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This study reports a critical development in the widely used combustion stability algorithm used by propulsion industries as a predictive tool for the design of large combustors. It has been recently demonstrated that, by incorporating unsteady rotational sources and sinks in the acoustic energy assessment, a more precise formulation of the acoustic instability in rocket motors can be achieved. The new algorithm, when applied to the linear stability formulation, leads to ten growth rate terms. In this article, we convert these ten stability corrections from volumetric to surface integral form. We further convert them to an acoustic form that is directly amenable to implementation in the Standard Stability Prediction code. The reduction to surface form greatly facilitates the evaluation of individual stability growth rates as they become function of quantities distributed along the chamber's control surface. This will preclude the need to carry out a rotational flow analysis inside the motor. Only surface quantities will be needed and these will be converted to acoustic form whenever possible using the no slip condition or other applicable response functions. Effectively, all needed information will be obtainable directly from the acoustic field. By precluding the need to evaluate the rotational field (which can be highly uncertain in arbitrary geometry), the evaluation of acoustic stability integrals is made possible in practical motors with variable grain perforation. In this article the conversion process is carefully detailed. The analysis entails acquiring and applying several vortico-acoustic and vector identities, the most notable of which being the Gaussian divergence theorem.

Nomenclature

A_p	= unsteady pressure amplitude
$A_b^{(r)}$	= burning surface admittance
$A_S^{(r)}$	= inert surface admittance
$A_N^{(r)}$	= nozzle entrance plane admittance
a_0	= mean speed of sound
E	= time averaged unsteady system energy
E_m^2	= energy normalization function for mode m
$\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z$	= unit vectors in r, θ and z directions
k_m	= wave number for axial mode m
L, R	= enclosure length and radius, $l = L/R$
m	= oscillation mode shape number
M_b	= surface Mach number, V_b/a_0
$\boldsymbol{n}$	= outward pointing unit normal vector
p_0	= mean pressure
r, z, t	= radial, axial, and temporal coordinates
$\boldsymbol{u}$	= total velocity vector
U_r, U_z	= mean flow velocities normalized by V_b
x	= action coordinate, $\frac{1}{2}\pi r^2$
y	= radial distance from the wall, $1-r$
α	= growth rate (dimensional, $\sec^{-1}$)
δ	= viscous number, $[\nu/(a_0 R)]^{1/2}$
ε	= wave amplitude, $A_p/(\gamma p_0)$
$\phi(r)$	= function defined in Eq. (96)
γ	= ratio of specific heats
ν	= kinematic viscosity, μ/ρ
ρ	= density
ω, Ω	= unsteady and mean vorticity magnitudes
$\psi(r)$	= exponential argument defined in Eq. (95)

Subscripts

b	= refers to the burning/transpiring surface
i, r	= irrotational or rotational
m	= for a given mode number
N, S	= nozzle or inert surface

Superscripts

*	= dimensional quantity
~ , ^	= rotational or acoustical part
r, i	= part of a complex variable

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I. Introduction

BY incorporating unsteady rotational terms into the standard energy equation, a more complete, all-inclusive formulation of the progression of internal energy in rocket motors has been recently proposed by Flandro and Majdalani.[1,2] This energy assessment involves both rotational and irrotational contributions to pressure (p) and velocity (u). The new formulation comprises a total of ten volumetric integrals representing several acoustico-vortical mechanisms affecting stability. These integrals are classified and characterized according to their physical significance;[3,4] in summary, they may be expanded in a series of the form

$$\alpha_m = \alpha_1 + \alpha_2 + \alpha_3 + \cdots = \sum_{i=1}^{N} \alpha_i$$

In this article the conversion of all stability corrections from volumetric to surface integral form will be carried out carefully and systematically. This course of action will involve the acquisition of several applicable vector theorems and their application to the ten stability factors.

II. Analysis

A. Energy Assessment

The evolution of the system energy was determined by Flandro and Majdalani.[1,3] By revisiting the standard acoustic energy balance with both rotational and irrotational terms retained, one can write

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \iiint_V \Big\langle \overbrace{-\nabla\cdot(\hat{p}\hat{\boldsymbol{u}}) - \tfrac{1}{2}M_b\left(\boldsymbol{U}\cdot\nabla\hat{p}^2\right) - M_b\left[\hat{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\hat{\boldsymbol{u}}\right)\right]}^{\text{irrotational}}$$
$$+\tfrac{4}{3}\delta^2\hat{\boldsymbol{u}}\cdot\nabla\left(\nabla\cdot\hat{\boldsymbol{u}}\right) + M_b\left[\hat{\boldsymbol{u}}\cdot\left(\hat{\boldsymbol{u}}\times\boldsymbol{\Omega}\right) + \hat{\boldsymbol{u}}\cdot\left(\boldsymbol{U}\times\boldsymbol{\omega}\right)\right]$$
$$\overbrace{-\tilde{\boldsymbol{u}}\cdot\nabla\hat{p} - M_b\begin{bmatrix}\tilde{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\hat{\boldsymbol{u}}\right) + \hat{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}\right) + \tilde{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}}\right) \\ -\tilde{\boldsymbol{u}}\cdot\left(\boldsymbol{U}\times\boldsymbol{\omega}\right) - \tilde{\boldsymbol{u}}\cdot\left(\tilde{\boldsymbol{u}}\times\boldsymbol{\Omega}\right)\end{bmatrix}}^{\text{rotational}}$$
$$+\tfrac{4}{3}\delta^2\tilde{\boldsymbol{u}}\cdot\nabla\left(\nabla\cdot\hat{\boldsymbol{u}}\right) - \delta^2\left[\hat{\boldsymbol{u}}\cdot\left(\nabla\times\boldsymbol{\omega}\right) + \tilde{\boldsymbol{u}}\cdot\left(\nabla\times\boldsymbol{\omega}\right)\right]$$
$$-\hat{\boldsymbol{u}}\cdot\nabla\tilde{p} - \tilde{\boldsymbol{u}}\cdot\nabla\tilde{p}\Big\rangle \mathrm{d}V \qquad (1)$$

The linear stability integrals to be converted will often refer to Eq. (1).

B. First Factor: Extended Pressure Coupling

The first correction factor combines the first three irrotational integrals representing pressure coupling and nozzle damping due to the acoustic energy carried out by the mean flow. The corresponding energy growth rate is expressible by

$$\alpha_1 = \frac{1}{\exp\left(2\alpha_m t\right)E_m^2}\iiint_V \Big\langle -\nabla\cdot\left[\hat{p}\hat{\boldsymbol{u}} + \tfrac{1}{2}M_b\boldsymbol{U}\left(\hat{p}\right)^2\right]$$
$$-M_b\left[\hat{\boldsymbol{u}}\cdot\nabla\left(\boldsymbol{U}\cdot\hat{\boldsymbol{u}}\right)\right]\Big\rangle \mathrm{d}V \qquad (2)$$

The first quantity between brackets is amenable to surface extraction using Gauss's theorem, specifically,

$$\iiint_V \nabla\cdot\boldsymbol{D}\,\mathrm{d}V = \iint_S \boldsymbol{D}\cdot\boldsymbol{n}\,\mathrm{d}S \qquad (3)$$

Subsequently, it is possible to transform the triple integral into a simpler double integral. At the outset Eq. (2) becomes

$$\alpha_1 = \underbrace{-\frac{1}{\exp(2\alpha_m t)E_m^2}\iint_S \Big\langle \boldsymbol{n}\cdot\left[\hat{p}\hat{\boldsymbol{u}} + \tfrac{1}{2}M_b\boldsymbol{U}\hat{p}^2\right]\Big\rangle \mathrm{d}S}_{\mathrm{I}}$$
$$\underbrace{+\frac{1}{\exp(2\alpha_m t)E_m^2}\iiint_V \Big\langle -M_b\left[\hat{\boldsymbol{u}}\cdot\nabla(\boldsymbol{U}\cdot\hat{\boldsymbol{u}})\right]\Big\rangle \mathrm{d}V}_{\mathrm{II}} \qquad (4)$$

As demonstrated by Flandro and Majdalani,[1,2] the next step is for vector projections to be carefully implemented along different sections where pressure coupling is manifested. These sections include the control surfaces delineating the idealized rocket motor chamber. Along burning surfaces, one must have

$$\boldsymbol{n}\cdot\hat{\boldsymbol{u}} = -M_b A_b^{(r)}\hat{p}\ ,\ \boldsymbol{n}\cdot\boldsymbol{U} = -1 \qquad (5)$$

Similarly, along the inert surface, one has

$$\boldsymbol{n}\cdot\hat{\boldsymbol{u}} = -M_b A_S^{(r)}\hat{p}\ ,\ \boldsymbol{n}\cdot\boldsymbol{U} = 0 \qquad (6)$$

and so, along the nozzle entrance plane

$$\boldsymbol{n}\cdot\hat{\boldsymbol{u}} = M_b A_N^{(r)}\hat{p}\ ,\ \boldsymbol{n}\cdot\boldsymbol{U} = U_N \qquad (7)$$

where U_N is the mean axial velocity crossing the nozzle entrance plane at $z = l$.

Assuming that $A_S^{(r)}$ is small compared to other terms, Eqs. (5)–(7) may be substituted back into Eq. (4). The first integral becomes

$$\mathrm{I} = \frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iint_S \Big\langle -M_b A_b^{(r)}\hat{p}^2 + M_b A_N^{(r)}\hat{p}^2$$
$$+\tfrac{1}{2}M_b\hat{p}^2\left(-1+U_N\right)\Big\rangle \mathrm{d}S \qquad (8)$$

Grouping and rearranging, Eq. (8) simplifies into the general surface integral

$$\mathrm{I} = \frac{-M_b E_m^{-2}}{\exp(2\alpha_m t)}\iint_S \Big\langle \hat{p}^2\left[-A_b^{(r)} - \tfrac{1}{2}\right]$$
$$+\hat{p}^2\left[A_N^{(r)} + \tfrac{1}{2}U_N\right]\Big\rangle \mathrm{d}S \qquad (9)$$

where $\hat{p}^2$ is defined from[3]

$$\hat{p} = \hat{p}_m \exp\left(\alpha_m t\right)\cos\left(k_m t\right) \qquad (10)$$

$$\hat{p}_m = \cos(k_m z) \quad (11)$$

At this juncture, Flandro and Majdalani[1,2] insert the value of $\hat{p}^2$ and carry out the time averaging; this leads to

$$\mathrm{I} = \tfrac{1}{2} M_b E_m^{-2} \left(\iint_{S_b} \left\{ \cos^2(k_m z) \left[A_b^{(r)} + \tfrac{1}{2} \right] \right\} \mathrm{d}S \right.$$

$$\left. - \iint_{S_N} \left\{ \cos^2(k_m z) \left[A_N^{(r)} + \tfrac{1}{2} U_N \right] \right\} \mathrm{d}S \right) \quad (12)$$

where k_m represents the dimensionless wave number which, for closed-end boundaries, is given by[5]

$$k_m = m\pi R / L = m\pi / l \quad (13)$$

As usual, m is the mode shape number and $l \equiv L/R$ is the aspect ratio for the motor.

