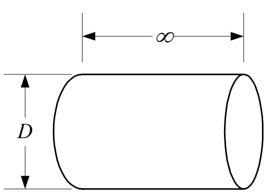
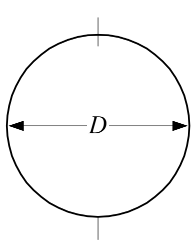
where the maximum error remains bounded within ±9.1% for $n \le 0.4$. The delimiting value for the small range is $k_{//} = (0.08 + 0.727m - 0.314m^2)A$. For $0 < n \le 0.4$, the error remains bounded within $\pm 2.5\%$; furthermore, the range for $k_{//}$ reduces to $0.40 \le k_{//}/A < 0.50$.

5 CONCLUSION

In this article, several analytical expressions were derived for the effective thermal conductivity (k_e) in forced convection applications. The final results are posted in Table 1. In addition to the final expressions and equations used to calculate the effective thermal conductivity for a specific geometry, we have provided a detailed description of the methodology used. Every attempt was made to make these methods general enough so that they could be applied to future correlations that are bound to arise. The compilation of the Nusselt number correlations that can be found in Table 1 is supplemented with a discussion of the Churchill and Bernstein equations, for which no direct solution exists. Asymptotic solutions for them are listed in Table 3.

The presentation of our asymptotic solution is kept general again in the hopes that as new correlations arise they can be transformed into ones that are of use to the compact modeling community. Since all of the results are self-contained and require no knowledge of advanced mathematics to implement they can be readily implemented in a compact simulation. Yet, for the interested researcher, we have also provided the tools that may be necessary to handle new equations in the future.

Table 3 Constants for forced convection asymptotic solutions

	Description	Forced convection correlation	Effective thermal conductivity
1	 horizontal cylinder	$\mathrm{Nu}_D = 0.3 + \dfrac{0.62\mathrm{Re}_D^{1/4}\mathrm{Pr}^{1/3}}{\left(1+0.543\,\mathrm{Pr}^{-2/3}\right)^{1/4}}$ Churchill and Bernstein [14]	$k_e \to$ Eq. (31), $(m,n,p) = (\frac{1}{3},\frac{2}{3},-\frac{1}{4})$, $a_i = (0.3, 0.62, 0.543)$, $A_0 = \mu C_p \mathrm{Re}_D^{1/4}$ $s_i = (0.9842, -0.1678, 0.0861, -0.0297, 0.0034)$ $10^4 < \mathrm{Re}_D \le 4\times10^5$, $\mathrm{Pr} > 0.5$
2		$\mathrm{Nu}_D = 0.3 + \dfrac{0.62\mathrm{Re}_D^{1/4}\mathrm{Pr}^{1/3}\left(1+\frac{1}{531}\mathrm{Re}_D^{1/2}\right)}{\left(1+0.543\,\mathrm{Pr}^{-2/3}\right)^{1/4}}$ Churchill and Chu [14]	$k_e \to$ Eq. (31), $(m,n,p) = (\frac{1}{3},\frac{2}{3},-\frac{1}{4})$ $a_i = (0.3, 0.62, 0.543)$, $A_0 = \mu C_p \mathrm{Re}_D^{1/4}\left(1+\frac{1}{531}\mathrm{Re}_D^{1/2}\right)$ $s_i = (0.9842, -0.1678, 0.0861, -0.0297, 0.0034)$ $4\times10^5 < \mathrm{Re}_D \le 5\times10^6$, $\mathrm{Pr} > 0.5$
3		$\mathrm{Nu}_D = 0.3 + \dfrac{0.62\mathrm{Re}_D^{1/4}\mathrm{Pr}^{1/3}\left(1+\frac{1}{2550}\mathrm{Re}_D^{5/8}\right)^{0.8}}{\left(1+0.543\,\mathrm{Pr}^{-2/3}\right)^{1/4}}$ Churchill and Chu [14]	$k_e \to$ Eq. (31), $(m,n,p) = (\frac{1}{3},\frac{2}{3},-\frac{1}{4})$ $a_i = (0.3, 0.62, 0.543)$, $A_0 \equiv \mu C_p \mathrm{Re}_D^{1/4}\left(1+\frac{1}{2550}\mathrm{Re}_D^{5/8}\right)^{0.8}$ $s_i = (0.9842, -0.1678, 0.0861, -0.0297, 0.0034)$ $3.5 \le \mathrm{Re}_D \le 7.6\times10^4$, $0.7 \le \mathrm{Pr} \le 380$
4	 sphere	$\mathrm{Nu}_D = 2 +$ $(0.4\mathrm{Re}_D^{1/2} + 0.06\mathrm{Re}_D^{2/3})(\frac{\mu}{\mu_s})^{1/4}\mathrm{Pr}^n$ Whitaker[18]	$k_e \to$ Eq. (37), $C_0 = 2$, $C_1 = (0.4\mathrm{Re}_D^{1/2} + 0.06\mathrm{Re}_D^{2/3})(\frac{\mu}{\mu_s})^{1/4}$, $m = 0.4$ $3.5 \le \mathrm{Re}_D \le 7.6\times10^4$, $0.7 \le \mathrm{Pr} \le 380$

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