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COMPACT THERMAL REPRESENTATIONS FOR SEVERAL FUNDAMENTAL SHAPES IN NATURAL CONVECTION

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In this article, general canonical forms for the effective thermal conductivities of compact heat sink models are derived using perturbation tools. The resulting approximations apply to a large number of fundamental heat sink shapes used in natural convection applications. The effective thermal conductivity is a property that can be assigned to the porous block (i.e., volume of fluid) above the heat sink base that was once occupied by the fins. The increased thermal conductivity of the fluid entering the porous block produces a reduced thermal resistance that matches that of the original heat sink. The use of a compact representation is accompanied by substantial computational savings that promote faster optimization and communication between simulation analysts and design engineers. The generalized approximations for the effective thermal conductivity presented here are numerically verified.

1 INTRODUCTION

Modern electronic devices exhibit a clear trend towards miniaturization and process acceleration. These two desirable trends of smaller and more powerful components are constantly producing devices that generate increasingly higher heat dissipation rates per unit area. Since the reliability of electronic components decreases exponentially with operating temperatures, methods for cooling electronic components are constantly resorted to. One such method relies on the use of heat sinks in a natural convection environment. In this context, the implementation of heat sinks has been proven to be an effective tool for reducing the junction-to-air thermal resistance and, thereby, maintaining operationally safe junction temperatures.

The popularity of heat sinks may be attributed to their relative simplicity and effectiveness as vehicles for heat removal in electronic devices. However, the design and optimization of their use is becoming increasingly more complex as electronic devices are requiring denser and more populated chip layouts. Generally, as conventional simulation methods are applied to small-scale board architectures, adequate thermal modeling results can be expected while using reasonable resolutions in discrete differencing schemes. However, using sufficiently resolved grids in many of today's larger-scale layouts often leads to exceedingly long computation times. This problem can often lead to critical project delays during prototype development. For thermal engineers involved in the product design cycle, the quest for faster simulation capabilities has recently become an important target in its own right.

Naturally, minimizing the thermal simulation complexity of electronic layouts has constituted the principal motivation behind several research investigations. Most approaches towards reducing the complexity of these systems employ discretizations that center around the lumped, coarse, or compact models of heat sinks. The use of these methods often reduces CPU time without significantly reducing the accuracy, a combination that can lead to faster design turnaround time. One such approach is described

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NOMENCLATURE

A_0 = $\mu C_p \mathrm{Gr}_L$
C_p = constant pressure specific heat
g = acceleration due to gravity
Gr_L = Grashoff number, $g\beta\Delta T L^3\rho^2\mu^{-2}$
k = thermal conductivity
k_e = effective thermal conductivity
L = characteristic length
Pr = Prandtl number, $\mu C_p / k$
Ra_L = Rayleigh number, $g\beta\Delta T L^3\rho^2\mu^{-2}\mathrm{Pr}$
T_f = film temperature, $(T_s + T_b)/2$
T_s = surface temperature
T_∞ = ambient coolant temperature
U = overall heat transfer coefficient, $\dot{Q}/(A\Delta T)$

α = thermal diffusivity, $k/\rho C_p$
β = volumetric thermal expansion coefficient, $1/T_f$
ΔT = $T_s - T_b$
μ, ν = dynamic and kinematic viscosities
ρ = density

by Narasimhan and Majdalani [1] who employ a three-dimensional lumped thermal resistance concept according to which the actual heat sink is replaced by a 'porous volumetric fluid block.' The thermal conductivity of this fluid block is altered so that it will exhibit the same equivalent thermal resistance as the replaced element. Other strategies have also been proposed and utilized by Bar-Cohen [2], Bar-Cohen, Elperin and Eliasi [3, 4], Culham, Yovanovich and Lee [5], Gauche, Coetzer and Visser [6], Boyalakuntla and Murthy [7], Lasance, Vinke and Rosten [8], and Linton and Agonafer [9].

Some of the stated methods utilize the same basic strategy of replacing the actual heat sink with a simpler model that exhibits similar thermal and flow resistance properties. Due to the replacement of the actual component with a less complex but nearly equivalent representation, the computational complexity and run-time required for thermal characterization are often reduced significantly. For instance, the fluidic block substitution method used by Narasimhan and Majdalani [1] has generated results that differed from detailed simulations by 5% despite being 10 times faster to obtain.

The computational expediency associated with the fluidic block method justifies the practical advantages of using simplified heat sink models in thermal analysis. Although methods such as the fluidic block have proven to be dependable, they are by no means optimal. An inherent problem in these methods is determining the thermal properties of the simplified/compact heat sinks. In the fluidic block method, a remaining obstacle is the determination of the effective thermal conductivity k_e that must be assigned to the fluidic block above the base to accurately mimic the enhanced heat transfer coefficient of the original heat sink. This value is typically extracted from the transcendental Nusselt number correlations associated with the heat sink configuration. In the past, this thermal conductivity was found through an iterative process which often required user intervention to obtain convergence. Obviating the need for iteration and user intervention was previously accomplished by Brucker, Ressler and Majdalani [10]. The need for this iteration can be seen by examining the widely used Churchill and Chu correlation for flow over a flat plate:

$$0 = -\frac{UL}{k_e} + 0.68 + \frac{0.67\mathrm{Ra}_L^{1/4}}{\left[1 + \left(0.492/\mathrm{Pr}\right)^{9/16}\right]^{4/9}} \tag{1}$$

In the past, the presence of the thermal conductivity k_e in the Nusselt, Prandtl, and Rayleigh numbers has made an explicit solution unlikely. This is due to these dimensionless numbers being raised to fractional powers, a fact that prevents simple algebraic extraction of the thermal conductivity.