In much the same way, the second integral of Eq. (4) can be converted. Starting with

$$\mathrm{II} = \frac{-E_m^{-2}}{\exp(2\alpha_m t)} \iiint_V \left\langle M_b \left[\hat{\boldsymbol{u}} \cdot \nabla(\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}V \quad (14)$$

one may take advantage of the well known vector identity $\boldsymbol{A} \cdot \nabla f = \nabla \cdot (f\boldsymbol{A}) - f \nabla \cdot \boldsymbol{A}$ (see Eq. (A6)) to expand the integrand into

$$\hat{\boldsymbol{u}} \cdot [\nabla(\boldsymbol{U} \cdot \hat{\boldsymbol{u}})] = \nabla \cdot [(\boldsymbol{U} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] - (\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \quad (15)$$

Equation (14) can now be represented as

$$\mathrm{II} = \frac{-M_b E_m^{-2}}{\exp(2\alpha_m t)} \iiint_V \left\langle \nabla \cdot [(\boldsymbol{U} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] - (\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \right\rangle \mathrm{d}V \quad (16)$$

By using the divergence theorem, the first term of the volumetric integral is converted into surface form viz.

$$\iiint_V \left\langle \nabla \cdot [(\boldsymbol{U} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] \right\rangle \mathrm{d}V = \iint_S \left\langle \boldsymbol{n} \cdot [(\boldsymbol{U} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}}] \right\rangle \mathrm{d}S \quad (17)$$

as shown in Eqs. (5)–(7), the normal projections of $\hat{\boldsymbol{u}}$ are $O(M_b)$, making the surface integral of $O(M_b^2)$. Equation (16) becomes

$$\mathrm{II} = \frac{M_b E_m^{-2}}{\exp(2\alpha_m t)} \iiint_V \left\langle (\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \right\rangle \mathrm{d}V \quad (18)$$

At this point time averaging can be applied to obtain

$$\mathrm{II} = \tfrac{1}{2} M_b E_m^{-2} \iiint_V (\boldsymbol{U} \cdot \hat{\boldsymbol{u}}_m) \nabla \cdot \hat{\boldsymbol{u}}_m \, \mathrm{d}V \quad (19)$$

where

$$\hat{\boldsymbol{u}}_m = -\nabla \hat{p}_m / k_m \quad (20)$$

$$\nabla \cdot \hat{\boldsymbol{u}}_m = -\nabla \cdot \nabla \hat{p}_m / k_m = k_m \hat{p}_m \quad (21)$$

Subsequently the integrand becomes

$$(\boldsymbol{U} \cdot \hat{\boldsymbol{u}}_m) \nabla \cdot \hat{\boldsymbol{u}}_m = -(\boldsymbol{U} \cdot \nabla \hat{p}_m) \hat{p}_m \quad (22)$$

The vector identity in Eq. (A6) may now be invoked alongside

$$\begin{cases} \nabla \cdot (\boldsymbol{U}\hat{p}_m) = \boldsymbol{U} \cdot \nabla \hat{p}_m + \hat{p}_m \nabla \cdot \boldsymbol{U} \\ \nabla \cdot \boldsymbol{U} = 0 \\ \nabla \cdot (\boldsymbol{U}\hat{p}_m) = \boldsymbol{U} \cdot \nabla \hat{p}_m \end{cases} \quad (23)$$

to transform Eq. (22) into

$$-(\boldsymbol{U} \cdot \nabla \hat{p}_m) \hat{p}_m = -(\nabla \cdot \boldsymbol{U}\hat{p}_m) \hat{p}_m = -\tfrac{1}{2}\left(\nabla \cdot \boldsymbol{U}\hat{p}_m^2\right) \quad (24)$$

Equation (24) may, in turn, be substituted into Eq. (19) to render

$$\mathrm{II} = -\tfrac{1}{4} M_b E_m^{-2} \iiint_V \left(\nabla \cdot \boldsymbol{U}\hat{p}_m^2\right) \mathrm{d}V \quad (25)$$

The form obtained is now suitable for transformation by way of the divergence theorem; the result is

$$\mathrm{II} = -\tfrac{1}{4} M_b E_m^{-2} \iint_S \left(\boldsymbol{n} \cdot \boldsymbol{U}\hat{p}_m^2\right) \mathrm{d}S \quad (26)$$

For the case of a cylindrical motor

$$\nabla \cdot \hat{\boldsymbol{u}} = \frac{\partial \hat{u}_z}{\partial z} = k_m \cos(k_m z) \exp(\alpha_m t) \sin(k_m t) \quad (27)$$

and

$$(\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) = U_z \sin(k_m z) \exp(\alpha_m t) \sin(k_m t) \quad (28)$$

so that

$$(\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} = U_z k_m \cos(k_m z) \sin(k_m z)$$

$$\times \exp(2\alpha_m t) \sin^2(k_m t) \quad (29)$$

The time average of Eq. (29) yields

$$\left\langle (\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \right\rangle = U_z k_m \cos(k_m z) \sin(k_m z) \quad (30)$$

At this point, one recalls from Flandro and Majdalani[1,3] that

$$\hat{p}_m = \cos(k_m z) \quad (31)$$

and

$$\nabla \cdot \left(\boldsymbol{U}\hat{p}_m^2\right) = \frac{\partial U_r}{\partial r} \hat{p}_m^2 + \frac{\partial U_z}{\partial z} \hat{p}_m^2 + \frac{\partial \hat{p}_m^2}{\partial r} U_r + \frac{\partial \hat{p}_m^2}{\partial z} U_z \quad (32)$$

Clearly the first two terms cancel because the divergence of the mean flow is equal to zero and $\hat{p}_m \neq \hat{p}_m(r)$; specifically

$$\nabla \cdot \left(\boldsymbol{U}\hat{p}_m^2\right) = 2\boldsymbol{U}\hat{p}_m \frac{\partial \hat{p}_m}{\partial z} \quad (33)$$

hence

$$\tfrac{1}{2} \nabla \cdot \left(\boldsymbol{U}\hat{p}_m^2\right) = U_z k_m \cos(k_m z) \sin(k_m z)$$

$$= \left\langle (\boldsymbol{U} \cdot \hat{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \right\rangle \quad (34)$$

so that

$$\mathrm{II} = -\tfrac{1}{2} E_m^{-2} M_b \iiint_V \nabla \cdot \left(\boldsymbol{U}\, \hat{p}_m^2 / 2 \right) \mathrm{d}V \qquad (35)$$

Using Gauss's theorem, the volumetric integral can be transformed into a surface integral

$$\mathrm{II} = -\tfrac{1}{2} M_b E_m^{-2} \iint_S \left(\tfrac{1}{2} \boldsymbol{n} \cdot \boldsymbol{U} \hat{p}_m^2 \right) \mathrm{d}S$$

$$= \tfrac{1}{2} M_b E_m^{-2} \left(\iint_{S_b} \tfrac{1}{2} \hat{p}_m^2 \mathrm{d}S - \iint_{S_N} \tfrac{1}{2} U_N \hat{p}_m^2 \mathrm{d}S \right) \qquad (36)$$

Combining Eqs. (36) and (9), Eq. (2) becomes

$$\alpha_1 = \tfrac{1}{4} M_b E_m^{-2} \left\{ \iint_{S_b} \cos^2(k_m z)\,\mathrm{d}S - \iint_{S_N} U_N \cos^2(k_m z)\,\mathrm{d}S \right\}$$

$$+ \tfrac{1}{2} M_b E_m^{-2} \left(\iint_{S_b} \left\{ \cos^2(k_m z) \left[A_b^{(r)} + \tfrac{1}{2} \right] \right\} \mathrm{d}S \right.$$

$$\left. + \iint_{S_N} \left\{ \cos^2(k_m z) \left[-A_N^{(r)} - \tfrac{1}{2} U_N \right] \right\} \mathrm{d}S \right) \qquad (37)$$

Collecting similar integrals, one obtains, at length,

$$\alpha_1 = \tfrac{1}{2} M_b E_m^{-2} \iint_{S_b} \cos^2(k_m z) \left[A_b^{(r)} + 1 \right] \mathrm{d}S$$

$$- \tfrac{1}{2} M_b E_m^{-2} \iint_{S_N} \cos^2(k_m z) \left[A_N^{(r)} + U_N \right] \mathrm{d}S \qquad (38)$$

In more general form, this can be expressed as

$$\alpha_1 = \tfrac{1}{2} M_b E_m^{-2} \iint_{S_b} \hat{p}_m^2 \left[A_b^{(r)} + 1 \right] \mathrm{d}S$$

$$- \tfrac{1}{2} M_b E_m^{-2} \iint_{S_N} \hat{p}_m^2 \left[A_N^{(r)} + U_N \right] \mathrm{d}S \qquad (39)$$

C. Second Factor: Dilatational Energy Correction

The dilatational energy term is the fourth of the irrotational terms. It has been proven in previous studies that this term is $O(M_b^3)$ so it may be ignored in the current analysis. For further confirmation,

$$\alpha_2 = \frac{E_m^{-2}}{\exp(2\alpha_m t)} \iiint_V \left\langle \tfrac{4}{3} \delta^2 \hat{\boldsymbol{u}} \cdot \nabla (\nabla \cdot \hat{\boldsymbol{u}}) \right\rangle \mathrm{d}V \qquad (40)$$

one may apply time averaging to get

$$\alpha_2 = \tfrac{2}{3} E_m^{-2} \iiint_V \delta^2 \hat{\boldsymbol{u}}_m \cdot \nabla (\nabla \cdot \hat{\boldsymbol{u}}_m) \mathrm{d}V \qquad (41)$$

Subsequently, one may employ $\nabla \cdot \hat{\boldsymbol{u}}_m = k_m \hat{p}_m$ and $\nabla \hat{p}_m = -k_m \hat{\boldsymbol{u}}_m$ to write

$$\alpha_2 = \tfrac{2}{3} E_m^{-2} \iiint_V \delta^2 \left(-\nabla \hat{p}_m / k_m \right) \cdot \nabla \left(k_m \hat{p}_m \right) \mathrm{d}V$$

$$= -\tfrac{2}{3} \delta^2 E_m^{-2} \iiint_V \left(\nabla \hat{p}_m \cdot \nabla \hat{p}_m \right) \mathrm{d}V \qquad (42)$$

Additionally, one may use the vector identity

$$\nabla \cdot \left(\hat{p}_m \nabla \hat{p}_m \right) = \nabla \hat{p}_m \cdot \nabla \hat{p}_m + \hat{p}_m \nabla^2 \hat{p}_m$$

$$= \nabla \hat{p}_m \cdot \nabla \hat{p}_m - k_m^2 \hat{p}_m^2 \qquad (43)$$

to solve for

$$\nabla \hat{p}_m \cdot \nabla \hat{p}_m = \nabla \cdot \left(\hat{p}_m \nabla \hat{p}_m \right) + k_m^2 \hat{p}_m^2 \qquad (44)$$

This transforms Eq. (42) into

$$\alpha_2 = -\tfrac{2}{3} \delta^2 E_m^{-2} \iiint_V \left[\nabla \cdot \left(\hat{p}_m \nabla \hat{p}_m \right) + k_m^2 \hat{p}_m^2 \right] \mathrm{d}V \qquad (45)$$

In SSP where acoustic modes are assumed to dominate, one may use the one-dimensional approximation of $E_m^2 = \tfrac{1}{2} \iiint_V \hat{p}_m^2 \, \mathrm{d}V$. At the outset, Eq. (45) becomes

$$\alpha_2 = -\tfrac{2}{3} \delta^2 E_m^{-2} \left[\iiint_V \nabla \cdot \left(\hat{p}_m \nabla \hat{p}_m \right) \mathrm{d}V + k_m^2 E_m^2 \right] \qquad (46)$$

