# Acoustic streaming in simplified liquid rocket engines with transverse mode oscillations

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This study considers a simplified model of a liquid rocket engine in which uniform injection is imposed at the faceplate. The corresponding cylindrical chamber has a small length-to-diameter ratio with respect to solid and hybrid rockets. Given their low chamber aspect ratios, liquid thrust engines are known to experience severe tangential and radial oscillation modes more often than longitudinal ones. In order to model this behavior, tangential and radial waves are superimposed onto a basic mean-flow model that consists of a steady, uniform axial velocity throughout the chamber. Using perturbation tools, both potential and viscous flow equations are then linearized in the pressure wave amplitude and solved to the second order. The effects of the headwall Mach number are leveraged as well. While the potential flow analysis does not predict any acoustic streaming effects, the viscous solution carried out to the second order gives rise to steady secondary flow patterns near the headwall. These axisymmetric, steady contributions to the tangential and radial traveling waves are induced by the convective flow motion through interactions with inertial and viscous forces. We find that suppressing either the convective terms or viscosity at the headwall leads to spurious solutions that are free from streaming. In our problem, streaming is initiated at the headwall, within the boundary layer, and then extends throughout the chamber. We find that nonlinear streaming effects of tangential and radial waves act to alter the outer solution inside a cylinder with headwall injection. As a result of streaming, the radial wave velocities are intensified in one-half of the domain and reduced in the opposite half at any instant of time. Similarly, the tangential waves are either enhanced or weakened in two opposing sectors that are at 90° angle to the radial velocity counterparts. The second-order viscous solution that we obtain clearly displays both an oscillating and a steady flow component. The steady part can be an important contributor to wave steepening, a mechanism that is often observed during the onset of acoustic instability. © 2010 American Institute of Physics. [doi:10.1063/1.3407663]

# I. INTRODUCTION

Combustion instability in liquid rocket engines is characterized by large amplitude pressure fluctuations, elevated mean pressures, and frequencies that closely match linear chamber acoustics.<sup>1,2</sup> Owing to this fact, analytical methodologies put forward to describe flow oscillations lean heavily on the assumption of small acoustic disturbances.<sup>3–17</sup> Contrary to this assumption, however, a vast body of experimental evidence conveys a dissimilar picture, specifically, one involving large amplitude oscillations with steep gradients in flow variables. For example, in the extensive experimental studies of Clayton, Sotter, and co-workers,<sup>18-21</sup> a heavily instrumented, laboratory scale, 20 000 lbf thrust engine was used to investigate high amplitude tangential oscillations. Their measurements exhibited sustained, steep-fronted pressure fluctuations with peak-to-peak amplitudes that were an order of magnitude larger than the chamber's operating pressure. The pressure transducers available at the time could not record data rapidly enough to determine if a true discontinuity was present, but the acquired wave forms displayed large amplitude spikes followed by long and shallow pressure segments.

Theoretical work attributed to Maslen and Moore<sup>22</sup> suggested that tangential waves could not steepen as in the case of plane waves. In their 1956 paper, these investigators studied the effects of secondary flows on tangential wave patterns. A circular cylinder with a zero mean flow was utilized to detail the interaction between tangential waves and the chamber's sidewall. Specifically, the secondary flow induced by viscous forces at the sidewall was described. Their analysis yielded a streaming profile that acted in the direction opposite to the wave spinning motion. As a result, it was speculated that steep fronted, shocklike waves could not be produced due to sidewall scattering and viscous dissipation. Later, a study by Flandro<sup>23</sup> that incorporated a mean flow along with mass transpiration from the sidewall predicted a streaming flow in the same direction as the first-order wave. This result was found to be dependent on the magnitude of the injection Mach number. Given the dissimilar views in the role of acoustic streaming on the production of transverse traveling waves, its origination, manifestation, and influence on wave steepening will be the chief focus of this study.

With extensive work already in place for the treatment of radial boundary layers forming over an injecting

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FIG. 1. Chamber geometry and coordinate system.

sidewall,<sup>24–28</sup> the present article also seeks to investigate the structure of the unsteady axial boundary layers forming over an injecting headwall in the presence of transverse waves. The motivation to tackle the axial boundary layer analog is inspired by experimental observations suggesting that the highest pressure amplitudes and therefore most severe waves often occur near the injector face.<sup>18–21</sup>

The mechanisms that cause a plane wave to steepen are well understood. At the pressure peak the local speed of sound is elevated, thus increasing the local wave propagation rate. At the outset, the crest of the wave overtakes the depressed pressure portion. The curled-up wave continues to steepen until the solution becomes multivalued when nonlinear forces act to reverse this trend. The present study will demonstrate how secondary streaming flows induced at a liquid engine's injector face can stimulate a similar steepening process for tangentially traveling waves. In order to model this behavior, tangential and radial waves will be superimposed onto a simple mean-flow model. Considerable effort will then be given to satisfy the no-slip condition at the engine's injector face. The representative geometry, displayed in Fig. 1, will correspond to that of a semi-infinite cylinder of radius R with a suitable coordinate system anchored at the chamber's headwall. The z-coordinate will be located along the chamber's centerline.

#### **II. FORMULATION**

For simplicity, we begin the analysis by normalizing all standard variables bearing an asterisk according to

$$\begin{cases} p = p^*/P_0 & \mathbf{u} = \mathbf{u}^*/a_0 & r = r^*/R & T = T^*/T_0 \\ \rho = \rho^*/\rho_0 & t = t^*/(R/a_0) & z = z^*/R & \boldsymbol{\varpi} = \boldsymbol{\varpi}^*/(a_0/R) \end{cases}$$
(1)

where  $\boldsymbol{\varpi}$  and  $a_0$  denote the circular frequency and speed of sound, respectively; as usual, the zero subscript is used to denote a mean flow property. Using  $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$  to denote the

vorticity, the dimensionless equations written for a viscous compressible fluid consist of

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$
 (continuity), (2)

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla \mathbf{u}^2 - \mathbf{u} \times \boldsymbol{\omega} \right)$$
  
=  $-\frac{1}{\gamma} \nabla p - \delta^2 \nabla \times (\nabla \times \mathbf{u}) + \delta_d^2 \nabla (\nabla \cdot \mathbf{u})$   
+ **F** (momentum) (3)

$$p = \rho T \quad \text{(state)},\tag{4}$$

$$p = \rho^{\gamma}$$
 (isentropic relation). (5)

Here  $\mathbf{F}$  is the body force whereas the viscous and dilatational parameters that appear in Eq. (3) are defined as

$$\delta = \sqrt{\nu/(a_0 R)}; \quad \delta_d = \delta \sqrt{\mu'/\mu + \frac{4}{3}}.$$
 (6)

The symbols  $\nu = \mu/\rho$ ,  $\mu'$ , and  $\gamma$  represent the kinematic viscosity, the second coefficient of viscosity, and the ratio of specific heats, respectively. The energy and species diffusion equations are not listed due to the analysis being based on a nonreactive, single-phase, homogeneous, calorically perfect gas assumption.

#### A. Unsteady flow equations

Decomposing the flow variables into steady and unsteady parts can be achieved by setting

$$\mathbf{u} = \overline{\mathbf{U}} + \mathbf{u}'; \quad \boldsymbol{\omega} = \overline{\mathbf{\Omega}} + \boldsymbol{\omega}'; \quad p = \overline{P} + p';$$

$$\boldsymbol{\rho} = \overline{\boldsymbol{\rho}} + p'; \quad T = \overline{T} + T',$$
(7)

where overbars denote mean flow properties and primes represent unsteady variables. Having normalized the velocity by the speed of sound, the mean flow component may be related to the headwall injection Mach number  $M_b = V_b/a_0$  using  $\overline{U} = M_b U$ . Vorticity is similarly expressed as  $\overline{\Omega} = M_b \Omega$ . Direct substitution of Eq. (7) into the governing equations enables us to isolate two sets of steady and unsteady equations. Subsequently, a perturbation expansion may be implemented to linearize the unsteady equations. This is accomplished by expanding each fluctuation a' in terms of a sequence in the pressure wave parameter,

$$a' = \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \varepsilon^3 a^{(3)} + \cdots .$$
(8)

Here *a* represents a generic flow variable, and  $\varepsilon$  is the wave parameter, the ratio of the unsteady pressure amplitude and the mean pressure. Retaining terms to the second order in  $\varepsilon$  enables us to capture the acoustic streaming effect. As elegantly described by Schlichting<sup>29</sup> (p. 431), the secondary, streaming flow "has its origin in the convective terms and is due to the interaction between inertia and viscosity."

TABLE I. Boundary conditions for both potential and viscous flow analyses.