Although direct algebraic manipulation fails at extracting k_e, a perturbation method can provide a closed-form solution, albeit approximate [10]. It is the purpose of this article to illustrate the extraction of the thermal conductivity from the corresponding Nusselt number relation not only for rectangular heat sinks, but for other fundamental heat sinks exhibiting diverse geometrical shapes that are often cited in the literature [2-5, 11-15].

2 GENERAL FREE CONVECTION

Empirical correlations designed to predict buoyancy and diffusive heat transfer from a heated surface to a quiescent fluid are abound in the technical literature. A sample of those can be found in [3-5, 12-22]. The simplest form of the average free-convection heat transfer coefficient can be represented in the following functional form

$$\mathrm{Nu}_L = C\mathrm{Ra}_L^m . \tag{2}$$

A slightly more general form can be functionally represented by [23]

$$\mathrm{Nu}_L = a_0 + a_1\mathrm{Ra}_L^m \tag{3}$$

The latter extends over a wider range of Rayleigh numbers due to the addition of the leading-order constant a_0. This constant accounts for thermal conduction effects which become dominant in the limit as $\mathrm{Ra}_L \to 0$. The exponent m is usually taken to be $1/4$ or $1/3$ depending on whether the flow is laminar or turbulent [24]. If the cooling fluid is not air, the simplest correlations are no longer adequate and the functional form of the equation must be modified to include the universal Prandtl number function. This is given by

$$F(\mathrm{Pr}) = \left[1 + \left(a_2/\mathrm{Pr}\right)^n\right]^r \tag{4}$$

The form of F for the laminar flow over a flat plate is given by Churchill and Churchill [25] as $[1+(0.492/\mathrm{Pr})^{9/16}]^{-16/9}$. The inclusion of the universal Prandtl number function in Eq. (3) yields

$$\mathrm{Nu}_L = a_0 + a_1\mathrm{Ra}_L^m\left[1 + \left(a_2/\mathrm{Pr}\right)^n\right]^{mr} \tag{5}$$

Equation (5) is very general as it can be used to simultaneously represent free convection over a vertical plate, a horizontal, vertical or inclined cylinder, a cube in several orientations, a sphere, a bisphere, a prolate spheroid and an oblate spheroid. The constants a_0, a_1, a_2, m, n and p are specific to the geometry and flow regime and are listed in Table 1.

In addition to Eq. (5) (which is restricted to laminar heat transfer), Churchill and Chu [23] have proposed the following correlation which stretches over the entire range of Rayleigh numbers:

$$\mathrm{Nu}_L = \left\{ a_0 + a_1 \mathrm{Ra}_L^{m/q} \left[1 + \left(a_2 / \mathrm{Pr} \right)^n \right]^{mr/q} \right\}^q \tag{6}$$

Equation (6) is quite general in nature as it can be used to represent the heat transfer from vertically flat plates, inclined flat plates, vertical or horizontal cylinders, vertical cones, spheres, inclined disks, and spherelike surfaces. Corresponding constants for use with Eq. (6) are compiled in Table 1 as well.

The remainder of the paper will be devoted to determining an explicit solution for k_e depending on the equation type. Every attempt is made to present a systematic methodology that could later be used to determine solutions for k_e that are not covered in the present paper. The format will be to proceed from simple to complex, in hopes that the simple solutions will provide a solid basis from which one can build upon in determining the more complex solutions.

3 EXACT k_e FOR FREE CONVECTION

For a correlation of the type given by Eq. (2), namely, $\mathrm{Nu}_L = C\mathrm{Ra}_L^m$, an exact solution k_e can be determined algebraically to be

$$k_e = \mathrm{Gr}_L \mu Cp [UL / (C \mathrm{Gr}_L \mu Cp)]^{1/(1-m)} \tag{7}$$

Many examples of such correlations are listed in Table 2 for specific geometries and flow conditions. This type, however, appears to be the last for which a universal algebraic solution can be determined. Even in the slightly more complicated case given by Eq. (3), $\mathrm{Nu}_L = a_0 + a_1 \mathrm{Ra}_L^m$, a general solution cannot be determined and a specific solution for a given value of m must be settled for. This specific solution can be obtained by solving for the meaningful root of

$$a_0 k_e + a_1 (\mathrm{Gr}_L \mu C_p)^m k_e^{1-m} - UL = 0 \tag{8}$$

There is only a finite number of cases where the exponent is such that an exact solution can be determined. This becomes apparent when one considers Abel's Impossibility theorem; this theorem affirms that the no polynomial above order four can be solved by a finite number of additions, subtractions, multiplications, divisions and root extractions. Since m is usually taken to be a positive number less than 1. It must be $0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1$ in order to have a direct solution.

For the case of $m = 0$ the equation becomes devoid of any convective terms; heat transfer occurs due purely to conduction. The solution to this equation takes the simple form of $k_e = UL / a_0$. This can be viewed as being a superfluous solution as we are interested in convective heat transfer. For the case of $m = 1/2$, Eq. (8) yields a quadratic equation for k_e; this is

$$k_e = \frac{1}{2} \Big(a_1^2 \mathrm{Gr}_L \mu C_p - a_1 \left(\mathrm{Gr}_L \mu C_p \right)^{1/2} \sqrt{a_1^2 \mathrm{Gr}_L \mu C_p + 4 a_0 UL} + 2 a_0 UL \Big) / a_0^2 \tag{9}$$

For the case of $m = 1$ we find that the equation is linear and easily solvable. It yields

$$k_e = \left(UL - \mathrm{Gr}_L \mu C_p \right) / a_0 \tag{10}$$

Most of the equations in the technical literature contain either $m = 1/3$ or $m = 1/4$ corresponding to the laminar and turbulent cases, respectively. The solution to these cases is essential but far more complicated than the cases presented in Eqs. (9)-(10); the latter result in third and fourth order polynomials. For the case of $m = 1/3$ the following third order polynomial must be solved

$$k_e^3 + (a_1/a_0)^3 \mathrm{Gr}_L \mu C_p k_e^2 - (UL / a_0)^3 = 0 \,. \tag{11}$$