The divergence theorem can now be applied to produce

$$\alpha_2 = -\tfrac{2}{3} \delta^2 E_m^{-2} \iint_S \boldsymbol{n} \cdot \left(\hat{p}_m \nabla \hat{p}_m \right) \mathrm{d}S - \tfrac{4}{3} \delta^2 k_m^2$$

$$= \tfrac{2}{3} \delta^2 k_m E_m^{-2} \iint_S \boldsymbol{n} \cdot \left(\hat{p}_m \hat{\boldsymbol{u}}_m \right) \mathrm{d}S - \tfrac{4}{3} \delta^2 k_m^2 \qquad (47)$$

Finally, evaluating the normal projection returns

$$\alpha_2 = -\tfrac{2}{3} \delta^2 k_m E_m^{-2} \left[\iint_{S_b} M_b A_b^{(r)} \hat{p}_m^2 \mathrm{d}S - \iint_{S_N} M_b A_N^{(r)} \hat{p}_m^2 \mathrm{d}S \right]$$

$$- \tfrac{4}{3} \delta^2 k_m^2 \qquad (48)$$

D. Third Factor: Acoustic Mean Flow Correction

It has been shown by Flandro and Majdalani[1,3] that the acoustic mean flow correction vanishes for the full-length circular-port motor whose internal flowfield can be adequately approximated by Culick's profile. Starting with

$$\alpha_3 = \frac{1}{E_m^2 \exp(2\alpha_m t)} \iiint_V \left\langle M_b \left\{ \hat{\boldsymbol{u}} \cdot \left(\hat{\boldsymbol{u}} \times \boldsymbol{\Omega} \right) \right\} \right\rangle \mathrm{d}V \qquad (49)$$

One may use Culick's profile to express the mean flow vorticity, $\boldsymbol{\Omega}$, as

$$\boldsymbol{\Omega} = \Omega \hat{\boldsymbol{e}}_\theta = \pi^2 r z \sin(x) \boldsymbol{e}_\theta \qquad (50)$$

thus giving

$$\hat{\boldsymbol{u}} \times \boldsymbol{\Omega} = -\Omega \hat{u}_z \boldsymbol{e}_r + \Omega \hat{u}_r \boldsymbol{e}_z \qquad (51)$$

and, hence

$$\hat{\boldsymbol{u}} \cdot \left(\hat{\boldsymbol{u}} \times \boldsymbol{\Omega} \right) = \left(\hat{u}_r \boldsymbol{e}_r + \hat{u}_z \boldsymbol{e}_z \right) \cdot \left(-\Omega \hat{u}_z \boldsymbol{e}_r + \Omega \hat{u}_r \boldsymbol{e}_z \right)$$

$$= -\hat{u}_r \Omega \hat{u}_z + \hat{u}_z \Omega \hat{u}_r = 0 \qquad (52)$$

One can see from Eq. (49) that α_3 vanishes for Culick's profile. It can be easily shown that this result is generally true and will vanish for any flow profile because $\hat{\boldsymbol{u}}\times\boldsymbol{\Omega}$ is always perpendicular to $\hat{\boldsymbol{u}}$, $\forall\boldsymbol{\Omega}$.

E. Fourth Factor: Flow Turning Correction

The fourth factor is a function of unsteady vorticity. Nonetheless, this term has often been dubbed 'the flow turning correction' in the standard stability formulation. Starting with

$$\alpha_4 = E_m^{-2} e^{-2\alpha_m t} \iiint_V \left\langle M_b \hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\right\rangle \mathrm{d}V \qquad (53)$$

The integrand may be expanded by recognizing that the vorticity is a function of the unsteady rotational velocity, $\boldsymbol{\omega} = \nabla\times\tilde{\boldsymbol{u}}$. Thus

$$\hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega}) = \hat{\boldsymbol{u}}\cdot\left[\boldsymbol{U}\times(\nabla\times\tilde{\boldsymbol{u}})\right] \qquad (54)$$

where

$$\nabla\times\tilde{\boldsymbol{u}} = \left(\frac{1}{r}\frac{\partial\tilde{u}_z}{\partial\theta} - \frac{\partial\tilde{u}_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial\tilde{u}_\theta}{\partial r} - \frac{\partial\tilde{u}_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z \qquad (55)$$

The first and third terms on the left-hand side of Eq. (55) vanish due to the unsteady rotational velocity being independent of θ (i.e. axisymmetric) and comprising no $\theta-$ component; this leaves

$$\nabla\times\tilde{\boldsymbol{u}} = \left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta \qquad (56)$$

Recalling from Flandro and Majdalani[1] that $\partial\tilde{u}_r/\partial z = O\left(M_b^2\right)$, Eq. (56) becomes

$$\nabla\times\tilde{\boldsymbol{u}} = -\frac{\partial\tilde{u}_z}{\partial r}\hat{\boldsymbol{e}}_\theta \qquad (57)$$

Consequently, Eq. (54) collapses into

$$\hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega}) = \hat{\boldsymbol{u}}\cdot\left[\boldsymbol{U}\times\left(-\frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta\right] \qquad (58)$$

which can be further expanded as

$$\hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega}) = \hat{\boldsymbol{u}}\cdot\left[\left(U_z\frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_r + \left(-U_r\frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_z\right] \qquad (59)$$

Taking advantage of the fact that the acoustic velocity does not possess radial or tangential components, Eq. (59) simplifies to

$$\hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega}) = -\hat{u}_z U_r\frac{\partial\tilde{u}_z}{\partial r} \qquad (60)$$

At this juncture it would be advantageous to insert the values of $\tilde{u}_z$, U_r and $\hat{u}_z$; one gets

$$\hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega}) = \sin(k_m z)e^{2\alpha_m t}\sin(k_m t)r^{-1}\sin(x) \times\left[\frac{\partial\tilde{u}_m^r}{\partial r}\cos(k_m t) + \frac{\partial\tilde{u}_m^i}{\partial r}\sin(k_m t)\right] \qquad (61)$$

In order to simplify Eq. (61), the time averaging must be performed. At the outset, one reaps

$$\left\langle\hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\right\rangle = \tfrac{1}{2}r^{-1}\sin(k_m z)\sin(x)e^{2\alpha_m t}\frac{\partial\tilde{u}_m^i}{\partial r} \qquad (62)$$

making α_4

$$\alpha_4 = \tfrac{1}{2}M_b E_m^{-2}k_m^{-1}\iiint_V r^{-1}\sin(x)k_m\sin(k_m z)\frac{\partial\tilde{u}_m^i}{\partial r}\mathrm{d}V \qquad (63)$$

Recalling from Flandro and Majdalani[3] that $k_m\sin(k_m z) = -\nabla\hat{p}_m$ for a full-length circular-port motor, Eq. (63) becomes

$$\alpha_4 = -\tfrac{1}{2}M_b E_m^{-2}k_m^{-1}\iiint_V r^{-1}\sin(x)\frac{\partial\tilde{u}_m^i}{\partial r}\nabla\hat{p}_m\mathrm{d}V \qquad (64)$$

The integrand can hence be represented by

$$r^{-1}\sin(x)\frac{\partial\tilde{u}_m^i}{\partial r}\nabla\hat{p}_m \approx \frac{\partial}{\partial r}\left(-U_r\tilde{\boldsymbol{u}}_m^i\cdot\nabla\hat{p}_m\right) \qquad (65)$$

This approximation is valid due to the unsteady vorticity being $O(M_b^{-1})$,[6] hence dominating over adjacent terms. The radial part of the above integral has been shown to be entirely determined by the upper limit at $r=1$. The volumetric integral, noting that $U_r(1) = -1$, can now be replaced by a surface integral of the form

$$\alpha_4 = -\tfrac{1}{2}M_b E_m^{-2}k_m^{-1}\iint_{S_b}\tilde{\boldsymbol{u}}_m^i\cdot\nabla\hat{p}_m\mathrm{d}S \qquad (66)$$

or, equivalently, using $\hat{\boldsymbol{u}}_{\boldsymbol{m}} = -\nabla p_m/k_m$

$$\alpha_4 = \tfrac{1}{2}M_b E_m^{-2}\iint_{S_b}\tilde{\boldsymbol{u}}_m^i\cdot\hat{\boldsymbol{u}}_m\mathrm{d}S \qquad (67)$$

The non-time averaged form of Eq. (67) can be easily seen to be

$$\alpha_4 = \exp(-2\alpha_m t)M_b E_m^{-2}\iint_{S_b}\left\langle\tilde{\boldsymbol{u}}^i\cdot\hat{\boldsymbol{u}}\right\rangle\mathrm{d}S \qquad (68)$$

Note that this integral is only defined over the burning surface: it is not to be evaluated over inert sections or the nozzle entrance region.

F. Fifth Factor: Rotational Flow Correction

The rotational flow correction is the first of the new terms introduced by Flandro and Majdalani.[1] This correction factor arises when retaining the important unsteady rotational terms. From Eq. (1), the first of the rotational integrals gives

$$\alpha_5 = \frac{-1}{E_m^2\exp(2\alpha_m t)}\iiint_V\left\langle\tilde{\boldsymbol{u}}\cdot\nabla\hat{p}\right\rangle\mathrm{d}V \qquad (69)$$

where, by use of vector identities

$$\begin{cases} \nabla\cdot(\tilde{\boldsymbol{u}}\hat{p}) = \tilde{\boldsymbol{u}}\cdot\nabla\hat{p} + \hat{p}\nabla\cdot\tilde{\boldsymbol{u}} \\ \nabla\cdot\tilde{\boldsymbol{u}} = 0; \quad \nabla\cdot(\tilde{\boldsymbol{u}}\hat{p}) = \tilde{\boldsymbol{u}}\cdot\nabla\hat{p} \end{cases} \tag{70}$$

transforms into

$$\alpha_5 = \frac{-1}{E_m^2 \exp(2\alpha_m t)} \iiint_V \left\langle \nabla\cdot(\tilde{\boldsymbol{u}}\hat{p}) \right\rangle \mathrm{d}V \tag{71}$$

α_5 can be readily converted to surface form by direct application of the divergence theorem; one gets

$$\alpha_5 = \frac{-1}{E_m^2 \exp(2\alpha_m t)} \iint_S \left\langle \boldsymbol{n}\cdot(\tilde{\boldsymbol{u}}\hat{p}) \right\rangle \mathrm{d}S \tag{72}$$

Time averaging can be subsequently applied to produce

$$\alpha_5 = \frac{-1}{2E_m^2} \iint_S \boldsymbol{n}\cdot(\tilde{\boldsymbol{u}}_m \hat{p}_m)\,\mathrm{d}S \tag{73}$$

which, given that at the burning surface,

$$\boldsymbol{n}\cdot\tilde{\boldsymbol{u}} = \tilde{u}_r\ (r=1) = -M_b\hat{p} \tag{74}$$

Equation (73) collapses into

$$\alpha_5 = \tfrac{1}{2}E_m^{-2}\iint_{S_b} M_b \hat{p}_m^2\,\mathrm{d}S \tag{75}$$