		Boundary		
	r=R	<i>z</i> =0	z=L	
Potential flow	$\mathbf{n} \cdot \nabla p' = 0 \longrightarrow u_r' = 0$	$\mathbf{n} \cdot \nabla p' = 0 \longrightarrow u_z' = 0$	$\mathbf{n} \cdot \nabla p' = 0 \longrightarrow u_z' = 0$	
Viscous flow	No condition imposed	No slip: $u'_r = u'_{\theta} = u'_z = 0$	Potential flow (matching)	

Furthermore, "simplifications in which the convective terms have been omitted lead to solutions which are free from streaming and may, therefore, give a misleading representation of the flow. Streaming does, in general, appear only when the solution is carried out to at least the second-order approximation." Bearing this requirement in mind, we perform some algebra and collect the first and second-order sets of equations, specifically,

$$\begin{cases} \frac{\partial \rho^{(1)}}{\partial t} = -\nabla \cdot \mathbf{u}^{(1)} - M_b \nabla \cdot [\rho^{(1)} \mathbf{U}] \\ \frac{\partial \mathbf{u}^{(1)}}{\partial t} + M_b \rho^{(1)} \frac{\partial \mathbf{U}}{\partial t} - M_b^2 \rho^{(1)} \mathbf{U} \times \mathbf{\Omega} = -\frac{\nabla p^{(1)}}{\gamma} - M_b [\mathbf{u}^{(1)} \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{u}^{(1)}] + \mathbf{F}^{(1)} + M_b \mathbf{u}^{(1)} \times \mathbf{\Omega} + M_b \mathbf{U} \times \boldsymbol{\omega}^{(1)} , \qquad (9) \\ - \delta^2 \nabla \times \boldsymbol{\omega}^{(1)} + \delta_d^2 \nabla [\nabla \cdot \mathbf{u}^{(1)}] \end{cases}$$

and

$$\frac{\partial \rho^{(2)}}{\partial t} = -\nabla \cdot \mathbf{u}^{(2)} - \nabla \cdot [\rho^{(1)} \mathbf{u}^{(1)}] - M_b \nabla \cdot [\rho^{(2)} \mathbf{U}]$$

$$\frac{\partial \mathbf{u}^{(2)}}{\partial t} + M_b \rho^{(2)} \frac{\partial \mathbf{U}}{\partial t} - M_b^2 \rho^{(2)} \mathbf{U} \times \mathbf{\Omega} = -\frac{\nabla p^{(2)}}{\gamma} - M_b \{ \mathbf{u}^{(2)} \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{u}^{(2)} + \rho^{(1)} [\mathbf{u}^{(1)} \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{u}^{(1)}] \}$$

$$- \frac{1}{2} \rho^{(2)} M_b^2 \mathbf{U} \cdot \nabla \mathbf{U} + \rho^{(1)} M_b \mathbf{U} \times \boldsymbol{\omega}^{(1)} + M_b \rho^{(1)} \mathbf{u}^{(1)} \times \mathbf{\Omega} + M_b \mathbf{U} \times \boldsymbol{\omega}^{(2)} \quad . \quad (10)$$

$$+ \mathbf{u}^{(1)} \times \boldsymbol{\omega}^{(1)} - \rho^{(1)} \frac{\partial \mathbf{u}^{(1)}}{\partial t} - \mathbf{u}^{(1)} \cdot \nabla \mathbf{u}^{(1)} + \mathbf{F}^{(2)} - \delta^2 \nabla \times \boldsymbol{\omega}^{(2)} + \delta_d^2 \nabla [\nabla \cdot \mathbf{u}^{(2)}]$$

$$\sqrt{p^{(2)}} = T^{(2)} + T^{(1)} \rho^{(1)} + \rho^{(2)}$$

To set the stage, our plan is to apply a procedure that will require the construction of both a potential (outer) solution and a corresponding viscous (inner) solution. The boundary conditions that will be employed in each part of the analysis are cataloged in Table I. Accordingly, the potential motion will be subject to the rigid wall boundary condition through which the flow velocity is required to vanish at the chamber walls. The viscous flow solution, on the other hand, will make use of the no-slip condition at the injector face, at z=0, where acoustic streaming is triggered. In applying the concepts of matched asymptotic theory, the potential flow solution will be used as the outer, farfield boundary for the viscous expansion. Finally, the thin boundary layer forming along the lateral wall, at r=R, will be ignored, consistently with the assumption of a proportionately large circular faceplate relative to a thin viscous region.

#### B. Headwall injection flow field

It may be instructive to note that Eqs. (9) and (10) represent the interaction equations that prescribe the unsteady wave motion in an idealized liquid rocket thrust engine. Both expressions of mass conservation and momentum balance are strongly influenced by the headwall injection Mach number  $M_b$  and the mean flow field velocity function U. In the context of a liquid rocket chamber, we recognize that the injection process at the headwall can be superbly complex. However, we also realize that despite the inherent complexity of the injection patterns, a streamtube motion is quickly established. Bearing these factors in mind, we adopt a simple representation of the mean flow field that consists of a uniform stream with constant velocity. This basic approximation will be necessary to simplify the problem and, in the process, help elucidate the underpinning physical mechanisms with

minimal algebraic encumbrance. Further complexity in the mean flow definition can be pursued at a later time. It should be kept in mind, however, that the uniform flow assumption is accompanied by certain limitations; these will be brought to light later in the analysis. With the near injector faceplate as the principal region of interest, we assume steady injection. We then introduce the nondimensional mean flow  $\overline{\mathbf{U}}=M_b\mathbf{U}$  where

$$\mathbf{U} = \mathbf{e}_{z}.\tag{11}$$

This basic representation is illustrated in Fig. 1.

# **III. POTENTIAL FLOW SOLUTION**

Away from the headwall region, viscous effects are confined to a thin boundary layer along the lateral, noninjecting sidewall. At the outset, a potential inviscid field may be assumed in the downstream region that is sufficiently removed from the injectors. Such a potential flow representation plays the role of an outer solution with respect to the flow adjacent to the headwall. By discounting viscosity, one is left with a set of wavelike equations that are described next.

# A. First-order potential solution

A combination of the first-order momentum and continuity equations delivers an expression for the unsteady wave motion to  $\mathcal{O}(\varepsilon)$ . Then, given the isentropic flow assumption, linearization of the pressure and density relation given by Eq. (5) yields

$$\gamma \rho^{(1)} = p^{(1)}.$$
 (12)

As usual, constructing the wave equation requires taking the time derivative of the continuity equation and subtracting from it the divergence of the momentum equation. One readily obtains

$$\nabla^2 p^{(1)} - p_{tt}^{(1)} = -\frac{\partial}{\partial t} \{ M_b \,\nabla \,\cdot [p^{(1)} \mathbf{U}] \} + \gamma M_b \nabla^2 [\mathbf{U} \cdot \mathbf{u}^{(1)}].$$
(13)

Given that the right-hand side in the above is of  $\mathcal{O}(M_b)$ , the first-order pressure can be represented by a dual perturbation expansion in  $M_b$ . Then to seek general solutions for  $p^{(1)}$  and  $\mathbf{u}^{(1)}$ , we first derive general expressions for the expanded subcomponents,  $p^{(1,0)}$  and  $\mathbf{u}^{(1,0)}$ . Thus using a' to denote a generic fluctuating variable, each level in the pressure wave parameter may be extended successively as

$$\begin{cases} a' = \varepsilon [a^{(1,0)} + M_b a^{(1,1)} + M_b^2 a^{(1,2)} + \cdots] + \varepsilon^2 [a^{(2,0)} + M_b a^{(2,1)} + M_b^2 a^{(2,2)} + \cdots] + \cdots \\ p' = \varepsilon [p^{(1,0)} + M_b p^{(1,1)} + M_b^2 p^{(1,2)} + \cdots] + \varepsilon^2 [p^{(2,0)} + M_b p^{(2,1)} + M_b^2 p^{(2,2)} + \cdots] + \cdots \\ \mathbf{u}' = \varepsilon [\mathbf{u}^{(1,0)} + M_b \mathbf{u}^{(1,1)} + M_b^2 \mathbf{u}^{(1,2)} + \cdots] + \varepsilon^2 [\mathbf{u}^{(2,0)} + M_b \mathbf{u}^{(2,1)} + M_b^2 \mathbf{u}^{(2,2)} + \cdots] + \cdots \end{cases}$$
(14)

or

$$p^{(1)} = p^{(1,0)} + M_b p^{(1,1)} + M_b^2 p^{(1,2)} + \cdots$$

$$p^{(2)} = p^{(2,0)} + M_b p^{(2,1)} + M_b^2 p^{(2,2)} + \cdots$$

$$\mathbf{u}^{(1)} = \mathbf{u}^{(1,0)} + M_b \mathbf{u}^{(1,1)} + M_b^2 \mathbf{u}^{(1,2)} + \cdots$$

$$\mathbf{u}^{(2)} = \mathbf{u}^{(2,0)} + M_b \mathbf{u}^{(2,1)} + M_b^2 \mathbf{u}^{(2,2)} + \cdots$$
(15)