To solve for the physical root k_e we turn to Cardano's method for solving a cubic equation [26]. There are two possibilities that emerge depending on the sign of Cardano's discriminant,

$$\Delta = (UL / a_0)^6 - \left[(a_1 / a_0)^3 \mathrm{Gr}_L \mu C_P \right]^3 (UL / a_0)^3 \,. \tag{12}$$

If $\Delta < 0$, the trigonometric root is given by

$$k_e = \tfrac{1}{3} b_0 \left(2\cos\theta - 1 \right), \quad \theta \equiv \tfrac{1}{3} \cos^{-1}\left(\tfrac{27}{2} b_1 b_0^{-3} - 1 \right),$$
$$b_0 \equiv \left(a_1 / a_0 \right)^3 \mathrm{Gr}_L \mu C_P, \quad b_1 \equiv \left(UL / a_0 \right)^3 \tag{13}$$

In the case of $\Delta \ge 0$, the ordinary root becomes

$$k_e = \tfrac{1}{6} b_2 + \tfrac{2}{3} b_0^2 b_2^{-1} - \tfrac{1}{3} b_0 \,;$$
$$b_2 \equiv \left(108 b_1 - 8 b_0^3 + 12 \sqrt{81 b_1^2 - 12 b_0^3 b_1} \right)^{1/3} \tag{14}$$

If the characteristic length L is on the order of $\mathrm{Gr}_L \mu C_P / U$ or larger, the trigonometric root given by Eq. (13) may be safely used. Equation (14) should be defaulted to under the rare circumstances in which Eq. (13) fails.

For the turbulent case ($m = 1/4$), Eq. (8) yields a fourth order polynomial. The solution for k_e is the meaningful root of the quartic equation, which can be written as

$$k_e = \left(\sqrt{2 + 2(1 + c_1)^{-1/2} - c_1} - 1 - \sqrt{1 - c_1} \right) c_0^{-1} \tag{15}$$

where

$$\begin{cases} c_0 \equiv (a_1 / a_0)^4 \mathrm{Gr}_L \mu C_P, \; c_1 \equiv 2 \left(\tfrac{2}{3} c_2 c_3 \right)^{1/3} \\ \qquad \times \left[\left(\tfrac{2}{3} \right)^{1/3} - 8 c_2^{1/3} c_3^{-2/3} \right] c_0^{-2} \\ c_2 \equiv (UL / a_0)^4, \; c_3 \equiv \sqrt{81 c_0^4 + 768 c_2} - 9 c_0^2 \end{cases} \tag{16}$$

For the case of $m = 2/3$, a cubic equation is obtained while for $m = 3/4$, one is left with a quartic equation. These can be solved, but their solutions are omitted because the authors are not aware of any physical correlation where they could be applied.

Table 1. Constants in the asymptotic solutions for k_e

Case	a_0	a_1	a_2	m	n	p	Eq.	s_0	s_1	s_2	s_3	s_4	Range Ra_L	Range Pr	Notes
Vertical Plate [23]	0.68	0.670	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^9	$\forall$	
Vertical Plate [23]	0.825	0.387	0.492	$1/3$	$9/16$	$-16/27$	(61)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0463	1.0304	10^0-10^{13}	$\forall$	
Inclined Plate [27] (heated surface up)	0.825	0.387	0.492	$1/3$	$9/16$	$-16/27$	(61)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0463	1.0304	$\forall$	$\forall$	$g \to g\cos\theta$ $\theta \le 60°$
Vertical Cylinder [23]	0.68	0.670	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^9	$\forall$	$D/L \ge 35/\mathrm{Gr}_L^{1/4}$
Vertical Cylinder [28]	3.44	0.645	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = A_w^{1/2}$
Vertical Cylinder [27]	0.825	0.387	0.492	$1/3$	$9/16$	$-16/27$	(61)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0463	1.0304	$\forall$	$\forall$	$D/L \ge 35/\mathrm{Gr}_L^{1/4}$
Horizontal Cylinder [29]	0.36	0.518	0.559	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^9	$\forall$	$L = D$
Horizontal Cylinder [28]	3.44	0.683	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^9	$\forall$	$L = \frac{1}{2}D\pi^{1/2}$
Horizontal Cylinder [29]	0.60	0.387	0.559	$1/3$	$9/16$	$-16/27$	(61)	2x10^{-6}	-0.00001	0.0042	-0.0465	1.0298	$10^{-6}-10^9$	$\forall$	$L = D$
Inclined Cylinder [28]	3.44	0.673	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^9	$\forall$	$\theta = 45°$, $L = A_w^{1/2}$
Vertical Cone [28]	0.735	0.387	0.492	$1/3$	$9/16$	$-16/27$	(61)	4x10^{-7}	-4x10^{-5}	0.0018	-0.0567	1.0283	$\forall$	$\forall$	$L = \frac{4}{5}H$
Sphere [27]	2.00	0.589	0.469	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	$\le 10^{11}$	> 0.5	$L = D$
Sphere [28]	3.545	0.685	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = \left(\frac{\pi}{3}\right)^{1/2} D$
Sphere [27]	1.77	0.387	0.492	$1/3$	$9/16$	$-16/27$	(61)	4x10^{-10}	-3x10^{-7}	6x10^{-5}	-0.0103	1.0336	$\forall$	$\forall$	$L = \frac{1}{2}\pi D$
Bi-Sphere [28]	3.475	0.622	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = \left(\frac{2\pi}{3}\right)^{1/2} D$
Oblate Spheroid [28] ($AR = 0.5$)	3.529	0.651	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = A_w^{1/2}$
Oblate Spheroid [28] ($AR = 0.1$)	3.342	0.515	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = A_w^{1/2}$
Prolate Spheroid [28] ($AR = 1.93$)	3.342	0.515	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = A_w^{1/2}$
Spherelike Surface* [28]	$\frac{3\pi}{9}A_S$	0.387	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	$\forall$	$\forall$	$L = A_S^{3/2}/(6\vartheta)$
Inclined Disk [27]	0.748	0.387	0.492	$1/3$	$9/16$	$-16/27$	(61)	3x10^{-7}	-3x10^{-5}	0.0017	-0.0553	1.027	$\forall$	$\forall$	$L = \frac{9}{11}D$
Cube [28] (Base in Plane)	3.388	0.637	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = \sqrt{6}H$
Cube [28] (45° to Base Plane)	3.388	0.663	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = \sqrt{6}H$
Cube [28] (Diagonal $\perp$ to Base Plane)	3.388	0.679	0.492	$1/4$	$9/16$	$-4/9$	(40)	2x10^{-7}	-2x10^{-5}	0.0012	-0.0465	1.0311	10^0-10^8	$\forall$	$L = \sqrt{6}H$