G. Sixth Factor: Mean Vortical Correction

The next rotational term of Eq. (1) can be written as

$$\alpha_6 = \frac{1}{E_m^2 \exp(2\alpha_m t)} \iiint_V \left\langle M_b \tilde{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\right\rangle \mathrm{d}V \tag{76}$$

This can be further simplified using $\boldsymbol{\omega} = \nabla\times\tilde{\boldsymbol{u}}$, namely,

$$\alpha_6 = \frac{M_b}{E_m^2 \exp(2\alpha_m t)} \iiint_V \left\langle \tilde{\boldsymbol{u}}\cdot\left[\boldsymbol{U}\times(\nabla\times\tilde{\boldsymbol{u}})\right]\right\rangle \mathrm{d}V \tag{77}$$

Upon expansion, one finds

$$\nabla\times\tilde{\boldsymbol{u}} = \left(\frac{1}{r}\frac{\partial\tilde{u}_z}{\partial\theta} - \frac{\partial\tilde{u}_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial\tilde{u}_\theta}{\partial r} - \frac{\partial\tilde{u}_r}{\partial\theta}\right)\hat{\boldsymbol{e}}_z \tag{78}$$

The first and third terms on the left-hand side of Eq. (78) vanish due to the fact that the unsteady rotational velocity does not have a $\theta-$component, nor is it a function of θ; this leaves us with

$$\nabla\times\tilde{\boldsymbol{u}} = \left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta \tag{79}$$

so that

$$\boldsymbol{U}\times(\nabla\times\tilde{\boldsymbol{u}}) = -U_z\left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_r + U_r\left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right)\hat{\boldsymbol{e}}_z \tag{80}$$

and

$$\tilde{\boldsymbol{u}}\cdot\left[\boldsymbol{U}\times(\nabla\times\tilde{\boldsymbol{u}})\right] = -\tilde{u}_r U_z\left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right) + \tilde{u}_z U_r\left(\frac{\partial\tilde{u}_r}{\partial z} - \frac{\partial\tilde{u}_z}{\partial r}\right) \tag{81}$$

Recalling from Flandro and Majdalani[3] that $\partial\tilde{u}_r/\partial z = O(M_b^2)$, Eq. (80) becomes

$$\tilde{\boldsymbol{u}}\cdot\left[\boldsymbol{U}\times(\nabla\times\tilde{\boldsymbol{u}})\right] = \tilde{u}_r U_z\frac{\partial\tilde{u}_z}{\partial r} - \tilde{u}_z U_r\frac{\partial\tilde{u}_z}{\partial r} \tag{82}$$

The first term on the right-hand side of Eq. (82) can be shown to be negligible: considering that $\tilde{u}_r = O(M_b)$ this term can be combined with the M_b on the outside of the volumetric integral to make it $O(M_b^2)$. Consequently, Eq. (82) becomes

$$\tilde{\boldsymbol{u}}\cdot\left[\boldsymbol{U}\times(\nabla\times\tilde{\boldsymbol{u}})\right] = -\tilde{u}_z U_r\frac{\partial\tilde{u}_z}{\partial r} \tag{83}$$

Next, we shift our attention to the term

$$\nabla\cdot\boldsymbol{U}\left(\tfrac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right) = \nabla\cdot\boldsymbol{U}\,\tfrac{1}{2}\left(\tilde{u}_r\tilde{u}_r + \tilde{u}_z\tilde{u}_z\right) \tag{84}$$

This can be expanded as

$$\nabla\cdot\boldsymbol{U}\,\tfrac{1}{2}\left(\tilde{u}_r\tilde{u}_r + \tilde{u}_z\tilde{u}_z\right) = \tfrac{1}{2}\nabla\cdot[U_r\left(\tilde{u}_r\tilde{u}_r + \tilde{u}_z\tilde{u}_z\right)\boldsymbol{e}_r + U_z\left(\tilde{u}_r\tilde{u}_r + \tilde{u}_z\tilde{u}_z\right)\boldsymbol{e}_z] \tag{85}$$

then manipulated using the divergence theorem

$$\begin{aligned} &\tfrac{1}{2}\nabla\cdot\left[U_r\left(\tilde{u}_r\tilde{u}_r + \tilde{u}_z\tilde{u}_z\right)\boldsymbol{e}_r + U_z\left(\tilde{u}_r\tilde{u}_r + \tilde{u}_z\tilde{u}_z\right)\boldsymbol{e}_z\right] \\ &= \tfrac{1}{2r}U_r\tilde{u}_r^2 + \frac{\partial U_r}{\partial r}\frac{\tilde{u}_r^2}{2} + U_r\tilde{u}_r\frac{\partial\tilde{u}_r}{\partial r} + \tfrac{1}{2r}U_r\tilde{u}_z^2 \\ &+ \tfrac{1}{2}\frac{\partial U_r}{\partial r}\tilde{u}_z^2 + U_r\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial r} + \tfrac{1}{2}\frac{\partial U_z}{\partial z}\tilde{u}_r^2 \\ &+ U_z\tilde{u}_r\frac{\partial\tilde{u}_r}{\partial z} + \tfrac{1}{2}\frac{\partial U_z}{\partial z}\tilde{u}_z^2 + U_z\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial z} \end{aligned} \tag{86}$$

Recalling that $\nabla\cdot\boldsymbol{U} = 0$, it is straightforward to show that six terms on the right-hand side of Eq. (86) will readily cancel. Equation (84) becomes

$$\nabla\cdot\boldsymbol{U}\left(\tfrac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right) = U_r\tilde{u}_r\frac{\partial\tilde{u}_r}{\partial r} + U_z\tilde{u}_r\frac{\partial\tilde{u}_r}{\partial z} + U_z\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial z} + U_r\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial r} \tag{87}$$

Then owing to $\partial\tilde{u}_r/\partial z = O(M_b^2)$, Eq. (87) reduces to

$$\nabla\cdot\boldsymbol{U}\left(\tfrac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right) = U_r\tilde{u}_r\frac{\partial\tilde{u}_r}{\partial r} + U_z\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial z} + U_r\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial r} \tag{88}$$

The first term on the right-hand side of Eq. (88) is also small because $\tilde{u}_r = O(M_b)$; one is left with

$$\nabla\cdot\boldsymbol{U}\left(\tfrac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right) = U_z\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial z} + U_r\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial r} \qquad (89)$$

The integral of the first term on the right-hand side of Eq. (89) yields

$$\iiint_V\left\langle U_z\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial z}\right\rangle\mathrm{d}V = \iiint_V\left\langle 2\pi z\cos(x)e^{2\alpha_m t}\left[\tilde{u}_m^r\cos(k_m t) + \tilde{u}_m^i\sin(k_m t)\right]\right.$$
$$\left.\times\left[\frac{\partial\tilde{u}_m^r}{\partial z}\cos(k_m t) + \frac{\partial\tilde{u}_m^i}{\partial z}\sin(k_m t)\right]\right\rangle\mathrm{d}V \qquad (90)$$

where $\tilde{u}_m^r$ and $\tilde{u}_m^i$ are defined from[3]

$$\tilde{u}_m^r = \sin(x)\exp(\phi)\sin(\psi)\sin\left[\sin(x)k_m z\right]\boldsymbol{e}_z \qquad (91)$$

$$\tilde{u}_m^i = -\sin(x)\exp(\phi)\cos(\psi)\sin\left[\sin(x)k_m z\right]\boldsymbol{e}_z \qquad (92)$$

Upon time averaging, Eq. (90) becomes

$$\iiint_V\left\langle U_z\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial z}\right\rangle\mathrm{d}V = \mathrm{I} = \iiint_V 2\pi\cos(x)e^{2\alpha_m t}\left[\tilde{u}_m^r\frac{\partial\tilde{u}_m^r}{\partial z} + \tilde{u}_m^i\frac{\partial\tilde{u}_m^i}{\partial z}\right]\mathrm{d}V \qquad (93)$$

Substituting the expressions for $\tilde{u}_m^r$ and $\tilde{u}_m^i$ renders

$$\mathrm{I} = 2k_m\pi e^{2\alpha_m t}\iiint_V z\cos(x)\sin^3(x)e^{2\phi}\sin\left[\sin(x)k_m z\right]\times\left\{\begin{array}{l}\sin^2(\psi)\cos\left[\sin(x)k_m z\right]\\ +\cos^2(\psi)\cos\left[\sin(x)k_m z\right]\end{array}\right\}\mathrm{d}V \qquad (94)$$

As shown in previous studies, the phase angle

$$\psi(r) = -[k_m/(\pi M_b)]\ln\tan\left(\tfrac{1}{2}x\right) \qquad (95)$$

controls the wavelength and spatial frequency of rotational shear waves. The real argument, $\phi(r)$, is responsible for viscous damping. It is found to be[7]

$$\phi(r) = \frac{\xi}{\pi^2}\left[1 - \frac{1}{\sin(x)} - x\frac{\cos(x)}{\sin^2(x)} + I(x) - I\left(\tfrac{1}{2}\pi\right)\right] \qquad (96)$$

$$I(x) = x + \tfrac{1}{18}x^3 + \tfrac{7}{1800}x^5 + \tfrac{31}{105840}x^7 + \ldots \qquad (97)$$

where S and ξ are dimensionless scaling factors

$$S \equiv \frac{k_m}{M_b} = \frac{m}{l}\frac{\pi}{M_b}, \quad \xi \equiv \frac{k_m^2\delta^2}{M_b^3} = \frac{S^2\delta^2}{M_b} = \frac{m^2}{l^2}\frac{\pi^2\delta^2}{M_b^3} \qquad (98)$$

Using the well known trigonometric identity $\cos^2(\psi) + \sin^2(\psi) = 1$, Eq. (94) simplifies to

$$\mathrm{I} = 2k_m\pi e^{2\alpha_m t}\iiint_V z\cos(x)\sin^3(x)e^{2\phi}\times\sin\left[\sin(x)k_m z\right]\cos\left[\sin(x)k_m z\right]\mathrm{d}V \qquad (99)$$

hence

$$\mathrm{I} = 2k_m\pi e^{2\alpha_m t}\int_0^l\int_0^{2\pi}\int_0^1 z\cos(x)\sin^3(x)e^{2\phi}\times\sin\left[\sin(x)k_m z\right]\cos\left[\sin(x)k_m z\right]r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \qquad (100)$$

It can be shown that this part is negligible, being of $O(M_b^2)$. Equation (89) collapses into

$$-\nabla\cdot\boldsymbol{U}\left(\tfrac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right) = -U_r\tilde{u}_z\frac{\partial\tilde{u}_z}{\partial r} \qquad (101)$$

Equation (83) can be replaced with Eq. (101); from the divergence theorem, one gathers

$$\alpha_6 = \frac{-M_b E_m^{-2}}{\exp(2\alpha_m t)}\iint_S\left\langle\boldsymbol{n}\cdot\left[\boldsymbol{U}\left(\tfrac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right)\right]\right\rangle\mathrm{d}S \qquad (102)$$