Subsequent expansions of the first-order wave equation yield

$$\begin{cases} \nabla^2 p^{(1,0)} - p_{tt}^{(1,0)} = 0\\ \mathbf{n} \cdot \nabla p^{(1,0)}|_{z=0} = 0; \quad \mathbf{n} \cdot \nabla p^{(1,0)}|_{r=1} = 0 \end{cases},$$
(16)

$$\begin{cases} \nabla^2 p^{(1,1)} - p_{tt}^{(1,1)} = -\mathbf{U} \cdot \nabla p_t^{(1,0)} + \gamma \nabla^2 [\mathbf{U} \cdot \mathbf{u}^{(1,0)}] \\ \mathbf{n} \cdot \nabla p^{(1,1)}|_{z=0} = 0; \quad \mathbf{n} \cdot \nabla p^{(1,1)}|_{r=1} = 0 \end{cases}, \quad (17)$$

and

$$\begin{cases} \nabla^2 p^{(1,2)} - p_{tt}^{(1,2)} = -\mathbf{U} \cdot \nabla p_t^{(1,1)} + \gamma \nabla^2 [\mathbf{U} \cdot \mathbf{u}^{(1,1)}] \\ \mathbf{n} \cdot \nabla p^{(1,2)}|_{z=0} = 0; \quad \mathbf{n} \cdot \nabla p^{(1,2)}|_{r=1} = 0 \end{cases}$$
(18)

Note that the appropriate boundary condition forces the normal projection of the pressure gradient to vanish at all chamber surfaces. To solve Eq. (16), separation of variables may be used to derive the first-order pressure in the form of  $p^{(1,0)} = F(r)G(\theta)H(z)\Gamma(t)$ . At the outset, the wave equation collapses into

$$\frac{d^2F}{dr^2}\frac{1}{F} + \frac{1}{r}\frac{dF}{dr}\frac{1}{F} + \frac{1}{r^2}\frac{d^2G}{d\theta^2}\frac{1}{G} - \frac{d^2\Gamma}{dt^2}\frac{1}{\Gamma} = -\frac{d^2H}{dz^2}\frac{1}{H} = k_l^2.$$
(19)

Equation (19) may be rearranged into

$$\begin{cases} \frac{d^2 F}{dr^2} \frac{1}{F} + \frac{1}{r} \frac{dF}{dr} \frac{1}{F} + \frac{1}{r^2} \frac{d^2 G}{d\theta^2} \frac{1}{G} - \frac{d^2 \Gamma}{dt^2} \frac{1}{\Gamma} - k_l^2 = 0\\ \frac{d^2 H}{dz^2} + k_l^2 H = 0 \end{cases}$$
(20)

On this basis, a longitudinal wave solution of the form  $H(z) = \cos(k_l z)$  may be realized. In the present work, the longitudinal wave number  $k_l$  is deliberately set to zero in order to isolate the tangential and radial wave contributions. One is left with the radial, azimuthal, and temporal ODEs,

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$$r^{2}\frac{d^{2}F}{dr^{2}}\frac{1}{F} + r\frac{dF}{dr}\frac{1}{F} - r^{2}\frac{d^{2}\Gamma}{dt^{2}}\frac{1}{\Gamma} = -\frac{d^{2}G}{d\theta^{2}}\frac{1}{G} = m^{2}.$$
 (21)

So on the one hand, knowing that the  $\theta$ -dependence cannot be multivalued,  $G(\theta)$  becomes

$$G = A_{\theta} e^{im\theta}.$$
 (22)

On the other hand, the radial and temporal dependence may be separated from

$$\frac{d^2F}{dr^2}\frac{1}{F} + \frac{1}{r}\frac{dF}{dr}\frac{1}{F} - \frac{m^2}{r^2} = \frac{d^2\Gamma}{dt^2}\frac{1}{\Gamma} = -K^2,$$
(23)

or

$$\begin{cases} \frac{d^2 F}{dr^2} \frac{1}{F} + \frac{1}{r} \frac{dF}{dr} \frac{1}{F} + \left(K^2 - \frac{m^2}{r^2}\right) = 0\\ \frac{d^2 \Gamma}{dt^2} + K^2 \Gamma = 0 \end{cases}$$
(24)

The classical form of the solution to Eq. (24) gives

$$F = A_r J_m(k_{mn}r) + B_r Y_m(k_{mn}r),$$
(25)

where  $k_{mn} \equiv K$  and  $(J_m, Y_m)$  represent the *m*th order Bessel functions of the first and second kind, respectively. In the time domain the solution to Eq. (24) can be represented by complex variables as

$$\Gamma = e^{-ik_{mn}t},\tag{26}$$

with  $k_{mn} = \varpi_0 R/a_0$ . For the assumed sinusoidal variation in time, initial conditions are immaterial to the character of the solution. Without loss of generality, the product of tangential, radial, and temporal contributions may be expressed as

$$p_{c}^{(1,0)} = [AJ_{m}(k_{mn}r)e^{im\theta} + BY_{m}(k_{mn}r)e^{im\theta}]e^{-ik_{mn}t}.$$
 (27)

Where the subscript *c* denotes a complex variable. Equation (27) admits a finite pressure at the centerline (i.e., B=0) and a vanishing pressure gradient at the impermeable wall,  $J'_m(k_{mn})=0$ . Using unit normalization Eq. (27) becomes

$$p_{c}^{(1,0)} = J_{m}(k_{mn}r)e^{i(m\theta - k_{mn}t)}.$$
(28)

Subsequently, care is exerted in extracting the real component of a given variable before substitution back into the governing equations that are constructed assuming real quantities. For example,  $p^{(1,0)}$ , the first order in  $\varepsilon$  and zeroth order in  $M_b$  approximation for the pressure becomes

$$p^{(1,0)} = \cos(m\theta - k_{mn}t)J_m(k_{mn}r);$$

$$m = 0, 1, 2, \dots; \quad n = 0, 1, 2, \dots,$$
(29)

where  $k_{mn}$  is determined by the roots of the first derivative of the Bessel function of order m,  $J'_m(k_{mn})=0$ . One finds

$$k_{01} \simeq 3.831\ 705\ 97, \ k_{10} \simeq 1.841\ 183\ 78,$$
  
 $k_{11} \simeq 5.331\ 442\ 77, \ k_{02} \simeq 7.0155\ 866\ 7,$  (30)  
 $k_{20} \simeq 3.054\ 236\ 93, \ \text{etc.}$ 

Being chiefly concerned with the effect of tangential wave motion at the headwall, the first spinning mode of interest is  $k_{10}$ . Note that Eq. (29) captures both tangential and radial oscillation modes. Using Eqs. (9) and (14) produces a set of equations representing the first-order potential velocity

$$\frac{\partial \mathbf{u}^{(1,0)}}{\partial t} = -\frac{\nabla p^{(1,0)}}{\gamma} , \qquad (31)$$
  
$$[\mathbf{n} \cdot \mathbf{u}^{(1,0)}]_{z=0} = 0; \quad \mathbf{n} \cdot \mathbf{u}^{(1,0)}|_{r=1} = 0$$

$$\begin{cases} \frac{\partial \mathbf{u}^{(1,1)}}{\partial t} = -\frac{\nabla p^{(1,1)}}{\gamma} - \mathbf{u}^{(1,0)} \cdot \nabla \mathbf{U} - \mathbf{U} \cdot \nabla \mathbf{u}^{(1,0)} \\ \mathbf{n} \cdot \mathbf{u}^{(1,1)}|_{z=0} = 0; \quad \mathbf{n} \cdot \mathbf{u}^{(1,1)}|_{r=1} = 0 \end{cases}, \quad (32)$$

and

$$\begin{cases} \frac{\partial \mathbf{u}^{(1,2)}}{\partial t} = -\frac{\nabla p^{(1,2)}}{\gamma} - \mathbf{u}^{(1,1)} \cdot \nabla \mathbf{U} - \mathbf{U} \cdot \nabla \mathbf{u}^{(1,1)} \\ \mathbf{n} \cdot \mathbf{u}^{(1,2)}|_{z=0} = 0; \quad \mathbf{n} \cdot \mathbf{u}^{(1,2)}|_{r=1} = 0 \end{cases}$$
(33)