*Parametric study needed with specific A_S for optimal performance. However, sphere data should give reasonable estimates especially if shape does not deviate significantly from a sphere.

4 FOR LAMINAR FREE CONVECTION

Although the exact solutions presented in Sec. 3 are suitable for use in compact models one must take great care in ensuring that the range of applicability is suitable. Often these simple correlations are only applicable over a very limited range. The more widely applicable and accurate correlations are usually those exhibiting more sophistication. Examples include those that exhibit the form given by Eq. (5), namely,

$$\mathrm{Nu}_L = UL/k_e = a_0 + a_1\mathrm{Ra}_L^m\left[1+\left(a_2/\mathrm{Pr}\right)^n\right]^p \tag{17}$$

This relation can be obtained by setting $p = mr$ in Eq. (5). By re-expressing Eq. (17) so that all occurrences of k_e are visible we find that

$$UL/k_e = a_0 + a_1\left(A_0/k_e\right)^m\left\{1+[a_2k_e/(\mu C_P)]^n\right\}^p\ ;\ A_0 \equiv \mu C_p \mathrm{Gr}_L \tag{18}$$

Table 2. Constants in the exact solutions for k_e given by Eq. (7)

Case	C	m	Range Ra_L	Range Pr	Notes
Vertical Plate [30]	0.59	$\frac{1}{4}$	$10^4 - 10^9$	$\forall$	
Vertical Plate [31]	0.1	$\frac{1}{3}$	$10^9 - 10^{13}$	$\forall$	
Vertical Plate [32]	0.021	$\frac{2}{5}$	$10^9 - 10^{13}$	$\forall$	
Horizontal Plate (heated side up) [33]	0.54	$\frac{1}{4}$	$10^4 - 10^7$	$\forall$	
Horizontal Plate (heated side down) [33]	0.27	$\frac{1}{4}$	$10^5 - 10^{11}$	$\forall$	
Horizontal Plate (heated side up) [33]	0.15	$\frac{1}{3}$	$10^7 - 10^{10}$	$\forall$	
Inclined Plate (heated side down) [34]	$0.56(\cos\theta)^{1/4}$	$\frac{1}{4}$	$10^5 - 10^{11}$	$\forall$	$T_f \to T_e = T_s - 0.25(T_s - T_\infty)$ $\beta = 1/T_\beta$, $T_\beta = T_\infty + 0.50(T_s - T_\infty)$, $\mathrm{Ra}_L = \mathrm{Gr}_L \mathrm{Pr} \cos\theta$
Inclined Plate (heated side up)	0.17	$\frac{1}{4}$	$10^{10} - 10^{15}$	$\forall$	$\mathrm{Ra}_L = \mathrm{Gr}_L^* \mathrm{Pr}$ $\mathrm{Gr}_L^* = g\beta U L^4 \cos\theta / (k\nu^2)$
Vertical Cylinder [30]	0.53	$\frac{1}{4}$	$10^4 - 10^9$	$\forall$	
Vertical Cylinder [31]	0.1	$\frac{1}{3}$	$10^9 - 10^{13}$	$\forall$	$D/L \geq 35/\mathrm{Gr}_L^{1/4}$
Horizontal Cylinder [35]	0.675	0.058	$10^{-9} - 10^2$	$\forall$	$L = D$
Horizontal Cylinder [35]	1.02	0.148	$10^{-2} - 10^2$	$\forall$	$L = D$
Horizontal Cylinder [35]	0.85	0.188	$10^2 - 10^4$	$\forall$	$L = D$
Horizontal Cylinder [35]	0.48	$\frac{1}{4}$	$10 - 10^7$	$\forall$	$L = D$
Horizontal Cylinder [35]	0.125	$\frac{1}{3}$	$10^7 - 10^{12}$	$\forall$	$L = D$
Inclined Cylinder [36]	$[0.47$ $-0.11(\sin\theta)^{4/5}]$ $\times \mathrm{Gr}_D^{-1/12}$	$\frac{1}{3}$	see notes	$\forall$	$(\mathrm{Gr}_L \mathrm{Pr})_{\mathrm{cr}} = 2.6\times 10^9 + 1.1\times 10^9 \tan\theta$ $(\mathrm{Gr}_L \mathrm{Pr})_{\mathrm{cr}} \leq \mathrm{Gr}_L \mathrm{Pr} \leq 2.95\times 10^{10}$ $1.08\times 10^4 \leq \mathrm{Gr}_D \mathrm{Pr} \leq 6.9\times 10^5$

The power-law embedment of k_e in the universal Prandtl number function given by (4) eliminates the possibility of obtaining an exact expression for k_e. In the most favorable scenario for which $p = 1$, any known correlation that would result would be order 16. This still could not be solved algebraically. Therefore, an asymptotic approximation of the form $k_e \approx k_0 + k_1$ will be sought instead.