This formulation can be further simplified by expanding the rotational unsteady velocity into a normal (i.e., radial in a cylindrical motor) and a tangential component. Using

$$\tilde{\boldsymbol{u}} = (\boldsymbol{n}\cdot\tilde{\boldsymbol{u}})\boldsymbol{n} + \left[\tilde{\boldsymbol{u}} - (\boldsymbol{n}\cdot\tilde{\boldsymbol{u}})\boldsymbol{n}\right] \qquad (103)$$

one recognizes that the tangential rotational component will satisfy the no-slip condition by identically offsetting the irrotational velocity at the surface; hence

$$\tilde{\boldsymbol{u}} - (\boldsymbol{n}\cdot\tilde{\boldsymbol{u}})\boldsymbol{n} = -\left[\hat{\boldsymbol{u}} - (\boldsymbol{n}\cdot\hat{\boldsymbol{u}})\boldsymbol{n}\right] \qquad (104)$$

This turns Eq. (103) into

$$\tilde{\boldsymbol{u}} = -M_b\hat{p}\boldsymbol{n} - \left[\hat{\boldsymbol{u}} - (\boldsymbol{n}\cdot\hat{\boldsymbol{u}})\boldsymbol{n}\right] = (-M_b\hat{p} + \boldsymbol{n}\cdot\hat{\boldsymbol{u}})\boldsymbol{n} - \hat{\boldsymbol{u}} \qquad (105)$$

and so

$$\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}} = (M_b\hat{p})^2 - (\boldsymbol{n}\cdot\hat{\boldsymbol{u}})^2 + \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}} \qquad (106)$$

One can substitute Eq. (106) into Eq. (102) and carry out the time averaging; this operation yields

$$\alpha_6 = \tfrac{1}{4}E_m^{-2}\iint_{S_b}M_b\left(M_b^2\left\{1 - \left[A_b^{(r)}\right]^2\right\}\hat{p}^2 + \hat{\boldsymbol{u}}_m\cdot\hat{\boldsymbol{u}}_m\right)\mathrm{d}S = \tfrac{1}{4}E_m^{-2}\iint_{S_b}M_b(\hat{\boldsymbol{u}}_m\cdot\hat{\boldsymbol{u}}_m)\mathrm{d}S + O(M_b^3) \qquad (107)$$

then using $\hat{\boldsymbol{u}}_m = -\nabla\hat{p}_m/k_m$, Eq. (107) becomes

$$\alpha_6 = \tfrac{1}{4}k_m^{-2}E_m^{-2}\iint_{S_b}M_b(\nabla\hat{p}_m)^2\,\mathrm{d}S \qquad (108)$$

H. Seventh Factor: Viscous Correction

The next two rotational groups in Eq. (1) involve viscous damping expressions. In the classical stability calculations, viscous effects are discounted. A correction to the dilatational effect is represented in the seventh rotational term. By the same method used before, this term can be transformed into a surface integral via

$$\tfrac{4}{3}\iiint_V \left\langle \delta^2 \tilde{\boldsymbol{u}} \cdot \nabla\left(\nabla \cdot \hat{\boldsymbol{u}}\right)\right\rangle \mathrm{d}V$$

$$= -\tfrac{4}{3}\delta^2 \iint_S \left\langle \boldsymbol{n} \cdot \tilde{\boldsymbol{u}} \partial \hat{p}^{(1)} / \partial t \right\rangle \mathrm{d}S \qquad (109)$$

Clearly, Eq. (109) must be negligible insofar as it scales with the product of δ^2 and the radial unsteady velocity at the boundaries. The eighth term with viscous damping is not so negligible; it leads to

$$\alpha_7 = \frac{E_m^{-2}}{\exp\left(2\alpha_m t\right)} \iiint_V \left\langle -\delta^2 \left(\hat{\boldsymbol{u}} + \tilde{\boldsymbol{u}}\right) \cdot \left(\nabla \times \boldsymbol{\omega}\right)\right\rangle \mathrm{d}V \qquad (110)$$

Having $\boldsymbol{u}^{(1)} = \hat{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$, one can put

$$\alpha_7 = \frac{-\delta^2 E_m^{-2}}{\exp\left(2\alpha_m t\right)} \iiint_V \left\langle \boldsymbol{u}^{(1)} \cdot \left(\nabla \times \boldsymbol{\omega}\right)\right\rangle \mathrm{d}V \qquad (111)$$

Equation (111) can be further simplified by the use of

$$\nabla \cdot \left(\boldsymbol{A} \times \boldsymbol{B}\right) = \boldsymbol{B} \cdot \left(\nabla \times \boldsymbol{A}\right) - \boldsymbol{A} \cdot \left(\nabla \times \boldsymbol{B}\right) \qquad (112)$$

The corresponding integrand becomes

$$\boldsymbol{u}^{(1)} \cdot \left(\nabla \times \boldsymbol{\omega}\right) = \nabla \cdot \left[\boldsymbol{\omega} \times \boldsymbol{u}^{(1)}\right] + \boldsymbol{\omega} \cdot \left[\nabla \times \boldsymbol{u}^{(1)}\right] \qquad (113)$$

Recalling that

$$\nabla \times \hat{\boldsymbol{u}} = 0;\ \nabla \times \tilde{\boldsymbol{u}} = \boldsymbol{\omega};\ \nabla \times \boldsymbol{u}^{(1)} = \boldsymbol{\omega} \qquad (114)$$

Equation (113) reduces to

$$\boldsymbol{u}^{(1)} \cdot \left(\nabla \times \boldsymbol{\omega}\right) = \nabla \cdot \left[\boldsymbol{\omega} \times \boldsymbol{u}^{(1)}\right] + \boldsymbol{\omega} \cdot \boldsymbol{\omega} \qquad (115)$$

α_7 then becomes

$$\alpha_7 = \frac{-\delta^2 E_m^{-2}}{\exp\left(2\alpha_m t\right)} \iiint_V \left\langle \nabla \cdot \left[\boldsymbol{\omega} \times \boldsymbol{u}^{(1)}\right] + \boldsymbol{\omega} \cdot \boldsymbol{\omega}\right\rangle \mathrm{d}V \qquad (116)$$

This volumetric integral can be separated and partially converted to a surface integral using the divergence theorem. The first term in Eq. (116) yields

$$\iiint_V \left\langle \nabla \cdot \left[\boldsymbol{\omega} \times \boldsymbol{u}^{(1)}\right]\right\rangle \mathrm{d}V = \iint_S \left\langle \boldsymbol{n} \cdot \left[\boldsymbol{\omega} \times \boldsymbol{u}^{(1)}\right]\right\rangle \mathrm{d}S \qquad (117)$$

At the surface, the component of $\boldsymbol{u}^{(1)}$ is parallel to $\boldsymbol{n}$ due to the no-slip boundary condition. It follows that Eq. (117) will be identically zero.

Now that the first surface integral has been shown to cancel, one is left with

$$\iiint_V \left\langle \boldsymbol{u}^{(1)} \cdot \nabla \times \boldsymbol{\omega}\right\rangle \mathrm{d}V = \iiint_V \left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}\right\rangle \mathrm{d}V \qquad (118)$$

and so

$$\alpha_7 = \frac{-\delta^2 E_m^{-2}}{\exp\left(2\alpha_m t\right)} \iiint_V \left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}\right\rangle \mathrm{d}V \qquad (119)$$

Note that the integrand is a scalar. The corresponding physical problem displays conventional boundary layer behavior: boundary layer ideas locally apply. To reduce this to surface integral form, one can use the Von Kàrmàn–Polhausen method and evaluate the part of the integral normal to the surface. Along the surface

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}^{(1)} = \omega \boldsymbol{e}_\theta = \boldsymbol{e}_\theta \left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right] = -\boldsymbol{e}_\theta \frac{\partial u_z}{\partial r} \qquad (120)$$

and

$$\boldsymbol{\omega} = \tilde{\omega} \exp\left(\alpha_m t\right) \exp\left(-i k_m t\right) \boldsymbol{e}_\theta \qquad (121)$$

where

$$\tilde{\omega} = r \frac{k_m}{M_b} \exp\left(\alpha_m t\right) \exp\left[i\psi\left(r\right)\right]$$

$$\times \sin\left[k_m z \sin\left(x\right)\right];\ x = \tfrac{1}{2}\pi r^2 \qquad (122)$$

The real component of $\boldsymbol{\omega}$ may be represented by

$$\Re\left(\boldsymbol{\omega}\right) = r \frac{k_m}{M_b} \exp\left(\alpha_m t\right) \exp\left[\phi^{(r)}\right]$$

$$\times \cos\left(\psi - k_m t\right) \sin\left[k_m z \sin\left(x\right)\right] \hat{\boldsymbol{e}}_\theta$$

$$= r \exp\left(\alpha_m t\right) \frac{k_m}{M_b} \exp\left(\phi^{(r)}\right) \sin\left[k_m z \sin\left(x\right)\right]$$

$$\times \left[\cos\left(\psi\right)\cos\left(k_m t\right) + \sin\left(\psi\right)\sin\left(k_m t\right)\right] e_\theta \qquad (123)$$

Therefore

$$\boldsymbol{\omega} \cdot \boldsymbol{\omega} = \exp\left(2\alpha_m t\right) \left\{ r \frac{k_m}{M_b} \exp\left[\phi^{(r)}\right] \sin\left[k_m z \sin\left(x\right)\right]\right\}^2$$

$$\times \left[\cos\left(\psi\right)\cos\left(k_m t\right) + \sin\left(\psi\right)\sin\left(k_m t\right)\right]^2 \qquad (124)$$

In order to further simplify Eq. (124), it is appropriate to apply time averaging; from

$$\left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}\right\rangle = \frac{1}{2\pi} k_m e^{\alpha_m t} \int_0^{2\pi/k_m} \boldsymbol{\omega} \cdot \boldsymbol{\omega} e^{-\alpha_m t} dt \qquad (125)$$

one can put

$$\left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}\right\rangle = e^{2\alpha_m t}\, \tfrac{1}{2} \left\{ r\left(k_m / M_b\right) e^{\phi^{(r)}} \sin\left[k_m z \sin\left(x\right)\right]\right\}^2$$

$$= \left(k_m / M_b\right)^2 e^{2\alpha_m t}\, \tfrac{1}{2} \left\{ r e^{\phi^{(r)}} \sin[k_m z \sin\left(x\right)\right\}^2$$

$$\times \left[\cos\left(\psi\right)^2 + \sin\left(\psi\right)^2\right] \qquad (126)$$

As shown by Flandro and Majdalani[1,3] the unsteady velocity exhibits the form

$$\tilde{u}_z = irU_r e^{\alpha_m t+\phi^{(r)}} e^{i\left[\psi(r)+\phi^{(i)}\right]} \sin\left[k_m z \sin(x)\right] e^{-ik_m t} \quad (127)$$

Since $e^{i\phi^{(i)}}(1) \approx 1$, the preceding equation simplifies to

$$\tilde{u}_z = irU_r e^{\alpha_m t} e^{\phi^{(r)}} e^{i\left[\psi(r)-k_m t\right]} \sin\left[k_m z \sin(x)\right] \quad (128)$$

Reverting off Euler's notation, Eq. (128) may be expanded into

$$\tilde{u}_z = irU_r e^{\alpha_m t} e^{\phi^{(r)}} \sin\left[k_m z \sin(x)\right]\left[\cos(\psi - k_m t) + i\sin(\psi - k_m t)\right] \quad (129)$$

By distributing the 'i' one obtains

$$\tilde{u}_z = rU_r e^{\alpha_m t} e^{\phi^{(r)}} \sin\left[k_m z \sin(x)\right]\left[i\cos(\psi - k_m t) - \sin(\psi - k_m t)\right] \quad (130)$$