The first order in  $\varepsilon$  and zeroth order in  $M_b$  inviscid velocity profile may be evaluated from Eq. (31). It gives

$$\mathbf{u}_{c}^{(1,0)} = -\frac{i}{\gamma K} J'_{m}(k_{mn}r) e^{i(m\theta - Kt)} \mathbf{e}_{r} + \frac{1}{\gamma K} \left(\frac{m}{r}\right) J_{m}(k_{mn}r) e^{i(m\theta - Kt)} \mathbf{e}_{\theta},$$
(34)

whence 
$$\mathbf{u}^{(1,0)} = \frac{1}{\gamma K} \sin(m\theta - k_{mn}t) J'_m(k_{mn}r) \mathbf{e}_r$$
  
  $+ \frac{1}{\gamma K} \left(\frac{m}{r}\right) \cos(m\theta - k_{mn}t) J_m(k_{mn}r) \mathbf{e}_{\theta}.$ 
(35)

To avoid the pitfalls of complex notation in the evaluation of nonlinear terms, the real part of the solution featured in Eqs. (29) and (35) will be used, as the solution is taken to second order, to represent the product of two oscillatory quantities. However, complex notation will be returned to in Sec. IV A, in the treatment of the first order viscous solution. By carrying the solution to higher orders in the injection Mach number, the same recursive formulation is obtained at every order. This behavior may be attributed to the assumptions that  $k_1=0$ , and U=1, thus leading to vanishing spatial derivatives of all unsteady flow variables in the *z*-direction. Then by summing all terms, one deduces

$$\begin{cases} \mathbf{u}_{c}^{(1)} = \left(\sum_{j=0}^{\infty} M_{b}^{j}\right) \mathbf{u}_{c}^{(1,0)} = \frac{1}{1 - M_{b}} \mathbf{u}_{c}^{(1,0)} \\ p_{c}^{(1)} = \left(\sum_{j=0}^{\infty} M_{b}^{j}\right) p_{c}^{(1,0)} = \frac{1}{1 - M_{b}} p_{c}^{(1,0)} = \frac{J_{m}(k_{mn}r)}{1 - M_{b}} e^{i(m\theta - k_{mn}t)} \end{cases}$$

and so, in the real domain,

$$\begin{cases} \mathbf{u}^{(1)} = \left(\sum_{j=0}^{\infty} M_b^j\right) \mathbf{u}^{(1,0)} = \frac{1}{1 - M_b} \mathbf{u}^{(1,0)} \\ p^{(1)} = \left(\sum_{j=0}^{\infty} M_b^j\right) p^{(1,0)} = \frac{1}{1 - M_b} p^{(1,0)} = \frac{J_m(k_{mn}r)}{1 - M_b} \cos(m\theta - k_{mn}t) \end{cases}$$

Note that the infinite series are reducible by use of the identity

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}.$$
(38)

By summing over an infinite series in the Mach number, the solution is captured exactly in  $M_b$ , specifically with a truncation error equal to

$$\lim_{j \to \infty} M_b^j = 0.$$
<sup>(39)</sup>

Therefore, for the remainder of the analysis, the highly accurate solution will be represented through the use of  $\mathbf{u}^{(1)}$  and  $p^{(1)}$ .



(36)

(37)

## B. Second-order potential solution

The second-order wave equation can be retrieved at  $\mathcal{O}(\epsilon^2)$ , with the outcome being

$$\nabla^{2} p^{(2)} - p_{tt}^{(2)} = \frac{1 - \gamma}{2\gamma} \{ [p^{(1)}]^{2} \}_{tt} + \nabla \cdot [p_{t}^{(1)} \mathbf{u}^{(1)}]$$
$$- \frac{1}{2} \gamma \nabla^{2} [\mathbf{u}^{(1)} \cdot \mathbf{u}^{(1)}] + M_{b} [\nabla \cdot p^{(2)} U]_{t}$$
$$- \gamma M_{b} \nabla \cdot \nabla [\mathbf{U} \cdot \mathbf{u}^{(2)}].$$
(40)

Parallel expansion in the Mach number can be performed using

$$p^{(2)} = p^{(2,0)} + M_b p^{(2,1)} + M_b^2 p^{(2,2)} + \cdots .$$
(41)

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This enables us to extract, at leading order in the Mach number and second order in the wave amplitude,

$$\begin{cases} \nabla^2 p^{(2,0)} - p_{tt}^{(2,0)} = \frac{1-\gamma}{2\gamma} \{ [p^{(1,0)}]^2 \}_{tt} + \nabla \cdot [p_t^{(1,0)} \mathbf{u}^{(1,0)}] - \frac{\gamma}{2} \nabla^2 [\mathbf{u}^{(1,0)} \cdot \mathbf{u}^{(1,0)}] \\ \mathbf{n} \cdot \nabla p^{(2,0)} |_{z=0} = 0; \quad \mathbf{n} \cdot \nabla p^{(2,0)} |_{r=1} = 0 \end{cases}$$
(42)

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Inserting the first-order flow field on the right-hand gives

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$$\nabla^2 p^{(2,0)} - p_{tt}^{(2,0)} = F(r) + B(r) \cos[2(m\theta - k_{mn}t)],$$
(43)

where

$$F(r) = \frac{1}{2\gamma k_{mn}^2} \begin{bmatrix} \left(\frac{3m^2}{r^3} + \frac{k_{mn}^2}{r}\right) J_m(k_{mn}r) J'_m(k_{mn}r) + \left(k_{mn}^2 - \frac{m^2}{r^2}\right) J'_m^2(k_{mn}r) + \left(k_{mn}^2 - \frac{m^2}{r^2}\right) J_m(k_{mn}r) J''_m(k_{mn}r) \\ - \frac{1}{r} J'_m(k_{mn}r) J''_m(k_{mn}r) - J''_m^2(k_{mn}r) - J''_m(k_{mn}r) J'''_m(k_{mn}r) - \frac{2m^2}{r^4} J_m^2(k_{mn}r) \end{bmatrix},$$
(44)

and

(48)

$$B(r) = \frac{1}{2\gamma k_{mn}^{2}} \left\{ 2 \left[ k_{mn}^{4}(\gamma - 1) + \frac{k_{mn}^{2}m^{2}}{r^{2}} - \frac{m^{2}(1 - m^{2})}{r^{4}} \right] J_{m}^{2}(k_{mn}r) + J_{m}^{\prime\prime\prime}^{\prime\prime}(k_{mn}r) \\ - \left( \frac{k_{mn}^{2}}{r} - \frac{3m^{2}}{r^{3}} \right) J_{m}(k_{mn}r) J_{m}^{\prime\prime}(k_{mn}r) - \left( k_{mn}^{2} + \frac{3m^{2}}{r^{2}} \right) J_{m}^{\prime\prime}^{\prime\prime}(k_{mn}r) \\ - \left( k_{mn}^{2} + \frac{m^{2}}{r^{2}} \right) J_{m}(k_{mn}r) J_{m}^{\prime\prime}(k_{mn}r) + \frac{J_{m}^{\prime\prime}(k_{mn}r) J_{m}^{\prime\prime\prime}(k_{mn}r)}{r} + J_{m}^{\prime\prime}(k_{mn}r) J_{m}^{\prime\prime\prime}(k_{mn}r) \right\}.$$

$$(45)$$

As in the previous section, the wave equation is further expanded in terms of the injection Mach number. The approximation to the set of second-order equations displays a pattern that is of familiar type. We find

$$p_p^{(2,0)} = H(r) + G(r)\cos[2(m\theta - k_{mn}t)], \qquad (47)$$

 $H(r) = -\frac{1}{4\gamma k_{mn}^2} \left\{ \left[ \left(\frac{m}{r}\right)^2 - k_{mn}^2 \right] J_m(k_{mn}r)^2 + J'_m(k_{mn}r)^2 \right\},\$ 

with

$$p^{(2)} = \sum_{j=0}^{\infty} M_b^j [(j+1)p_p^{(2,0)} + \cos(m\theta - k_{mn}t)J_m(k_{mn}r)]$$
  
= 
$$\frac{p^{(2,0)} + (1 - M_b)\cos(m\theta - k_{mn}t)J_m(k_{mn}r)}{(1 - M_b)^2},$$
 (46)

where the particular solution  $p_p^{(2,0)}$  is given by the juxtaposition of a steady and a time-dependent part,

and

$$\begin{cases} G(r) = \frac{1}{4\gamma k_{mn}^2} \left[ -\left(\frac{m_2}{r^2} + 3k_{mn}^2\right) J_m(k_{mn}r)^2 + J'_m(k_{mn}r)^2 + 4k_{mn}^4(\gamma - 1)f(r) \right]; \\ f'' + \frac{1}{r}f' - 4\left(\frac{m^2}{r^2} - k_{mn}^2\right) f = J_m(k_{mn}r)^2. \end{cases}$$
(49)