A Taylor series expansion is necessary to first remove the power law nature of the universal Prandtl number function. Knowing that a Taylor series is only convergent over some finite interval of the domain, two solutions must be sought for small and large values of $\left(a_2 k_e/(\mu C_P)\right)^n$. For convenience, we let $\kappa \equiv a_2/\left(\mu C_p\right)$.

4.1 Type-I: Laminar Regime, Small k_e

For small $\left(a_2 k_e/(\mu C_P)\right)^n$ (i.e. $\left(\kappa k_e\right)^n$), the Taylor series expansion of $F(\mathrm{Pr})$ becomes

$$\left[1+\left(\kappa k_e\right)^n\right]^p \simeq 1 + p\left(\kappa k_e\right)^n + \tfrac{p(p-1)}{2!}\left(\kappa k_e\right)^{2n} + \tfrac{p(p-1)(p-2)}{3!}\left(\kappa k_e\right)^{3n} + \tfrac{p(p-1)(p-2)(p-3)}{4!}\left(\kappa k_e\right)^{4n} + O\left(\kappa k_e\right)^{5n} \qquad (19)$$

The substitution of Eq. (19) into Eq. (18) yields

$$UL \simeq \underset{1}{a_0 k_e} + \underset{2}{B_0 k_e^{1-m}} + \underset{(-)}{C_0 k_e^{1-m+n}} + \underset{3}{D_0 k_e^{1-m+2n}} + \underset{(-)}{E_0 k_e^{1-m+3n}} + \underset{4}{F_0 k_e^{1-m+4n}} \qquad (20)$$

where

$$\begin{cases} B_0 \equiv a_1 A_0^m; \quad C_0 \equiv p B_0 \kappa^n \\ D_0 \equiv \frac{p(p-1)}{2!} B_0 \kappa^{2n}; \quad E_0 \equiv \frac{p(p-1)(p-2)}{3!} B_0 \kappa^{3n} \\ F_0 \equiv \frac{p(p-1)(p-2)(p-3)}{4!} B_0 \kappa^{4n} \end{cases} \qquad (21)$$

Equation (20) is a valid representation of Eq. (17) for small values of $\left(\kappa k_e\right)^n$. The convergence criteria are discussed in Sec. 4.3. To determine the most influential term in Eq. (20) we first neglect quantities that are negative because a negative thermal conductivity is unphysical. By considering the remaining terms we find that the leading-order solution may be dependent on any of the following terms

$$k_e = \left\{UL/a_0, (UL/B_0)^{1/(1-m)}, (UL/D_0)^{1/(1-m+2n)}, (UL/F_0)^{1/(1-m+4n)}\right\} \qquad (22)$$

To determine which of these terms is actually the leading-order solution, we first eliminate $k_e = UL/a_0$; as this term only accounts for conduction, it dominates in the limit of $\mathrm{Ra}_L \to 0$. It is then seen that all remaining terms are related to the expansion of the universal Prandtl number function.

Consequently, the leading-order term becomes the first non-negative term resulting from the Taylor expansion of this function. One deduces that

$$k_0 = (UL/B_0)^{1/(1-m)} . \tag{23}$$

The next step is to add the first-order correction term k_1, which results in the following:

$$k_e^{\mathrm{I}} \simeq k_0 + k_1 \tag{24}$$

where the superscript is used to denote a type-I (small) solution.

Using the notion of successive approximations, the first-order correction k_1 can be found as follows. Equation (24) is substituted back into Term 2 of Eq. (20) while Eq. (23) is used everywhere else; after effectuating these substitutions, one finds

$$a_0 k_0^2 + (1-m) ULk_1 + C_1 k_0^{2-m+n} + D_0 k_0^{2-m+2n} + E_0 k_0^{2-m+3n} + F_0 k_0^{2-m+4n} = 0 . \tag{25}$$

and so

$$k_1 = -\left[UL(1-m)\right]^{-1} \left(a_0 k_0^2 + C_0 k_0^{2-m+n} + D_0 k_0^{2-m+2n} + E_0 k_0^{2-m+3n} + F_0 k_0^{2-m+4n}\right). \tag{26}$$

Equation (24) is valid for all L and B_0 as long as $0 \le k_e^{\mathrm{I}} \le k_{//}$ where $k_{//}$ is a cut-off value separating the two solutions. This physical limitation is due to the convergence criteria of the Taylor series used in the expansion of the universal Prandtl number function.

4.2 Type-II: Laminar Regime, Large k_e

Considering the LHS of Eq. (17), we see that the thermal conductivity of the volumetric fluid block is commensurate with the overall heat transfer coefficient of the actual heat sink. This inter-coupling leads to the conclusion that the series expansion used to obtain Eq. (20) can become divergent as U is increased, since by definition $(\kappa k_e)^n$ is increased as well. Therefore, for cases where $(\kappa k_e)^n$ is no longer small, a Taylor series expansion in the reciprocal of $(\kappa k_e)^n$ is needed to ensure convergence. Equation (19) then becomes

$$(\kappa k_e)^{np}\left[1+(\kappa k_e)^{-n}\right]^p = (\kappa k_e)^{np}\left[1 + p(\kappa k_e)^{-n} + \tfrac{p(p-1)}{2!}(\kappa k_e)^{-2n} + \tfrac{p(p-1)(p-2)}{3!}(\kappa k_e)^{-3n} + O(\kappa k_e)^{-4n}\right] \tag{27}$$

After substituting Eq. (27) into Eq. (18), one gets

$$UL = \underset{1}{a_0 k_e} + \underset{2}{B_1 k_e^{1+np-m}} + \underset{(-)}{C_1 k_e^{1-n+np-m}} + \underset{3}{D_1 k_e^{1-2n+np-m}} + \underset{(-)}{E_1 k_e^{1-3n+np-m}} \tag{28}$$

where

$$\begin{cases} B_1 \equiv B_0 \kappa^{np}; \quad C_1 \equiv B_0 p \kappa^{np-n} \\ D_1 \equiv \frac{p(p-1)}{2!} B_0 \kappa^{np-2n}; \; E_1 \equiv \frac{p(p-1)(p-2)}{3!} B_0 \kappa^{np-3n} \end{cases} \tag{29}$$