The real component of $\tilde{u}_z$ may hence be represented by

$$\Re(\tilde{u}_z) = -rU_r e^{\alpha_m t} e^{\phi^{(r)}} \sin\left[k_m z \sin(x)\right] \times\left[\sin(\psi)\cos(k_m t) - \cos(\psi)\sin(k_m t)\right] \quad (131)$$

so that

$$\tilde{u}_z{}^2 = \left\{rU_r e^{\alpha_m t} e^{\phi^{(r)}} \sin\left[k_m z \sin(x)\right]\right\}^2 \times\begin{bmatrix}\sin^2(\psi)\cos^2(k_m t) \\ -2\sin(\psi)\cos(\psi)\sin(k_m t)\cos(k_m t) \\ +\cos^2(\psi)\sin^2(k_m t)\end{bmatrix} \quad (132)$$

When time averaging is performed on $\tilde{u}_z^2$, one gets

$$\left\langle \tilde{u}_z{}^2 \right\rangle = \tfrac{1}{2} e^{2\alpha_m t}\left\{rU_r e^{\phi^{(r)}} \sin\left[k_m z \sin(x)\right]\right\}^2 \quad (133)$$

Note the correlation between $\boldsymbol{\omega}\cdot\boldsymbol{\omega}$ and $\tilde{u}_z$

$$\langle \boldsymbol{\omega}\cdot\boldsymbol{\omega}\rangle \approx \left\langle (k_m / M_b)^2 \tilde{u}_z^2 \right\rangle \approx e^{2\alpha_m t}(k_m / M_b)^2 \tilde{u}_m \frac{\partial \tilde{u}_m}{\partial r} = \tfrac{1}{2}(k_m / M_b)^2 e^{2\alpha_m t} \frac{\partial \tilde{u}_m^2}{\partial r} \quad (134)$$

Therefore α_7 becomes

$$\alpha_7 = -\tfrac{1}{4}\delta^2 E_m^{-2}(k_m / M_b)^2 \iiint_V \frac{\partial \tilde{u}_m^2}{\partial r}\,\mathrm{d}V \quad (135)$$

For the circular-port motor, one has

$$\alpha_7 = -\tfrac{1}{4}\delta^2 E_m^{-2}(k_m / M_b)^2 \iint_S \left(\int_0^1 \frac{\partial \tilde{u}_m^2}{\partial r}\,\mathrm{d}r\right)\mathrm{d}S \quad (136)$$

This expression reduces to

$$\alpha_7 = -\tfrac{1}{4}\delta^2 E_m^{-2}(k_m / M_b)^2 \iint_S \tilde{u}_m^2\Big|_{r=1}\,\mathrm{d}S \quad (137)$$

which corresponds to the non-time averaged form

$$\alpha_7 = -\tfrac{1}{2}\frac{\delta^2}{\exp(2\alpha_m t)} E_m^{-2}(k_m / M_b)^2 \iint_S \left\langle \tilde{u}^2 \right\rangle\Big|_{r=1}\,\mathrm{d}S \quad (138)$$

At the surface the no-slip condition must be satisfied; this enables us to use $\tilde{u}_m = -\hat{u}_m$. Recalling that $\hat{u}_m = \sin(k_m z)$, one can put

$$= -\tfrac{1}{4}\delta^2 E_m^{-2}(k_m / M_b)^2 \iint_S \left[\sin^2(k_m z)\right]\mathrm{d}S \quad (139)$$

or, alternatively,

$$\alpha_7 = -\tfrac{1}{4}\delta^2 E_m^{-2} M_b^{-2} \iint_S \left(\frac{\partial \hat{p}_m}{\partial z}\right)^2 \mathrm{d}S \quad (140)$$

Using basic deduction, it can be proven that Eq. (140) will be true in all spatial directions; for the general case, one has

$$\alpha_7 = -\tfrac{1}{4}\delta^2 E_m^{-2} M_b^{-2} \iint_S (\nabla \hat{p}_m)^2\,\mathrm{d}S \quad (141)$$

I. Eighth Factor: Pseudo Acoustical Correction

The pseudo acoustical term is due to the pseudopressure coupling associated with the vortical field and either the unsteady acoustical or rotational velocities. It has been shown by Flandro and Majdalani[2,3] that this term is negligible, being $O\left(M_b{}^3\right)$. The first of these two terms can be expressed by

$$\alpha_8 = \frac{E_m^{-2}}{\exp(2\alpha_m t)} \iiint_V \left\langle -\hat{\boldsymbol{u}}\cdot\nabla\tilde{p}\right\rangle \mathrm{d}V \quad (142)$$

Traditionally, terms due to the pseudopressure $\tilde{p}$ (or pseudosound) are ignored because of their small contribution. In order to test the size of Eq. (142), one may follow the asymptotic approach used recently to evaluate α_8 for a full-length cylindrical motor.[3] Following the form used by Flandro and Majdalani,[1,3] one can write

$$\tilde{p} = \exp(\alpha_m t)\left[\tilde{p}_m^r \cos(k_m t) + \tilde{p}_m^i \sin(k_m t)\right] \quad (143)$$

with

$$\tilde{p}_m^r = \tfrac{-\pi}{2} M_b z \sin(\psi)\sin(2x)\exp(\phi)\sin\left[k_m z \sin(x)\right] \quad (144)$$

$$\tilde{p}_m^i = \tfrac{1}{2}\pi M_b z \cos(\psi)\sin(2x)\exp(\phi)\sin\left[k_m z \sin(x)\right] \quad (145)$$

Subsequently, one can evaluate

$$\nabla\tilde{p} = \exp(\alpha_m t)\left[\cos(k_m t)\nabla\tilde{p}_m^r + \sin(k_m t)\nabla\tilde{p}_m^i\right] \quad (146)$$

then expand both terms

$$\nabla\tilde{p}_m^r = \frac{\partial}{\partial r}(\tilde{p}_m^r)\boldsymbol{e}_r + \frac{\partial}{\partial z}(\tilde{p}_m^r)\boldsymbol{e}_z$$

$$\approx \tfrac{-\pi}{2}(k_m/\boldsymbol{U}_r)z\cos(\psi)\sin(2x)\exp(\phi)\sin\left[k_m z\sin(x)\right]\boldsymbol{e}_r$$
$$-\tfrac{1}{2}\pi M_b\sin(\psi)\sin(2x)\exp(\phi)$$
$$\times\left\{\sin\left[k_m z\sin(x)\right]+k_m z\sin(x)\cos\left[k_m z\sin(x)\right]\right\}\boldsymbol{e}_z \quad (147)$$

and

$$\nabla\tilde{p}_m^i=\frac{\partial}{\partial r}(\tilde{p}_m^i)\boldsymbol{e}_r+\frac{\partial}{\partial z}(\tilde{p}_m^i)\boldsymbol{e}_z$$
$$\approx -\tfrac{1}{2}\pi(k_m/U_r)z\sin(\psi)\sin(2x)\exp(\phi)\sin\left[k_m z\sin(x)\right]\boldsymbol{e}_r$$
$$+\tfrac{1}{2}\pi M_b\cos(\psi)\sin(2x)\exp(\phi)$$
$$\times\left\{\sin\left[k_m z\sin(x)\right]+k_m z\sin(x)\cos\left[k_m z\sin(x)\right]\right\}\boldsymbol{e}_z \quad (148)$$

Furthermore, one can put

$$\hat{\boldsymbol{u}}\cdot\nabla\tilde{p}\approx\tfrac{\pi}{2}M_b\exp(2\alpha_m t)\sin(k_m z)\sin(k_m t)\sin(2x)\exp(\phi)$$
$$\times\left\{\sin\left[k_m z\sin(x)\right]+k_m z\sin(x)\cos\left[k_m z\sin(x)\right]\right\}$$
$$\times\left[\cos(\psi)\sin(k_m t)-\sin(\psi)\cos(k_m t)\right] \quad (149)$$

Time-averaging later gives

$$\left\langle\hat{\boldsymbol{u}}\cdot\nabla\tilde{p}\right\rangle=\tfrac{\pi}{4}M_b\exp(2\alpha_m t+\phi)\sin(k_m z)\sin(2x)\cos(\psi)$$
$$\times\left\{\sin\left[k_m z\sin(x)\right]+k_m z\sin(x)\cos\left[k_m z\sin(x)\right]\right\} \quad (150)$$

and so

$$\alpha_8=-E_m^{-2}\iiint_V\tfrac{1}{4}\pi M_b\exp(\phi)\sin(k_m z)\sin(2x)\cos(\psi)$$
$$\times\left\{\sin\left[k_m z\sin(x)\right]+k_m z\sin(x)\cos\left[k_m z\sin(x)\right]\right\}\mathrm{d}V \quad (151)$$

Evaluating the volumetric integrals, one gets

$$\alpha_8=-\tfrac{1}{4}\pi M_b E_m^{-2}\int_0^{2\pi}\int_0^l\int_0^1 r\sin\left(k_m z\right)\sin\left(2x\right)e^{\phi}\cos\left(\psi\right)$$
$$\times\left\{\begin{matrix}\sin\left[k_m z\sin(x)\right]\\+k_m z\sin(x)\cos\left[k_m z\sin(x)\right]\end{matrix}\right\}\mathrm{d}r\mathrm{d}z\mathrm{d}\theta$$
$$=-\tfrac{1}{2}\pi^2 M_b E_m^{-2}\int_0^l\int_0^1 r\sin\left(k_m z\right)\sin(2x)\exp\left(\phi\right)$$
$$\times\cos\left(\psi\right)\left\{\sin\left[k_m z\sin\left(x\right)\right]\right.$$
$$\left.+k_m z\sin\left(x\right)\cos\left[k_m z\sin(x)\right]\right\}\mathrm{d}r\mathrm{d}z \quad (152)$$

Direct integration of Eq. (152) with respect to z yields

$$\alpha_8=-\tfrac{1}{4}\pi^2 M_b k_m^{-1}E_m^{-2}\int_0^1 Q(r)\mathrm{d}r \quad (153)$$

where

$$Q(r)=r\exp(\phi)\cos(\psi)\sin(2x)\sec^3(x)\left(\cos(k_m l)\right.$$
$$\times\left\{-2k_m l\cos(x)\cos\left[k_m l\sin(x)\right]\sin(x)+\left[\cos(2x)-3\right]\right.$$
$$\left.\left.\times\sin^2(x)\sin\left[k_m l\sin(x)\right]-2\cos\left[k_m l\sin(x)\right]\tan(x)\right\}\right) \quad (154)$$

Linearizing and integrating with respect to r, one obtains

$$\alpha_8=-\tfrac{1}{24}\pi^3 l M_b E_m^{-2}\int_0^1 \mathrm{e}^{-\xi y}\cos\left(k_m y/M_b\right)\left\{\left[2\left(\pi k_m l\right)^2\right.\right.$$
$$\left.\left.-\pi^2+3\right]y^3-9y^2+6y\right\}\mathrm{d}y \quad (155)$$

or

$$\alpha_8=\tfrac{1}{4}\pi^3 l M_b^3 k_m^{-2}E_m^{-2}(1+M_b^2\xi^2/k_m^2)^{-3}$$
$$\times\left\{1-M_b^2 k_m^{-2}\left[9\xi+(3-\pi^2+2\pi^2 k_m^2 l^2)(1+M_b^2\xi^2/k_m^2)^{-1}\right]\right\}$$
$$+O(M_b^7 k_m^{-6}) \quad (156)$$

Using Eqs. (9) and (29) from Flandro and Majdalani,[1,3] α_8 can be rearranged into

$$\alpha_8=\tfrac{2}{5}\frac{M_b^3 l^2}{m^2}\left[1+\frac{M_b^2\xi^2 l^2}{(m\pi)^2}\right]^{-3}\left(1-\frac{M_b^2 l^2}{(m\pi)^2}\right.$$
$$\left.\times\left\{9\xi+(3-\pi^2+2\pi^4 m^2)\left[1+M_b^2\xi^2 l^2/(m\pi)^2\right]^{-1}\right\}\right) \quad (157)$$

In most rocket motor applications exhibiting a relatively small ξ, α_8 may be given by

$$\alpha_8=\frac{2M_b^3 l^2}{5m^2}\left[1-3\xi^2\frac{M_b^2 l^2}{(m\pi)^2}\right]<<O\left(1\right) \quad (158)$$

Despite the applicability of Eq. (158) to full length circular-port motors only, its small order of magnitude suggests that pseudo acoustical corrections constitute insignificant contributions almost independently of the motor shape. This conclusion can be readily corroborated by numerical measurements.