The second-order momentum equation may be expanded along similar lines. One gets

$$\mathbf{u}^{(2)} = \sum_{j=0}^{\infty} M_b^j [(j+1)\mathbf{u}_p^{(2,0)} + \mathbf{u}^{(1,0)}] = \frac{(1-M_b)\mathbf{u}^{(1,0)} + \mathbf{u}^{(2,0)}}{(1-M_b)^2},$$
(50)

where

$$\mathbf{u}_{p}^{(2,0)} = \begin{cases} -\frac{1}{2\gamma^{2}k_{mn}} [2J_{m}(k_{mn})J_{m}'(k_{mn}r) + (1-\gamma)k_{mn}^{2}f'(r)]\sin[2(m\theta - k_{mn}t)]\mathbf{e}_{r} \\ -\frac{1}{\gamma^{2}k_{mn}} \left(\frac{m}{r}\right) [J_{m}^{2}(k_{mn}r) + k_{mn}^{2}(1-\gamma)f(r)]\cos[2(m\theta - k_{mn}t)]\mathbf{e}_{\theta} \end{cases}$$
(51)

Unlike the second-order pressure  $p^{(2)}$ , the velocity in Eq. (51) does not comprise a steady, second-order streaming component akin to the time-independent term H(r) that arises in Eq. (47). In other streaming studies,<sup>18</sup> one may see both steady and unsteady second-order contributions to the velocity. In such models, the second order pressure gradient is either assumed or ignored. In the present study, the use of such an assumption is not required. Instead, we recall Schlichting's description<sup>29</sup> (p. 430), namely, that "a potential flow which is periodic with respect to time induces a steady, secondary ('streaming') motion... as a result of viscous

forces." In brief, a viscous model is vitally needed to suitably capture the second-order interactions, as attempted in similar context by Maslen and Moore.<sup>22</sup>

# **IV. VISCOUS FLOW**

Attention is now turned to the region directly above the headwall, specifically to the viscous boundary layer that must develop as a result of transverse shear parallel to the injector faceplate. This boundary layer is necessary to bring the transverse components of the velocity, both tangential and radial, to vanish at the surface. Friction at the headwall permits the attainment of a more realistic representation of the adjacent fluid motion. The ensuing flow field must, on the one hand, satisfy the no-slip condition at the headwall and, on the other hand, merge with the outer solution in the farfield. In the present study, we ignore the sidewall boundary layers and assume that all viscous effects are confined to a small region near the headwall.

# A. First-order viscous solution

In our attempt to unravel the acoustic streaming motion induced by viscous effects at the injector faceplate, the boundary layer equations at the headwall must be established. Following standard perturbation practices, a coordinate transformation is introduced such that the *z*-coordinate is rescaled by the square root of the inverse acoustic Reynolds number  $\delta = \sqrt{\nu/(a_0 R)}$ . The corresponding inner, slow variable becomes

$$\zeta = \frac{z}{\delta}.$$
 (52)

Starting with the first-order momentum equation,

$$\mathbf{u}_{t}^{(1)} = -M_{b} \nabla [\mathbf{U} \cdot \mathbf{u}^{(1)}] + M_{b} [\mathbf{U} \times \boldsymbol{\omega}^{(1)} + \mathbf{u}^{(1)} \times \boldsymbol{\Omega}^{0}] - (1/\gamma) \nabla p^{(1)} - \delta^{2} \nabla \times \boldsymbol{\omega}^{(1)} + \delta_{d}^{2} \nabla [\nabla \cdot \mathbf{u}^{(1)}], \quad (53)$$

an expansion in terms of  $\zeta$  leads to a set of three linear second-order partial differential equations (PDEs). Expanding the right-hand side of Eq. (53) gives, term-by-term,

$$\delta_{d}^{2} \nabla \left( \nabla \cdot \mathbf{u}^{(1)} \right) = \begin{cases} \delta_{d}^{2} \left[ \frac{\partial^{2} \mathbf{u}_{r}^{(1)}}{\partial r^{2}} + \frac{1}{r} \frac{\partial \mathbf{u}_{r}^{(1)}}{\partial r} - \frac{\mathbf{u}_{r}^{(1)}}{r^{2}} - \frac{1}{r^{2}} \frac{\partial \mathbf{u}_{\theta}^{(1)}}{\partial \theta} + \frac{1}{r} \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^{2} \mathbf{u}_{z}^{(1)}}{\partial \theta \partial r \partial \zeta} \right] \mathbf{e}_{r} \\ + \delta_{d}^{2} \left[ \frac{1}{r} \frac{\partial^{2} \mathbf{u}_{r}^{(1)}}{\partial \theta \partial r} + \frac{1}{r^{2}} \frac{\partial \mathbf{u}_{r}^{(1)}}{\partial \theta} + \frac{1}{r^{2}} \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r\delta} \frac{\partial^{2} \mathbf{u}_{z}^{(1)}}{\partial \theta \partial \zeta} \right] \mathbf{e}_{\theta} \\ + \delta_{d}^{2} \left[ \frac{1}{\delta} \frac{\partial^{2} \mathbf{u}_{r}^{(1)}}{\partial \zeta \partial r} + \frac{1}{r\delta} \frac{\partial \mathbf{u}_{r}^{(1)}}{\partial \zeta} + \frac{1}{r\delta} \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial \zeta \partial \theta} + \frac{1}{\delta^{2}} \frac{\partial^{2} \mathbf{u}_{z}^{(1)}}{\partial \zeta^{2}} \right] \mathbf{e}_{z} \end{cases}$$

$$(54)$$

$$-\delta^{2} \nabla \times \boldsymbol{\omega}^{(1)} = \begin{cases} -\delta^{2} \left[ \frac{1}{r} \left( \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial \mathbf{u}_{\theta}^{(1)}}{\partial \theta} - \frac{1}{r} \frac{\partial^{2} \mathbf{u}_{r}^{(1)}}{\partial \theta^{2}} \right) - \frac{1}{\delta} \left( \frac{1}{\delta} \frac{\partial^{2} \mathbf{u}_{r}^{(1)}}{\partial \zeta \partial r} - \frac{\partial^{2} \mathbf{u}_{z}^{(1)}}{\partial \zeta \partial r} \right) \right] \mathbf{e}_{r} \\ -\delta^{2} \left[ \frac{1}{\delta} \left( \frac{1}{r} \frac{\partial^{2} \mathbf{u}_{z}^{(1)}}{\partial \zeta \partial \theta} - \frac{1}{\delta} \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial \zeta^{2}} \right) - \left( \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial r^{2}} - \frac{\mathbf{u}_{\theta}^{(1)}}{r^{2}} + \frac{1}{r} \frac{\partial \mathbf{u}_{\theta}^{(1)}}{\partial r} - \frac{1}{\sigma^{2} \mathbf{u}_{r}^{(1)}} \right) \right] \mathbf{e}_{\theta} \\ - \frac{\delta^{2}}{r} \left[ \left( \frac{1}{\delta} \frac{\partial \mathbf{u}_{r}^{(1)}}{\partial \zeta} + \frac{r}{\delta} \frac{\partial^{2} \mathbf{u}_{r}^{(1)}}{\partial r \partial \zeta} - \frac{\partial \mathbf{u}_{z}^{(1)}}{\partial r} - \frac{\partial^{2} \mathbf{u}_{z}^{(1)}}{\partial r^{2}} \right) - \left( \frac{1}{r} \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial \theta^{2}} - \frac{1}{\delta} \frac{\partial^{2} \mathbf{u}_{\theta}^{(1)}}{\partial \theta \partial \zeta} \right) \right] \mathbf{e}_{z} \end{cases}$$
(55)

$$M_{b}\mathbf{U}\times\boldsymbol{\omega}^{(1)} = -M_{b}\left(\frac{1}{\delta}\frac{\partial\mathbf{u}_{r}^{(1)}}{\partial\zeta} - \frac{\partial\mathbf{u}_{z}^{(1)}}{\partial r}\right)\mathbf{e}_{r} + M_{b}\left(\frac{1}{r}\frac{\partial\mathbf{u}_{z}^{(1)}}{\partial\theta} - \frac{1}{\delta}\frac{\partial\mathbf{u}_{\theta}^{(1)}}{\partial\zeta}\right)\mathbf{e}_{\theta},\tag{56}$$

$$-M_b \nabla \left(\mathbf{U} \cdot \mathbf{u}^{(1)}\right) = -M_b \frac{\partial u_z^{(1)}}{\partial r} \mathbf{e}_r - \frac{M_b}{r} \frac{\partial u_z^{(1)}}{\partial \theta} \mathbf{e}_\theta - \frac{M_b}{\delta} \frac{\partial u_z^{(1)}}{\partial \zeta} \mathbf{e}_z,\tag{57}$$

and

$$-\frac{1}{\gamma}\nabla p^{(1)} = -\frac{1}{\gamma}\frac{\partial p^{(1)}}{\partial r}\mathbf{e}_r - \frac{1}{\gamma r}\frac{\partial p^{(1)}}{\partial \theta}\mathbf{e}_{\theta} - \frac{1}{\gamma\delta}\frac{\partial p^{(1)}}{\partial z}\mathbf{e}_z.$$
(58)