By dismissing negative terms that cannot possibly dominate the solution, three terms are identified in Eq. (28) as possible leading-order candidates. These are

$$k_e = \left\{UL/a_0, (UL/B_1)^{1/(1+np-m)}, \; (UL/D_1)^{1/(1-2n+np-m)}\right\} \tag{30}$$

In determining which of these terms is the leader, we eliminate $k_e = UL/a_0$, being a non-convective term. Again, all of the remaining terms are related to the expansion of the universal Prandtl number function. The leading-order term should be the first non-negative term resulting from the Taylor expansion of this function. Subsequently, the leading term is deduced to be $k_e = (UL/B_1)^{1/(1+np-m)}$; although this is a valid leading-order term, it required 10 correction terms to converge the solution. Thus, the corresponding solution is quite lengthy and cumbersome. In light of this, a new method is utilized. In this method, a composite leading-order term is found by considering the conduction term as well as the first term in the expansion. This results in the following hybrid combination

$$\text{Root of}\left\{-UL + a_0 K_0 + B_1 K_0^{1+np-m} = 0\right\} \tag{31}$$

Fortuitously, all cases presented in this paper are characterized by a single exponent, $1+np-m = 1/2$. This simplifying power leads to a quadratic equation for the leading term. Based on the meaningful quadratic root, one can put

$$K_0 = \left[UL + \tfrac{1}{2} B_1\left(B_1 - \sqrt{B_1^2 + 4a_0 UL}\right)/a_0\right]/a_0 \tag{32}$$

Next we proceed as before and add a first-order correction term K_1; this results in

$$k_e^{\mathrm{II}} \simeq K_0 + K_1 \tag{33}$$

where the superscript is used to denote a type-II (large) solution. Using successive approximations, Eq. (33) is now substituted into Eq. (28) in both Terms 1 and 2. However, only Eq. (32) is substituted into all other terms. Once completed, K_1 is found to be

$$K_1 = 16k_0\left(E_1 + D_1 K_0^{9/16} + C_1 K_0^{9/8}\right) / \left(19E_1 + 10D_1 K_0^{9/16} + C_1 K_0^{9/8} - 8B_1 K_0^{27/16} - 16a_0 K_0^{35/16}\right) \tag{34}$$

For the more general case of an arbitrary exponent in Eq. (31), the correction K_1 exhibits the form

$$K_1 = K_0\left(E_1 K_0^{2+2np-2m-3n} + D_1 K_0^{2+2np-2m-4n} + C_1 K_0^{2+2np-2m-5n} + B_1 K_0^{4+4np-4m-6n}\right) / \Big(E_1(1+np-m-3n)K_0^{2+2np-2m-3n} + D_1(1+np-m-2n)K_0^{2+2np-2m-4n} + B_1 C_1(1-n+np-m)K_0^{2+2np-2m-5n} - a_0 K_0^{5+4np-4m-6n} - B_1 K_0^{4+4np-4m-6n}\Big) \tag{35}$$

Equation (32) is valid for all L and B_1 as long as $k_e^{\mathrm{II}} > k_{//}$. Again, the range of applicability is limited by the divergence of the Taylor series expansion involved in the solution.

4.3 Cut-off Value $k_{//}$

The proposed asymptotic solution depends on two Taylor series expansions, each valid over only part of the domain. Therefore, we define $k_{//}$ to be the value that optimally separates the type-I and type-II solutions for k_e. This value is as important as the expansions themselves, since the Taylor series rapidly diverge outside of their region of validity. This divergence is characterized by an extremely rapid progression to $\pm\infty$. The optimal patching value can be determined by carefully examining the convergence criteria for the two cases at hand. Based on Eq. (19), the requirement for the type-I expansion can be seen to be $|p(\kappa k_e)^n|<1$. Accordingly, the small series expansion is divergent when

$$k_e < k_e^+ ;\, k_e^+ \equiv | p^{-1/n}\kappa^{-1} | \tag{36}$$

The large series expansion in Eq. (27) becomes divergent when $|p(\kappa k_e)^{-n}|<1$ or

$$k_e > k_e^- ;\, k_e^- \equiv | p^{1/n}\kappa^{-1} | \tag{37}$$

These two asymptotic bounds can be interpreted as the simultaneous upper and lower limits of the range of validity for the type-I and type-II approximations, respectively. The optimum patching value $k_{//} \in [k_e^-, k_e^+]$ is found by setting

$$k_{//} = x k_e^- + (1-x) k_e^+ ;\quad x\in[0,1] \tag{38}$$

Here x is chosen so as to minimize the maximum asymptotic error in both the type-I and type-II solutions. The value of x takes the form of a fourth order polynomial, namely,

$$x = s_0 + s_1\bar{k} + s_2\bar{k}^2 + s_3\bar{k}^3 + s_4\bar{k}^4 ;\; \bar{k} = 2UL/(k_e^- + k_e^+). \tag{39}$$

The constants (s_0, s_1, s_2, s_3) are determined using a parametric study involving 2,000 runs per geometry. The correlation coefficient in all cases exceeds 0.998. The resulting numeric values of s_0, s_1, s_2, s_3 are posted in Table 1.