J. Ninth Factor: Pseudo Rotational Correction

As alluded to earlier, the last term by Flandro and Majdalani[1] is due to the less obvious coupling that is formed between vorticity-induced pseudopressure and the unsteady rotational velocity. The significance of this term can be captured by examining

$$\alpha_9=\frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\left\langle\tilde{\boldsymbol{u}}\cdot\nabla\tilde{p}\right\rangle\mathrm{d}V \quad (159)$$

By use of vector identities (see Eq. (A6)), one has

$$\left\{\begin{matrix}\nabla\cdot\left(\tilde{\boldsymbol{u}}\tilde{p}\right)=\tilde{\boldsymbol{u}}\cdot\nabla\tilde{p}+\tilde{p}\nabla\cdot\tilde{\boldsymbol{u}}\\ \nabla\cdot\tilde{\boldsymbol{u}}=0;\quad\nabla\cdot\left(\tilde{\boldsymbol{u}}\tilde{p}\right)=\tilde{\boldsymbol{u}}\cdot\nabla\tilde{p}\end{matrix}\right. \quad (160)$$

α_9 becomes

$$\alpha_9=\frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\left\langle\nabla\cdot\left(\tilde{\boldsymbol{u}}\tilde{p}\right)\right\rangle\mathrm{d}V \quad (161)$$

Immediate application of the divergence theorem yields

$$\alpha_9 = \frac{-E_m^{-2}}{\exp(2\alpha_m t)} \iint_S \left\langle \boldsymbol{n} \cdot (\tilde{\boldsymbol{u}} \tilde{p}) \right\rangle \mathrm{d}S \tag{162}$$

Furthermore, time averaging changes Eq. (162) into

$$\alpha_9 = -\tfrac{1}{2} E_m^{-2} \iint_{S_N} \left[\tilde{u}_m^{(r)} \tilde{p}_m^{(r)} + \tilde{u}_m^{(i)} \tilde{p}_m^{(i)} \right] \mathrm{d}S \tag{163}$$

Unlike α_8, this term can be expanded for two simple geometric shapes and shown to be large.

K. Tenth Factor: Unsteady Nozzle Correction

It was shown previously that retention of unsteady rotational energy gives rise to a term at the downstream chamber boundary.[3] This growth rate combines the third and fourth rotational terms in Eq. (1) such that

$$\alpha_{10} = -\frac{M_b E_m^{-2}}{\exp(2\alpha_m t)} \iiint_V \left\langle (\hat{\boldsymbol{u}} + \tilde{\boldsymbol{u}}) \cdot \nabla (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right\rangle \mathrm{d}V \tag{164}$$

To convert Eq. (164) to a double integral, we first let $\boldsymbol{u}^{(1)} = \hat{\boldsymbol{u}} + \tilde{\boldsymbol{u}}$; the integrand in Eq. (164) yields

$$\boldsymbol{u}^{(1)} \cdot \left[\nabla (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] = \nabla \cdot \left[\boldsymbol{u}^{(1)} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] - (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \boldsymbol{u}^{(1)} \tag{165}$$

The first term on the right-hand side can be written as

$$\iiint_V \left\langle \nabla \cdot \left[\boldsymbol{u}^{(1)} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}V =$$

$$\iiint_V \left\langle \nabla \cdot \left[\tilde{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] + \nabla \cdot \left[\hat{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}V \tag{166}$$

These integrals are easily converted to surface form via the divergence theorem. For example, on can put

$$\iiint_V \left\langle \nabla \cdot \left[\tilde{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] + \nabla \cdot \left[\hat{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}V$$

$$= \iint_S \left\langle \boldsymbol{n} \cdot \left[\tilde{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] + \boldsymbol{n} \cdot \left[\hat{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}S \tag{167}$$

As shown in Eqs. (5)–(7), the normal projections of $\hat{\boldsymbol{u}}$ are (M_b); the second surface integral on the right-hand side of Eq. (167) is (M_b^2); it can be dropped for asymptotic consistency. This turns Eq. (167) into

$$\iiint_V \left\langle \nabla \cdot \left[\tilde{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}V = \iint_S \left\langle \boldsymbol{n} \cdot \left[\tilde{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}S \tag{168}$$

In like fashion, the second term on the right-hand side of Eq. (165) can be written as

$$-(\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \boldsymbol{u}^{(1)} = -(\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} - (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \tilde{\boldsymbol{u}} \tag{169}$$

Since, $\nabla \cdot \tilde{\boldsymbol{u}} = 0$, Eq. (169) simplifies to

$$(\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \boldsymbol{u}^{(1)} = (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \tag{170}$$

At the outset, α_{10} becomes

$$\alpha_{10} = \frac{-M_b E_m^{-2}}{\exp(2\alpha_m t)} \left\{ \iint_S \left\langle \boldsymbol{n} \cdot \left[\tilde{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}S \right.$$

$$\left. - \iiint_V \left\langle (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \right\rangle \mathrm{d}V \right\} \tag{171}$$

Evaluating the second term on the right-hand side of Eq. (171) gives

$$\iiint_V \left\langle (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \nabla \cdot \hat{\boldsymbol{u}} \right\rangle \mathrm{d}V$$

$$= \iiint_V \left\langle \pi z \cos(x) \sin(x) e^{\phi} \sin(\psi) \sin\left[\sin(x) k_m z \right] \right.$$

$$\left. \times \cos(k_m z)\, e^{2\alpha_m t} \cos(k_m t) \sin(k_m t) \mathrm{d}V \right\rangle \tag{172}$$

The radial integral yields a value of $O(M_b)$; although Eq. (172) corresponds to the case of a full-length circular-port motor, it can be shown that this component is small for several geometric shapes. For the general case of an arbitrary motor, one is left with

$$\alpha_{10} = \frac{-M_b}{E_m^2 \exp(2\alpha_m t)} \iint_S \left\langle \boldsymbol{n} \cdot \left[\tilde{\boldsymbol{u}} (\boldsymbol{U} \cdot \tilde{\boldsymbol{u}}) \right] \right\rangle \mathrm{d}S \tag{173}$$

Finally, time averaging of Eq. (173) reduces it to

$$\alpha_{10} = -\tfrac{1}{2} M_b E_m^{-2} \iint_{S_N} \left\{ \left[\tilde{u}_m^{(r)} \right]^2 + \left[\tilde{u}_m^{(i)} \right]^2 \right\} U_z \, \mathrm{d}S \tag{174}$$

III. Discussion

A summary of the transformed integrals is catalogued in Table 1 where the original and newly converted forms are posted. The surface converted integrals are given in a general form, before time averaging, and also in a form that is most suitable for direct implementation in the Standard Stability Prediction code. It must be noted that these integrals correspond to the linear growth rate regime preceding the onset of nonlinear oscillations. A similar analysis needs to be applied to the nonlinear growth rate integrals that control the system behavior during wave steepening and formation of limit cycle amplitudes. It is hoped that the tools presented here will be similarly employed in future studies aimed at converting the nonlinear growth rate corrections to surface-based quantities. From this standpoint, it may be seen that a key contribution of this study lies, perhaps, in its proof-of-concept demonstration for the linear stability growth rates. Here, they are shown to be amenable to surface transformation despite their relative complexity. The feasibility of this approach may be readily extended to other combustion instability mechanisms that are expressed in volumetric integral form.

Of the ten integrals posted in Table 1, it should be noted that only seven are important. These include i) pressure coupling α_1, ii) flow turning α_4, iii) rotational flow correction α_5, iv) mean vortical correction α_6, v)