Equation (53) is hence transformed into a group of PDEs, namely,

$$\begin{cases} \left[ u_{r}^{(1)} \right]_{t} = \frac{-M_{b}}{\delta} \frac{\partial u_{r}^{(1)}}{\partial \zeta} - \frac{1}{\gamma} \frac{\partial p^{(1)}}{\partial r} - \delta^{2} \begin{bmatrix} \frac{1}{r} \left( \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial u_{\theta}^{(1)}}{\partial \theta} - \frac{1}{r} \frac{\partial^{2} u_{r}^{(1)}}{\partial \theta^{2}} \right) \\ -\frac{1}{\delta} \left( \frac{1}{\delta} \frac{\partial^{2} u_{r}^{(1)}}{\partial \zeta^{2}} - \frac{\partial^{2} u_{r}^{(1)}}{\partial \zeta \partial r} \right) \end{bmatrix} + \delta^{2} \begin{bmatrix} \frac{\partial^{2} u_{r}^{(1)}}{\partial r} - \frac{u_{r}^{(1)}}{r} - \frac{u_{r}^{(1)}}{r^{2}} - \frac{1}{r^{2}} \frac{\partial u_{\theta}^{(1)}}{\partial \theta} \\ + \frac{1}{r} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial r \partial \theta} + \frac{1}{\delta} \frac{\partial^{2} u_{z}^{(1)}}{\partial r \partial \delta r} \\ + \frac{1}{r} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial r \partial \theta} + \frac{1}{\delta} \frac{\partial^{2} u_{z}^{(1)}}{\partial r \partial \theta} \\ + \frac{1}{r} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial r \partial \theta} + \frac{1}{\delta} \frac{\partial^{2} u_{z}^{(1)}}{\partial r \partial \theta} \\ + \frac{1}{r} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial r \partial \theta} + \frac{1}{\delta} \frac{\partial^{2} u_{z}^{(1)}}{\partial r \partial \theta} \\ + \frac{1}{r} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial r \partial \theta} + \frac{1}{\delta} \frac{\partial^{2} u_{z}^{(1)}}{\partial r \partial \theta} \\ + \frac{1}{r} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial r \partial \theta} + \frac{1}{\delta} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta \theta} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta \theta} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta \theta} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta^{2}} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \theta^{2}} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{z}^{(1)}}{\partial \zeta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \zeta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \zeta} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \xi} \\ + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \theta^{2}} + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(1)}}{\partial \xi} \\ + \frac{1}{r^{2$$

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In seeking a leading order inner approximation, only terms that appear at the zeroth order in  $\delta$  are retained. This enables us to simplify Eq. (59), for the region near the wall, into

$$\begin{cases} [u_r^{(1)}]_l = -\frac{M_b}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} - \frac{1}{\gamma} \frac{\partial p^{(1)}}{\partial r} + \frac{\partial^2 u_r^{(1)}}{\partial \zeta^2} \\ [u_\theta^{(1)}]_l = -\frac{M_b}{\delta} \frac{\partial u_\theta^{(1)}}{\partial \zeta} - \frac{1}{\gamma r} \frac{\partial p^{(1)}}{\partial \theta} + \frac{\partial^2 u_\theta^{(1)}}{\partial \zeta^2} \\ [u_z^{(1)}]_l = -\frac{M_b}{\delta} \frac{\partial u_z^{(1)}}{\partial \zeta} - \frac{1}{\gamma \delta} \frac{\partial p^{(1)}}{\partial z} + \frac{\delta_d^2}{\delta^2} \frac{\partial^2 u_z^{(1)}}{\partial \zeta^2} \end{cases}$$
(60)

According to classic acoustic boundary layer theory, the viscous layer has a minimal bearing on the oscillatory pressure distribution. Thus in seeking solutions within the acoustic boundary layer region, pressure from the outer, potential flow solution will be used.

#### 1. Solution for the first order tangential velocity

To illustrate the procedural steps needed to solve this set, we start with the tangential equation,

$$\frac{\partial^2 u_{\theta}^{(1)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} - [u_{\theta}^{(1)}]_t = \frac{1}{\gamma r} \frac{\partial p^{(1)}}{\partial \theta}.$$
 (61)

Owing to the fact that  $M_b/\delta$  is not small, all terms on the left-hand side of Eq. (61) appear at the same order. At this time the complex variable representation of the outer pressure from the potential flow field is used to represent  $p^{(1)}$ ; one collects

$$\frac{\partial^2 u_{\theta,c}^{(1)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial u_{\theta,c}^{(1)}}{\partial \zeta} + ik_{mn} u_{\theta,c}^{(1)} = \frac{i}{\gamma} \left(\frac{m}{r}\right) \frac{J_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)}.$$
(62)

The return to complex notation is done in the interest of simplicity. The particular integral for Eq. (62) may be readily evaluated such that a compact solution is deduced. One gets

$$[u_{\theta,c}^{(1)}]_{p} = \frac{1}{\gamma k_{mn}} \left(\frac{m}{r}\right) \frac{J_{m}(k_{mn}r)}{(1-M_{b})} e^{i(m\theta - k_{mn}t)}.$$
(63)

In turn, the homogenous solution takes the form

$$[u_{\theta,c}^{(1)}]_h = A_\theta(r,\theta,t)e^{X_1\zeta} + B_\theta(r,\theta,t)e^{X_2\zeta},$$
(64)

with

$$(X_1, X_2) = \frac{M_b}{2\delta} (1 \pm \sqrt{1 - 4ik_{mn}\delta^2 M_b^{-2}})$$
$$= \frac{V_b}{2} \sqrt{\frac{R}{a_0\nu}} (1 \pm \sqrt{1 - 4ik_{mn}\delta^2 M_b^{-2}}), \tag{65}$$

or

$$X_{1} = \frac{M_{b}}{2\delta} \left( 1 + \sqrt{\frac{1 + \sqrt{1 + 16k_{mn}^{2}\delta^{4}M_{b}^{-4}}}{2}} - i\sqrt{\frac{-1 + \sqrt{1 + 16k_{mn}^{2}\delta^{4}M_{b}^{-4}}}{2}} \right),$$
(66)

$$X_{2} = \frac{M_{b}}{2\delta} \left( 1 - \sqrt{\frac{1 + \sqrt{1 + 16k_{mn}^{2}\delta^{4}M_{b}^{-4}}}{2}} + i\sqrt{\frac{-1 + \sqrt{1 + 16k_{mn}^{2}\delta^{4}M_{b}^{-4}}}{2}} \right).$$
(67)

Note that the real parts  $X_1^r > 0$  and  $X_2^r < 0$ . The total solution for the first-order boundary layer approximation becomes

$$u_{\theta,c}^{(1)} = A_{\theta}(r,\theta,t)e^{X_{1}\zeta} + B_{\theta}(r,\theta,t)e^{X_{2}\zeta} + \frac{1}{\gamma k_{mn}} \left(\frac{m}{r}\right) \frac{J_{m}(k_{mn}r)}{(1-M_{b})} e^{i(m\theta - k_{mn}t)}.$$
(68)

Knowing that the velocity cannot increase unboundedly as  $\zeta \rightarrow \infty$ , one must set  $A_{\theta}(r, \theta, t) = 0$  because  $X_1^r > 0$ . This leaves the second constant in Eq. (68) to satisfy the no-slip condition at the headwall. Thereafter, one puts

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$$u_{\theta,c}^{(1)}(r,\theta,0,t) = B_{\theta}(r,\theta,t) + \frac{1}{\gamma k_{mn}} \left(\frac{m}{r}\right) \frac{J_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)}$$
$$= 0, \tag{69}$$

whence

$$B_{\theta}(r,\theta,t) = -\frac{1}{\gamma k_{mn}} \left(\frac{m}{r}\right) \frac{J_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)},\tag{70}$$

and so

$$u_{\theta,c}^{(1)}(r,\theta,\zeta,t) = \frac{1}{\gamma k_{mn}} \left(\frac{m}{r}\right) \frac{J_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta-k_{mn}t)} (1-e^{X_2\zeta}).$$
(71)

It may be useful to remark that  $u_{\theta,c}^{(1)}(1, \theta, \zeta, t) \neq 0$ . The radial velocity fluctuation does not observe the velocity adherence condition at the sidewall. As stated earlier, this outcome is due to our deliberate dismissal of the sidewall boundary layer.