At the delineation point, $k_e = k_{//}$, both solutions are slightly divergent, resulting in the maximum error. This is a useful quantity to know since it will provide an upper bound on the error involved in a specific calculation. This can be accomplished by comparing either k_e^{I} or k_e^{II} to an iterative solution. The attendant error remains under 2%, so long as k_e is at least 5% away from $k_{//}$. In practice, once the cut-off $k_{//}$ is calculated for a given heat sink application, one may safely use the appropriate correlation depending on the operating range viz.

$$k_e = \begin{cases} k_e^{\mathrm{I}}; & 0 < k_e \le k_{//} \\ k_e^{\mathrm{II}}; & k_e > k_{//} \end{cases} \text{(laminar regime)} \tag{40}$$

where both approximations are equivalent at the cut-off point.

5 FOR LAMINAR AND TURBULENT FREE CONVECTION

Given that Eq. (17) is limited to the laminar regime Churchill and Chu [23] have also proposed a broader correlation that is applicable under turbulent conditions. This can be generically written as

$$\mathrm{Nu}_L = UL/k_e = \left\{ a_0 + a_1 \mathrm{Ra}^{m/2}\left[1+(a_2/\mathrm{Pr})^n\right]^{p/2}\right\}^2 \tag{41}$$

Although the originally introduced correlation is for flow over a flat plate, it has also been adopted in diverse physical settings. Equation (41) is a special case of Eq. (6) with $q=2$.

In solving, the first step is to expand the equation; one uses a quadratic expansion to expose individual exponents via

$$UL/k_e = a_0^2 + 2a_0a_1\,\mathrm{Ra}^{m/2}\left[1+(a_2/\mathrm{Pr})^n\right]^{p/2}$$
$$+\left\{a_1\,\mathrm{Ra}^{m/2}\left[1+(a_2/\mathrm{Pr})^n\right]^{p/2}\right\}^2 ;\, A_2 \equiv (g\beta\Delta T L^3 \rho^2 C_p/\mu)^{m/2} \tag{42}$$

Although Eq. (42) is similar in form to Eq. (18), the laminar solution is not applicable; this is due to the universal Prandtl number function appearing twice. Therefore, a new solution is sought. To that end, Eq. (42) is rearranged so that

$$UL = a_0^2 k_e + 2a_0a_1 k_e^{1-m/2} A_2\left\{1+\left[a_2k_e/(\mu C_P)\right]^n\right\}^{p/2} + a_1^2 k_e^{1-m} A_2^2$$
$$\times\left\{1+\left[a_2k_e/(\mu C_P)\right]^n\right\}^p. \tag{43}$$

The universal Prandtl number function must be expanded lest a direct solution is precluded. This is accomplished in the following two sections.

5.1 Type-I: Dual Regime, Small k_e

To start we set $w=[a_2k_e/(\mu C_p)]^n$, then if w is assumed to be small, two Taylor series can be used to expand both occurrences of the universal Prandtl number function; this strategy yields

$$UL = a_0^2 k_e + 2a_0a_1A_2k_e^{1-m/2}T_1 + a_1^2A_2^2k_e^{1-m}T_2 \tag{44}$$

where T_1 and T_2 are Taylor series defined by

$$T_1 = 1-\tfrac{1}{2}pw + \tfrac{1}{8}p(p-2)w^2 + O\left(w^3\right),$$
$$T_2 = 1 - pw + \tfrac{1}{2}p(p-1)w^2 + O\left(w^3\right) \tag{45}$$

At this point Eq. (45) is inserted into Eq. (44) to expose the powers of k_e

$$UL = a_0^2k_e + 2a_0a_1A_2\left[k_e^{1-m/2} - \tfrac{1}{2}puk_e^{1-m/2+n}\right.$$
$$\left.+\tfrac{1}{8}p(p-2)u^2k_e^{1-m/2+2n}\right] + a_1^2A_2^2$$
$$\times\left[k_e^{1-m} - puk_e^{1-m+n} + \tfrac{1}{2}p(p-1)u^2k_e^{1-m+2n}\right];$$
$$u \equiv \left(a_2/\mu C_P\right)^n \tag{46}$$

The leading-order term can be obtained following Brucker, Ressler and Majdalani [10]. The result is

$$k_0 = a_1^{-2/(1-m)}A_2^{-2/(1-m)}(UL)^{1/(1-m)} \tag{47}$$

Using $m=1/3$ and the specific constants (m,n,p) listed in Table 1, one obtains

$$k_0 = a_1^{-3}A_2^{-3}(UL)^{3/2} \tag{48}$$

Adding correction terms to the leading-order solution yields the final solution

$$k_e^{\mathrm{I}} \simeq a_1^{-3}A_2^{-3}(UL)^{3/2} + k_1 + k_n \tag{49}$$

where

$$k_1 = -\tfrac{3}{2}\Big[a_0^2 k_0^2 + 2a_0 a_1 A_2 \Big(k_0^{11/6} - \tfrac{8}{27} u k_0^{115/48}$$
$$+ \tfrac{140}{729} u^2 k_0^{71/24} \Big) + a_1^2 A^2 u \Big(\tfrac{344}{729} u k_0^{67/24} - \tfrac{16}{27} k_0^{107/48} \Big) \Big] / (UL) \qquad (50)$$

The higher-order corrections are defined by the recurrence relation

$$k_n = -\tfrac{3}{2} k_0 k_{n-1} \Big[a_0^2 + a_0 a_1 A_2 \Big(\tfrac{5}{3} k_0^{-1/6} - \tfrac{67}{81} u k_0^{19/48}$$
$$+ \tfrac{1645}{2187} u^2 k_0^{23/24} \Big) + a_1^2 A_2^2 u \Big(\tfrac{1849}{2187} u k_0^{19/24} - \tfrac{59}{81} k_0^{11/48} \Big) \Big] / (UL) \qquad (51)$$

which is valid for $n = 2, 3, \ldots$

Brucker, Ressler and Majdalani [10] show that a two-term approximation $k_e^{\mathrm{I}} \simeq k_0 + k_1$ is sufficiently adequate for problems of practical interest. This was realized by comparing results from the compact model with experimental and detailed numerical simulations. Here too we find Eq. (49) to be valid so long as $0 < k_e^{\mathrm{I}} \le k_{//}$.

