

The Taylor-Culick profile with arbitrary headwall injection

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Taylor's incompressible and rotational profile is extended to a porous cylinder with arbitrary headwall injection. This profile, often referred to as Culick's mean flow, is now generalized to permit the imposition of reactive headwall conditions. Starting with Euler's steady equations, the solution that we derive is approximate, being exact only at the sidewall, the centerline, or for similarity-conforming inlet profiles. Furthermore, the approximation is quasiviscous, being observant of the no slip requirement at the sidewall. Based on numerical experiments under inviscid flow conditions, the closed-form approximation that we obtain appears to be well suited to describe the bulk flow field in basic models of solid and hybrid rockets where uniform sidewall injection is imposed at the propellant surface. For similarity-nonconforming profiles, the approximation becomes more accurate as we move away from the headwall. Results are verified using computational fluid dynamics for several headwall injection patterns. © 2007 American Institute of Physics. [DOI: 10.1063/1.2746003]

I. INTRODUCTION

Culick's solution for describing the gaseous motion in solid rocket motors (SRMs) was obtained under the contingencies of steady, incompressible, rotational, axisymmetric, and quasiviscous flow.¹ Despite being strictly inviscid, its streamlines observed the no slip requirement along the porous wall. It also coincided with Taylor's expression obtained a decade earlier, albeit in an entirely different physical context.² As noted by many specialists in the propulsion field, the corresponding mean flow was driven by inviscid pressure forces and did not need viscosity to exhibit vorticity or satisfy the apparent no-slip condition at the porous sidewall. Furthermore, the often cited Taylor-Culick profile was repeatedly verified in a number of investigations. These start with the inventive tests reported by Taylor² and continue to those carried out in later years by way of computation (Dunlap, Willoughby, and Hermsen;³ Baum, Levine, and Lovine;⁴ Sabnis, Gibeling, and McDonald⁵), laboratory experiments (Yamada, Goto, and Ishikawa;⁶ Dunlap *et al.*⁷), and theory (Clayton;⁸ Balachandar, Buckmaster, and Short;⁹ Majdalani and Roh;¹⁰ Majdalani and Flandro¹¹). In short, a collective body of research has confirmed the suitability of the Taylor-Culick model in approximating the bulk flow in a full-length cylindrical motor (Kuentzmann;¹² Traineau, Hervat, and Kuentzmann;¹³ Apte and Yang¹⁴). Due to its robustness, this profile has stood at the foundation of many theoretical studies, especially, those concerned with wave propagation (Majdalani and Flandro¹¹) and both hydrodynamic and combustion instability theories in porous chambers including those that account for particle interactions (Griffond and Casalis;^{15,16} Féraille and Casalis¹⁷).

The Taylor-Culick profile has also been extended to

simulate solid rocket motors with regressing walls. This was accomplished using a nozzleless, nonreactive, rotational, viscous, and incompressible approximation that employs similarity in time to model the expansion pattern of the porous wall.¹⁸ It has also been submitted by Majdalani and Vyas¹⁹ as a basic model for simulating the bulk motion in hybrid rockets exhibiting circular-port fuel grains. This was achieved by imposing a sinusoidal headwall injection profile to mimic oxidizer injection. In this article, we extend the solution by incorporating variable headwall injection in the context of steady, incompressible, axisymmetric, inviscid, and rotational flow. We first derive the solution for uniform headwall injection to the extent of making it applicable to solid and hybrid rockets in which the inflow at the headwall is nearly uniform. The headwall-to-sidewall injection velocity ratio will be considerably larger in the case of hybrids, thus leading to the onset of stream tube motion. The present solution, based on the vorticity stream function approach, will fully capture this behavior. The resulting formulation will provide an elemental approximation as it discounts the effects of compressibility, mixing, viscosity, and chemical reactions. However, by satisfying the no slip condition on the walls, the solution will exhibit a quasiviscous trait akin to that displayed by the Taylor-Culick profile. The same may be said of the solution that we then formulate for arbitrary headwall injection. The ensuing representation will permit the incorporation of practical injection scenarios that can be verified numerically.

II. MATHEMATICAL MODEL

As usual, a rocket motor can be idealized as a cylindrical chamber of porous length L_0 and radius a with both a reactive headwall and a nozzleless aft end (see Fig. 1). At the headwall, a fluid stream (which may denote an oxidizer or gaseous propellant mixture) is injected into the chamber at a prescribed velocity \vec{u}_0 ; this could be given by

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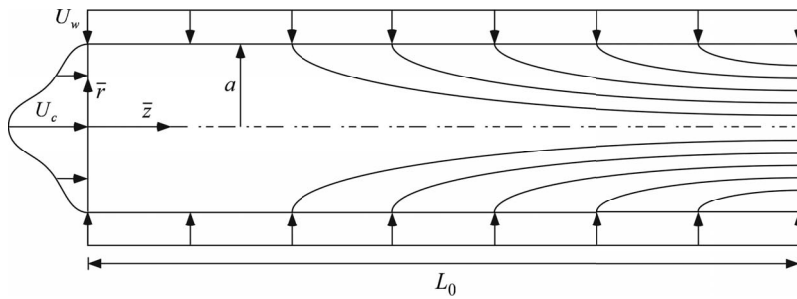