## 2. Solution for the first order radial velocity

In the radial direction, Eq. (60) yields

$$\frac{\partial^2 u_r^{(1)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} - [u_r^{(1)}]_t = \frac{1}{\gamma} \frac{\partial p^{(1)}}{\partial r}.$$
(72)

Substituting the complex notation pressure from the outer potential flow solution, we get

$$\frac{\partial^2 u_{r,c}^{(1)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial u_{r,c}^{(1)}}{\partial \zeta} + ik_{mn} u_{r,c}^{(1)} = \frac{1}{\gamma} \frac{J'_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)}.$$
 (73)

The particular integral delivers

$$[u_{r,c}^{(1)}]_p = -\frac{i}{\gamma k_{mn}} \frac{J'_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)},$$
(74)

with the homogenous solution being of the form

$$[u_{r,c}^{(1)}]_h = A_r(r,\theta,t)e^{X_1\zeta} + B_r(r,\theta,t)e^{X_2\zeta}.$$
(75)

Here one must set  $A_r(r, \theta, t) = 0$  to prevent unboundedness in the downstream direction. The complete solution for the first-order radial velocity approximation is therefore

$$u_{r,c}^{(1)} = B_r(r,\theta,t)e^{X_2\zeta} - \frac{i}{\gamma k_{mn}} \frac{J'_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)}.$$
 (76)

The no-slip condition at the headwall permits extracting the final unknown

$$u_{r,c}^{(1)}(r,\theta,0,t) = B_r(r,\theta,t) - \frac{i}{\gamma k_{mn}} \frac{J'_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)} = 0$$
(77)
or
$$B_r(r,\theta,t) = \frac{i}{\gamma k_{mn}} \frac{J'_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)}.$$

Backward substitution yields, at length,

$$u_{r,c}^{(1)}(r,\theta,\zeta,t) = \frac{i}{\gamma k_{mn}} \frac{J'_m(k_{mn}r)}{(1-M_b)} e^{i(m\theta - k_{mn}t)} (e^{X_2\zeta} - 1).$$
(78)

## 3. Solution for the first order axial velocity

The continuity equation can be used to extract the *z*-component of velocity to the first order. Inserting  $\gamma \rho^{(1)} = p^{(1)}$  into the first-order continuity expression in Eq. (9), one obtains

$$\frac{\partial \rho^{(1)}}{\partial t} = -\nabla \cdot \mathbf{u}^{(1)} - M_b \nabla \cdot [\rho^{(1)} \mathbf{U}] \quad \text{or}$$

$$\frac{\partial p^{(1)}}{\partial t} = -\gamma \nabla \cdot \mathbf{u}^{(1)} - \gamma M_b \nabla \cdot [p^{(1)} \mathbf{U}/\gamma] = -\gamma \nabla \cdot \mathbf{u}^{(1)}.$$
(79)

In terms of the slow boundary layer coordinate, one can put

$$\frac{1}{\gamma}p_t^{(1)} = \frac{\partial u_r^{(1)}}{\partial r} + \frac{u_r^{(1)}}{r} + \frac{1}{r}\frac{\partial u_{\theta}^{(1)}}{\partial \theta} + \frac{1}{\delta}\frac{\partial u_z^{(1)}}{\partial \zeta}.$$
(80)

Substituting Eqs. (36), (71), and (78) into Eq. (80), one can rearrange and retrieve

$$\frac{\gamma}{\delta} \frac{\partial u_{z,c}}{\partial \zeta} = -\frac{ie^{i(m\theta - k_{mn}t)}}{k_{mn}(1 - M_b)} \left\{ \left[ J_m''(k_{mn}r) + \frac{1}{r} J_m'(k_{mn}r) - \frac{m^2}{r^2} J_m(k_{mn}r) \right] (e^{X_2\zeta} - 1) - k_{mn}^2 J_m(k_{mn}r) \right\}.$$
 (81)

However,  $J_m(k_{mn}r)$  satisfies, by definition, the Bessel equation,

$$J_m''(k_{mn}r) + \frac{1}{r}J_m'(k_{mn}r) - \frac{m^2}{r^2}J_m(k_{mn}r) + k_{mn}^2J_m(k_{mn}r) = 0.$$
(82)

Equation (81) becomes

$$\frac{\partial u_{z,c}}{\partial \zeta} = \frac{i \, \delta k_{mn} e^{i(m\theta - Kt)}}{\gamma(1 - M_b)} J_m(k_{mn} r) e^{X_2 \zeta}.$$
(83)

Subsequent integration yields

$$u_{z,c}^{(1)}(r,\theta,\zeta,t) = \frac{i\,\delta k_{mn}e^{i(m\,\theta-k_{mn}t)}}{\gamma X_2(1-M_b)}J_m(k_{mn}r)e^{X_2\zeta} + A_z(r,\theta,t).$$
(84)

Note that the axial velocity is not obtained from Eq. (60) but rather deduced directly from the continuity equation. Based on strictly steady axial injection at the headwall, the *z*-component of velocity must not interfere with the uniform injection at  $z = \zeta = 0$ , This condition translates into

$$u_{z,c}^{(1)}(r,\theta,0,t) = \frac{i\,\partial k_{mn}e^{i(m\theta-k_{mn}t)}}{\gamma X_2(1-M_b)}J_m(k_{mn}r) + A_z(r,\theta,t) = 0$$
(85)
or
$$A_z(r,\theta,t) = -\frac{i\,\partial k_{mn}e^{i(m\theta-k_{mn}t)}}{\gamma X_2(1-M_b)}J_m(k_{mn}r).$$

In the end, one obtains

$$u_{z,c}^{(1)} = \frac{i\,\delta k_{mn} e^{i(m\theta - k_{mn}t)}}{\gamma X_2 (1 - M_b)} J_m(k_{mn}r) (e^{X_2\zeta} - 1).$$
(86)



FIG. 2. First-order approximations for (a) radial and (b) tangential velocities. The scale on the left-hand side is for injection Mach numbers of 0.3 and 0.03. The scale on the right-hand side is for  $M_b$ =0.003.

#### 4. Solution for the complete first order velocity

As was done during the potential flow derivation, the real part of the first order solution will be used in the derivation of the second order flow field. The real parts of the solution can be summarized as

$$\mathbf{u}^{(1)}(r,\theta,\zeta,t) = \frac{J_m(k_{mn}r)}{\gamma k_{mn}(1-M_b)} \left( \frac{J_m'(k_{mn}r)}{J_m(k_{mn}r)} \left[ \underbrace{\sin(m\theta - k_{mn}t)}_{\text{inviscid outer field}} - \underbrace{\sin(m\theta + X_i\,\zeta - k_{mn}t)}_{\text{wave propagation}} \underbrace{e^{X_r\zeta}}_{\text{decay}} \right] \mathbf{e}_r + \left(\frac{m}{r}\right) \left[ \cos(m\theta - k_{mn}t) - \cos(m\theta + X_i\,\zeta - k_{mn}t) e^{X_r\zeta} \right] \mathbf{e}_{\theta} + \frac{\delta k_{mn}^2}{X_r^2 + X_i^2} \left\{ X_r \left[ \sin(m\theta - k_{mn}t) - e^{X_r\zeta} \sin(m\theta + X_i\zeta - k_{mn}t) \right] + X_i \left[ \cos(m\theta - k_{mn}t) - e^{X_r\zeta} \cos(m\theta + X_i\zeta - k_{mn}t) \right] \right\} \mathbf{e}_z \right\},$$
(87)

where  $X_2 = X = X_r + iX_i$  may be synthesized from

$$\begin{cases} X_r = \frac{M_b}{2\delta} \left( 1 - \sqrt{\frac{1 + \sqrt{1 + 16k_{mn}^2 \delta^4 M_b^{-4}}}{2}} \right) \approx \frac{M_b}{2\delta} (1 - \sqrt{1 + 4k_{mn}^2 \delta^4 M_b^{-4}}) \approx -\frac{\delta^3 k_{mn}^2}{M_b^3} = -\frac{\delta}{S_p} \\ X_i = \frac{M_b}{2\delta} \sqrt{\frac{\sqrt{1 + 16k_{mn}^2 \delta^4 M_b^{-4}} - 1}{2}} \approx \frac{M_b}{2\delta} \sqrt{\frac{8k_{mn}^2 \delta^4 M_b^{-4}}{2}} \approx \frac{\delta k_{mn}}{M_b} = \delta S \end{cases}$$
(88)