5.2 Type-II: Dual Regime, Large k_e

In this case, the universal Prandtl number function must be expanded in the reciprocal of w or $[a_2 k_e / (\mu C_p)]^{1/n}$. Using two Taylor series expansions, Eq. (43) becomes

$$UL = \underset{1}{a_0^2 k_e} + \underset{2}{2a_0 a_1 A_2 u^{-p/2} k_e^{1-m/2+np/2}}$$
$$+ \underset{3}{a_1^2 A_2^2 u^p k_e^{1-m+np}} - \underset{4}{p a_1^2 A_2^2 u^{-p-1} k_e^{1-m-np-n}} \qquad (52)$$

Terms that are not shown are those having marginal contributions. Again, we utilize the concept of a hybrid leading-order solution; accordingly, all except the last term in Eq. (52) must be retained. The problem now becomes that of solving

$$-c_1 + c_2 K_0^{1-m/2+np/2} + c_3 K_0^{1-m+np} + c_4 K_0 = 0 \qquad (53)$$

where

$$c_1 \equiv UL; c_2 \equiv a_1^2 A_2^2 u^p; c_3 \equiv 2a_0 a_1 A_2 u^{p/2}; c_4 \equiv a_0^2 \qquad (54)$$

For the specific cases listed in Table 1, Eq. (53) leads to the cubic polynomial

$$-c_1 + c_2 K_0^{1/3} + c_3 K_0^{2/3} + c_4 K_0 = 0 \qquad (55)$$

whence

$$K_0 = \left(c_1 - c_3 p_1^2 - c_2 p_2 \right) / c_4 \, ;$$

$$\begin{cases} p_0 \equiv (36 c_2 c_3 c_4 + 108 c_1 c_4^2 - 8 c_3^3 + p_2)^{1/3} \\ p_1 \equiv \tfrac{1}{6} p_0 / c_4 - \tfrac{2}{3}(3 c_2 c_4 - c_3^2)/(c_4 p_0) - \tfrac{1}{3}(c_3 / c_4) \\ p_2 \equiv 12\sqrt{3} c_4 \left(27 c_1^2 c_4 - c_2^2 c_3^2 - 4 c_1 c_3^3 + 4 c_2^3 c_4 + 18 c_1 c_2 c_3 c_4 \right)^{1/2} \end{cases} \qquad (56)$$

Using the method of successive approximations to account for the small correction associated with the fourth term in Eq. (52), one sets

$$k_e^{\mathrm{II}} \approx K_0 + K_1 \, . \qquad (57)$$

Equation (57) is then substituted back into Terms 1-3 of Eq. (52). In the same vein, Eq. (56) is substituted into Term 4 of Eq. (52). The first-order correction term is then found to be

$$K_1 = -\tfrac{16}{9} a_1^2 A_2^2 K_0^{-11/48} u^{-43/27}$$
$$/ \left(3{a_0}^2 + 4 a_0 a_1 A_2 K_0^{-1/3} u^{-8/27} + a_1^2 A_2^2 K_0^{-2/3} u^{-16/27} \right) \qquad (58)$$

which is valid so long as $k_e^{\mathrm{II}} > k_{//}$.

5.3 Cut-off Value $k_{//}$

In the dual regime the value of $k_{//}$ can be obtained following the method described in Section 4.3. Considering Eq. (45) it can be seen that both expansions of type-I will diverge unless $|\tfrac{1}{2} p u k_e^n| < 1$ and $|p u k_e^n| < 1$. Both conditions are met when

$$k_e < k_e^+ ; \; k_e^+ \equiv |(pu)^{-1/n}| \qquad (59)$$

In like manner, the type-II will diverge unless $|\tfrac{1}{2} p u^{-1} k_e^{-n}| < 1$ and $|p u^{-1} k_e^{-n}| < 1$. This is true when

$$k_e > k_e^- ; \; k_e^- \equiv |(p/u)^{1/n}| \qquad (60)$$

The procedure summarized in Eqs. (38)–(39) may once again be used to determine the optimal $k_{//}$. A parametric study is again used, with 2000 runs per geometric shape. The coefficients of the best fit polynomial are determined with a correlation coefficient exceeding 0.998. The error associated with the solution is the same as that in Sec. 4.3. The final solution is expressible by the piecewise form given by Eq. (40), specifically,

$$k_e = \begin{cases} k_e^{\mathrm{I}}; & 0 < k_e \le k_{//} \\ k_e^{\mathrm{II}}; & k_e > k_{//} \end{cases}$$

(dual regime, laminar and turbulent) (61)

6 CONCLUSIONS

In this study we have presented the effective thermal conductivities of many common geometric shapes and these are catalogued in Tables 1 and 2. In addition to providing these correlations, we have attempted to present a complete methodology so that other Nusselt number correlations can be adapted for use in compact modeling. This will allow new geometries to be incorporated into compact modeling as well as new, more accurate correlations for existing shapes. Several direct solutions are provided for simpler Nusselt number correlations. It is, however, the asymptotic or approximate solutions that we believe will be the most beneficial. These should be more accurate in most cases than the direct solutions, and every attempt has been made to facilitate their incorporation into existing compact models.

In general the methods presented in this paper are applicable to all but the most complex correlation equations. Although the method used is fairly complex, the end result is simple enough that it may be directly used by a thermal engineer. These tools are hoped to allow for increased speed during critical phases of the design cycle. We would especially like to emphasize the general nature and the accuracy of approximate solutions. They are valid for all of the correlations proposed by Churchill, Yovanovich, and others. They are mostly within a few percent of the iterative solution which can require millions of clock cycles to compute. They also preclude negative roots and the need for initial guesses. If properly implemented, the methods presented here can lead to refined compact modeling capabilities appropriate of a variety of geometric configurations.

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