FIG. 1. Sketch of a porous tube with variable headwall injection.

$$\bar{u}_0(\bar{r}) = \begin{cases} U_c = \text{const}; & \text{uniform} \\ U_c \cos\left(\frac{1}{2}\pi\bar{r}^2/a^2\right); & \text{half-cosine} \\ U_c[1 - (\bar{r}/a)^m]; & \text{laminar and turbulent} \\ U_c(1 - \bar{r}/a)^{1/m}; & \text{turbulent} \end{cases} \quad (1)$$

where $U_c = \bar{u}_z(0, 0)$ is the centerline velocity at the headwall (a constant), m is some integer, and the overbar denotes dimensional variables. The incoming stream merges with the crossflow sustained by uniform mass addition along the porous sidewall. Naturally, the sidewall injection velocity $U_w = -\bar{u}_r(a, \bar{z})$ is commensurate with propellant or fuel regression rates. In hybrids, U_w can be appreciably smaller than U_c due to slow fuel pyrolysis; in SRM analysis, these two values are identical. As shown in Fig. 1, \bar{r} and \bar{z} stand for the radial and axial coordinates used to describe the solution from the headwall to the typical nozzle attachment point at the aft end. The solution that we seek applies, in particular, to simulated rocket motors and, in general, to injection-driven porous tubes with headwall injection.

A. Normalization

For expediency, it is helpful to normalize all recurring variables and operators. This can be done by setting

$$r = \frac{\bar{r}}{a}; \quad z = \frac{\bar{z}}{a}; \quad \nabla = a\nabla; \quad p = \frac{\bar{p}}{\rho U_w^2}; \quad \psi = \frac{\bar{\psi}}{a^2 U_w}; \quad (2)$$

$$u_r = \frac{\bar{u}_r}{U_w}; \quad u_z = \frac{\bar{u}_z}{U_w}; \quad \Omega = \frac{\bar{\Omega}a}{U_w}; \quad u_c = \frac{U_c}{U_w}; \quad L = \frac{L_0}{a}. \quad (3)$$

For steady inviscid motion, the vorticity transport equation reduces to

$$\nabla \times (\mathbf{u} \times \Omega) = 0; \quad \Omega = \nabla \times \mathbf{u}. \quad (4)$$

An assortment of four boundary conditions can be prescribed by writing

$$\begin{aligned} u_r(0, z) &= 0; & \text{no flow across centerline} \\ u_z(1, z) &= 0; & \text{no slip at sidewall} \\ u_r(1, z) &= -1; & \text{constant radial inflow at sidewall} \\ u_z(r, 0) &= u_0(r); & \text{axial inflow at headwall} \end{aligned} \quad (5)$$

$$u_0(r) = \begin{cases} u_c = \text{const} \\ u_c \cos\left(\frac{1}{2}\pi r^2\right) \\ u_c(1 - r^m) \\ u_c(1 - r)^{1/m}. \end{cases}$$

B. Vorticity-stream function approach

Continuity is fulfilled by the Stokes stream function when it is written as

$$u_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}; \quad u_z = \frac{1}{r} \frac{\partial \psi}{\partial r}. \quad (6)$$

Substitution into Eq. (4) requires

$$\Omega = rF(\psi), \quad (7)$$

so we follow tradition¹ and set

$$\Omega = C^2 r \psi. \quad (8)$$

Despite the nonuniqueness of this relation, it enables us to satisfy Eq. (4). Straightforward substitution into the vorticity equation yields the standard PDE,

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + C^2 r^2 \psi = 0 \quad (9)$$

with the particular set of constraints,

$$\lim_{r \rightarrow 0} \frac{1}{r} \frac{\partial \psi(r, z)}{\partial z} = 0 \text{ (a);} \quad \frac{\partial \psi(1, z)}{\partial r} = 0 \text{ (b);} \quad (10)$$

$$\frac{1}{r} \frac{\partial \psi(1, z)}{\partial z} = 1 \text{ (c);} \quad \frac{1}{r} \frac{\partial \psi(r, 0)}{\partial r} = u_0(r) \text{ (d).}$$

By virtue of L'Hôpital's rule, removing the singularity in Eq. (10)(a) requires that both

$$\frac{\partial \psi(0, z)}{\partial z} = 0 \text{ (a)} \quad \text{and} \quad \frac{\partial^2 \psi(0, z)}{\partial r \partial z} = 0 \text{ (b).} \quad (11)$$

Being linear, Eq. (9) is solvable by separation of variables; it yields