Table 1 Linear stability integrals in both volumetric and surface integral forms

	Rotational set in volumetric form	Rotational set in surface form	Standard Stability Prediction form
E_m^2	$E_m^2=\frac{1}{2}\iiint[(\hat{p}_m)^2+\hat{\boldsymbol{u}}_m\cdot\hat{\boldsymbol{u}}_m+2\hat{\boldsymbol{u}}_m\cdot\tilde{\boldsymbol{u}}_m^i+\tilde{\boldsymbol{u}}_m^r\cdot\tilde{\boldsymbol{u}}_m^r+\tilde{\boldsymbol{u}}_m^i\cdot\tilde{\boldsymbol{u}}_m^i]\mathrm{d}V$		
α_1	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\left\langle-\nabla\cdot\left[\hat{p}\hat{\boldsymbol{u}}+\frac{1}{2}M_b\boldsymbol{U}(\hat{p})^2\right]-M_b\left[\hat{\boldsymbol{u}}\cdot\nabla(\boldsymbol{U}\cdot\hat{\boldsymbol{u}})\right]\right\rangle\mathrm{d}V$	$\frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iint_s\left\langle\boldsymbol{n}\cdot\left[\hat{p}\hat{\boldsymbol{u}}+\frac{1}{2}M_b\boldsymbol{U}(\hat{p})^2\right]-M_b\left[\boldsymbol{n}\cdot\hat{\boldsymbol{u}}(\boldsymbol{U}\cdot\hat{\boldsymbol{u}})\right]\right\rangle\mathrm{d}S$	$\frac{1}{2}M_bE_m^{-2}\left\{\iint_{S_b}\hat{p}_m^2\left[A_b^{(r)}+1\right]\mathrm{d}S-\iint_{S_N}\hat{p}_m^2\left[A_N^{(r)}+U_N\right]\mathrm{d}S\right\}$
α_2	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\left\langle\frac{4}{3}\delta^2\hat{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\hat{\boldsymbol{u}})\right\rangle\mathrm{d}V$	$\frac{4k_m\delta^2E_m^{-2}}{3\exp(2\alpha_m t)}\iint_S\langle\boldsymbol{n}\cdot(\hat{p}\hat{\boldsymbol{u}})\rangle\mathrm{d}S-\frac{2}{3}\delta^2k_m^4$	$-\frac{2}{3}k_m\delta^2E_m^{-2}\left[\iint_{S_b}M_bA_b^{(r)}\hat{p}_m^2\,\mathrm{d}S-\iint_{S_N}M_bA_N^{(r)}\hat{p}_m^2\,\mathrm{d}S\right]-\frac{4}{3}\delta^2k_m^2$
α_3	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\left\langle M_b\{\hat{\boldsymbol{u}}\cdot(\hat{\boldsymbol{u}}\times\boldsymbol{\Omega})\}\right\rangle\mathrm{d}V$	0	0
α_4	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\langle M_b\hat{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\rangle\mathrm{d}V$	$\frac{M_bE_m^{-2}}{\exp(\alpha_m t)}\iint_{S_b}\langle\tilde{\boldsymbol{u}}^i\cdot\hat{\boldsymbol{u}}\rangle\mathrm{d}S$	$-\frac{1}{2}M_bk_m^{-1}E_m^{-2}\iint_{S_b}\tilde{\boldsymbol{u}}_m^i\cdot\nabla\hat{p}_m\,\mathrm{d}S$
α_5	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\langle-\tilde{\boldsymbol{u}}\cdot\nabla\hat{p}\rangle\mathrm{d}V$	$\frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iint_S\langle\boldsymbol{n}\cdot(\tilde{\boldsymbol{u}}\hat{p})\rangle\mathrm{d}S$	$\frac{1}{2}M_bE_m^{-2}\iint_{S_b}\hat{p}_m^2\,\mathrm{d}S$
α_6	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\langle M_b\tilde{\boldsymbol{u}}\cdot(\boldsymbol{U}\times\boldsymbol{\omega})\rangle\mathrm{d}V$	$\frac{-M_bE_m^{-2}}{\exp(2\alpha_m t)}\iint_S\left\langle\boldsymbol{n}\cdot\boldsymbol{U}\left(\frac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right)\right\rangle\mathrm{d}S$	$\frac{1}{4}k_m^{-2}M_bE_m^{-2}\iint_{S_b}(\nabla\hat{p}_m)^2\,\mathrm{d}S$
α_7	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\left\langle-\delta^2(\hat{\boldsymbol{u}}+\tilde{\boldsymbol{u}})\cdot(\nabla\times\boldsymbol{\omega})\right\rangle\mathrm{d}V$	$-\frac{\delta^2E_m^{-2}k_m^2M_b^{-2}}{2\exp(2\alpha_m t)}\iint_S\langle\tilde{\boldsymbol{u}}^2\rangle\big\vert_{r=1}\mathrm{d}S$	$-\frac{1}{4}\delta^2E_m^{-2}M_b^{-2}\iint_{S_b}(\nabla\hat{p}_m)^2\,\mathrm{d}S$
α_8	$\frac{E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\langle-\hat{\boldsymbol{u}}\cdot\nabla\tilde{p}\rangle\mathrm{d}V$	$\frac{2}{5}M_b^3l^2/m^2$ (negligible)	$\frac{2}{5}M_b^3l^2/m^2$ (negligible)
α_9	$\frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\langle\tilde{\boldsymbol{u}}\cdot\nabla\tilde{p}\rangle\mathrm{d}V$	$\frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iint_S\langle\boldsymbol{n}\cdot\tilde{\boldsymbol{u}}\tilde{p}\rangle\mathrm{d}S$	$-\frac{1}{2}E_m^{-2}\iint_{S_N}\left[\tilde{u}_m^{(r)}\tilde{p}_m^{(r)}+\tilde{u}_m^{(i)}\tilde{p}_m^{(i)}\right]\mathrm{d}S$
α_{10}	$\frac{-E_m^{-2}}{\exp(2\alpha_m t)}\iiint_V\langle M_b(\hat{\boldsymbol{u}}+\tilde{\boldsymbol{u}})\cdot\nabla(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}})\rangle\mathrm{d}V$	$\frac{-M_bE_m^{-2}}{\exp(2\alpha_m t)}\iint_S\langle\hat{\boldsymbol{n}}\cdot[\tilde{\boldsymbol{u}}(\boldsymbol{U}\cdot\tilde{\boldsymbol{u}})]\rangle\mathrm{d}S$	$-\frac{1}{2}E_m^{-2}M_b\iint_{S_N}\left[(\tilde{u}_m^i)^2+(\tilde{u}_m^r)^2\right]U_z\,\mathrm{d}S$

viscous damping α_7, vi) pseudo rotational correction α_9, and vii) unsteady nozzle correction α_{10}. Two additional corrections not covered here are due to particle damping and distributed combustion.[8] These are covered quite thoroughly by Culick[8] and others.[9-11] As for the velocity coupling correction that has often been cited in the literature,[12-20] it is accounted for systematically in the current (rotational) formulation. This will be explained next.

Considering that velocity coupling seeks a fundamental relation between the fluctuating radial component of velocity at the propellant surface and the longitudinal fluctuations in the axial velocity, such coupling is intrinsic to this analysis; here, velocity coupling is accommodated internally by virtue of unsteady mass conservation which entertains a well-defined relation between radial and axial velocity fluctuations.[5] In the previous, irrotational formulation, velocity coupling had to be introduced a posteriori (having been invented in the form of an empirical relation) to compensate for the inability of the one-dimensional acoustic model to permit a relationship between axial and radial velocity fluctuations at the propellant surface.[12-20] In the present analysis, velocity coupling is not only built into the model a priori, it is obtained from the fundamental requirement to satisfy mass conservation, thus obviating the need for guesswork, experimentation, curve fitting, or trial.

Having obtained the surface integral forms of the most significant growth rate corrections, the newly simplified expressions can be readily implemented into the Standard Stability Prediction (SSP) code. In fact, work in this direction is currently underway.[21] The newly developed SSP stands to provide numerous advantages to rocket motor designers. Within the code itself, the implementation of surface calculations will bring about many advantages. These include promoting better predictive capabilities and eliminating the need to estimate the acoustical and rotational wave components inside the motor chamber; instead, only local properties are needed along the chamber's control volume. This will not only simplify the evaluation of the stability integrals, but will greatly enhance the accuracy of SSP predictions. Subsequently, users of the code will only need to be concerned with providing accurate estimates of propellant properties and injection characteristics along the motor boundaries.

Another key aspect that this study addresses is the impact of retaining the pseudopressure which is often neglected in the literature. Being the unsteady pressure wave (or pseudosound) generated at solid boundaries, $\tilde{p}$ is ignored in stability assessments because of its small magnitude and its rapid decay away from the burning surface. However, considering that most important instability mechanisms occur in close vicinity to the propellant surface, it is not surprising that one of the two pseudo corrections is large (i.e., α_9).[3] This point is confirmed in the present analysis as pseudo-related corrections are carefully examined in both volumetric and surface form. In later studies, it may be shown that α_9 and α_{10} can cancel each other's contribution. The same can be said of flow turning and the rotational flow corrections as α_4 and α_5 will generally cancel as well. These ideas will be deferred to forthcoming work in which they will be fully discussed and verified using representative rocket motors.

IV. Concluding Remarks

The current study describes a major breakthrough in improving our modeling capabilities of acoustic instability growth for motors undergoing linear oscillations. The breakthrough lies in simplifying the growth rate expressions to equivalent, albeit more accurate and manageable identities and approximations. The translated surface integrals are obtained in conceptual forms that are nearly independent of chamber geometry. It would be helpful to evaluate these integrals for the full-length circular-port and slab motor configurations. Results could then be compared to predictions obtained either directly from SSP or by evaluating the triple integrals using parametric sets that are representative of actual motors. Another method of verification could be attempted by evaluating the stability integrals by computer. In that respect, comparisons to SSP predictions could prove to be instrumental. These tasks are hoped to be covered in a forthcoming article.

Appendix A: Useful Identities

In converting volumetric integrals to surface form, several vectorial and algebraic manipulations are required. Here we compile and catalogue vector identities and theorems that may be needed during the integral conversion process. Below is a compilation of vector identities and theorems written in standard notation, with bold letters to represent vectors.

A. Vector Identities

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = (\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A} \tag{A1}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \tag{A2}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C}) \tag{A3}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{D})\boldsymbol{C} - (\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{D} \tag{A4}$$

$$\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f \tag{A5}$$

$$\nabla\cdot(f\boldsymbol{A}) = f\nabla\cdot\boldsymbol{A} + \nabla f\cdot\boldsymbol{A} \tag{A6}$$

$$\nabla\times(f\boldsymbol{A}) = f\nabla\times\boldsymbol{A} + \nabla f\times\boldsymbol{A} \tag{A7}$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \tag{A8}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) \tag{A9}$$

$$\nabla\cdot(\boldsymbol{A}\boldsymbol{B}) = (\nabla\cdot\boldsymbol{A})\boldsymbol{B} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{A10}$$

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{B}) = (\nabla\boldsymbol{B})\cdot\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{A11}$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) \tag{A12}$$

$$\nabla^2 f = \nabla\cdot\nabla f \tag{A13}$$

$$\nabla\times(\nabla f) = 0 \tag{A14}$$

$$\nabla\cdot(\nabla\times\boldsymbol{A}) = 0 \tag{A15}$$

$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} \tag{A16}$$

B. Theorems

$$\int_V \nabla f\,\mathrm{d}V = \int_S \boldsymbol{n}\cdot f\,\mathrm{d}S \tag{A17}$$

$$\int_V \nabla\cdot\boldsymbol{F}\,\mathrm{d}V = \int_S \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S \tag{A18}$$

$$\int_V \nabla \times \boldsymbol{F} \, \mathrm{d}V = \int_S \boldsymbol{F} \times \boldsymbol{n} \, \mathrm{d}S \tag{A19}$$

$$\int_V \left(f \nabla^2 g - g \nabla^2 f \right) \mathrm{d}V = \int_S \boldsymbol{n} \cdot \left(f \nabla g - g \nabla f \right) \mathrm{d}S \tag{A20}$$

$$\int_V \left\{ \boldsymbol{A} \cdot \left[\nabla \times \left(\nabla \times \boldsymbol{B} \right) \right] - \boldsymbol{B} \cdot \left[\nabla \times \left(\nabla \times \boldsymbol{A} \right) \right] \right\} \mathrm{d}V$$

$$= \int_S \boldsymbol{n} \cdot \left[\boldsymbol{B} \times \left(\nabla \times \boldsymbol{A} \right) - \boldsymbol{A} \times \left(\nabla \times \boldsymbol{B} \right) \right] \mathrm{d}S \tag{A21}$$

$$\int_V \nabla^2 \boldsymbol{A} \, \mathrm{d}V = \int_S \left(\hat{\boldsymbol{n}} \cdot \nabla \right) \boldsymbol{A} \mathrm{d}S \tag{A22}$$

$$\int_V \left[\boldsymbol{B} \left(\nabla \cdot \boldsymbol{A} \right) + \left(\boldsymbol{A} \cdot \nabla \right) \boldsymbol{B} \right] \mathrm{d}V = \int_S \boldsymbol{B} \left(\hat{\boldsymbol{n}} \cdot \boldsymbol{A} \right) \mathrm{d}S \tag{A23}$$

C. Time Averaging

After all of the needed stability factors have been converted into surface integral form, time averaging needs to be applied. The angle brackets used in the energy assessment equations developed by Flandro and Majdalani[1,2] require time averaging of the enclosed function. This is accomplished via

$$\langle A \rangle = \tfrac{1}{2\pi} k_m e^{\alpha_m t} \int_0^{2\pi / k_m} A e^{-\alpha_m t} \, \mathrm{d}t \tag{A24}$$

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