Γ

It may be useful to remark that the tangential component of the velocity does not vanish at the sidewall. Its behavior in the vicinity of r=1 deteriorates to the extent of overshooting the expected value in the absence of fluid friction at the sidewall. Our domain of analysis is therefore limited to a large diameter chamber with the exclusion of the sidewall. Such a model may be deemed acceptable considering that the principal objective here lies in the treatment of the mean flow interactions with the wave motion directly above the headwall. To illustrate the solution that we obtain, Fig. 2 is used to display the first-order boundary layer approximation for the traveling wave at r=0.4,  $\theta=\pi/3$ , and  $\delta=0.000$  647. The wave evolutions in the streamwise direction are shown at three headwall injection Mach numbers and the first spinning mode number  $k_{10} \approx 1.841$  183 78. The axial velocity fluctuation is not shown due to its small relative magnitude. It is apparent that the viscous stresses have a more pronounced effect as the injection Mach number is decreased. Conversely, when the injection Mach number is increased, the boundary layer is more effectively blown off the surface (see Cole and Aroesty<sup>30</sup> or Majdalani).<sup>24,26</sup> Furthermore, the propagation wavelength measured by the peak-to-peak distance decreases as the Mach number is lowered. In this case, the decay of the wave is also seen to be more rapid. Physically, this behavior may be attributed to the increased dimensionless frequency, or Strouhal number, given by S  $=k_{mn}/M_b = \omega_0 R/V_b$ . As the dimensionless frequency is increased (or the Mach number is decreased), the transverse fluctuations undergo a larger number of reversals per unit time. In the presence of viscosity, the higher frequency at which oscillations occur enhances the effects of fluid friction. Mathematically, the same behavior may be extrapolated from the dependence of the exponential decay terms on  $M_b$ . As one may infer from Eq. (88), increasing the Mach number leads to a smaller  $X_r$  and, consequently, to a slower viscous damping in the axial direction. In actuality, the net damping is strongly dominated by

$$\exp(X_r \zeta) \approx \exp\left(-\frac{\delta^2 k_{mn}^2}{M_b^3} z\right) = \exp\left(-\frac{z}{S_p}\right),\tag{89}$$

where the effective penetration number  $S_p$  emerges in the form

$$S_p = \frac{M_b^3}{\delta^2 k_{mn}^2} = \frac{V_b^3}{a_0^3} \frac{a_0 R}{\nu} \frac{a_0^2}{\omega_0^2 R^2} = \frac{V_b^3}{\nu \omega_0^2 R}.$$
 (90)

This parameter was first discovered in work by Majdalani<sup>24</sup> and then reaffirmed by Flandro,<sup>31</sup> both in the context of an oscillating longitudinal wave over an injecting surface in a porous cylinder. The penetration number is further explored in porous cylinders<sup>25–27</sup> and channels<sup>32–35</sup> with various injection patterns. In the present study, a similar dimensionless group is found to control the depth of penetration of the headwall boundary layer. This can be clearly seen by letting  $\varphi = m\theta - Kt$  and recasting Eq. (87) into

$$\mathbf{u}^{(1)} = \frac{1}{\gamma k_{mn}(1 - M_b)} \left( J'_m(k_{mn}r) [\sin \varphi - \sin(\varphi + Sz)e^{-z/S_p}] \mathbf{e}_r + \frac{m J_m(k_{mn}r)}{r} [\cos \varphi - \cos(\varphi + Sz)e^{-z/S_p}] \mathbf{e}_\theta + \frac{k_{mn}^2 J_m(k_{mn}r)}{S^2 + S_p^2} \{S \cos \varphi - S_p \sin \varphi + [S_p \sin(\varphi + Sz) - S \cos(\varphi + Sz)]e^{-z/S_p}\} \mathbf{e}_z \right).$$
(91)

Note that as  $S_p$  is increased, a larger depth of penetration is realized. Conversely, for small penetration numbers, the exponential damping constant in Eq. (89) is increased, leading to rapid spatial damping of the wave envelope and a shorter penetration depth. Physically, the penetration number unraveled here renders visible the balance between two coexisting forces: unsteady inertia and viscous diffusion of the tangential (or radial) velocity in the axial direction. This dimensionless parameter reflects the ratio of

$$\frac{\text{unsteady inertial force}}{\text{viscous force}} \approx \frac{\frac{\partial u_{\theta}^{*}}{\partial t^{*}}}{\nu \frac{\partial^{2} u_{\theta}^{*}}{\partial z^{*2}}} \approx \frac{\overline{u}_{\theta}^{*}}{\nu \overline{\overline{u}}_{\theta}^{*}} = \frac{\overline{z}^{*2}}{\nu \overline{t}^{*}}$$
$$\approx \frac{(V_{b}/\omega_{0})^{2}}{\nu (R/V_{b})} = \frac{V_{b}^{3}}{\nu \omega_{0}^{2}R} = S_{p}. \quad (92)$$

former work on longitudinal waves is significant. While Majdalani and Flandro<sup>27</sup> considered an oscillating axial flow with steady radial mass flux at the porous sidewall, the present study addresses the motion of an oscillating transverse flow with steady axial flux at the headwall. By comparing these two problems, the blowing velocity  $V_b$  that appears in Eq. (90) will refer to either the transverse or axial mean flow values at the porous wall. The frequency of oscillations for a given mode shape will also be distinctly different, namely,

$$\omega_0 = \begin{cases} \frac{k_{mn}a_0}{R} & \text{transverse wave} \\ m\frac{\pi a_0}{L} & \text{axial wave} \end{cases}$$
(93)

In the above, we use  $\overline{z}^* \approx V_b / \omega_0$  to represent the lengthscale of a wave of frequency  $\omega_0$  being convected at an axial speed that is proportional to  $V_b$ . We also take  $\overline{t}^* \approx R / V_b$  to denote the timescale of a particle crossing the radius of the chamber at a characteristic speed equal to  $V_b$ . It is clear that the penetration number not only accounts for the influence of inertia and viscosity, but also embodies the effects of mean flow convection in the axial direction. The analogy with the Aside from the blowing velocity and dimensional frequency, the remaining parameters in Eq. (90) are the same in both models. At the outset, a full characterization of the headwall boundary layer may be systematically carried out using the steps delineated by Majdalani.<sup>26</sup> While such analysis may be useful in elucidating the structure of the transverse waves under different oscillatory mode configurations, our attention here remains focused on the streaming effects produced by these waves. To this end, a higher approximation is in order.

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# B. Second-order viscous solution

In what follows, we show that extending the boundary layer analysis to the second order in the wave parameter gives rise to a steady flow component that has its origin in the interaction between viscosity and inertia. To this end, the second-order momentum equation, defined in Eq. (10), is recast using the stretched inner coordinate  $\zeta$ :

$$\frac{\partial \mathbf{u}^{(2)}}{\partial t} = -\frac{\nabla p^{(2)}}{\gamma} - \rho^{(1)} \frac{\partial \mathbf{u}^{(1)}}{\partial t} - \frac{1}{2} \nabla (\mathbf{u}^{(1)} \cdot \mathbf{u}^{(1)}) + \mathbf{u}^{(1)} \times \boldsymbol{\omega}^{(1)} + \delta_d^2 \nabla [\nabla \cdot \mathbf{u}^{(2)}] - \delta^2 \nabla \times \boldsymbol{\omega}^{(2)} - M_b \bigg\{ \nabla [\mathbf{U} \cdot \mathbf{u}^{(2)}] - \mathbf{u}^{(2)} \times \boldsymbol{\Omega}^{(0)} - \mathbf{U} \times \boldsymbol{\omega}^{(2)} - \frac{1}{\gamma} p^{(1)} [\nabla (\mathbf{U} \cdot \mathbf{u}^{(1)}) - \mathbf{u}^{(1)} \times \boldsymbol{\Omega}^{(0)} - \mathbf{U} \times \boldsymbol{\omega}^{(1)}] \bigg\}.$$
(94)

Using a suitable boundary layer coordinate transformation, terms on the right-hand side of Eq. (94) become

$$\delta_{d}^{2} \nabla \left[ \nabla \cdot \mathbf{u}^{(2)} \right] = \delta_{d}^{2} \left\{ \begin{bmatrix} \frac{\partial^{2} u_{r}^{(2)}}{\partial r^{2}} + \frac{1}{r} \frac{\partial u_{r}^{(2)}}{\partial r} - \frac{u_{r}^{(2)}}{r^{2}} - \frac{1}{r^{2}} \frac{\partial u_{\theta}^{(2)}}{\partial \theta} + \frac{1}{r} \frac{\partial^{2} u_{\theta}^{(2)}}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^{2} u_{z}^{(2)}}{\partial \sigma r + \frac{1}{r^{2}} \frac{\partial u_{\theta}^{(2)}}{\partial \theta}} \right] \mathbf{e}_{r} \\ + \left[ \frac{1}{r} \frac{\partial^{2} u_{r}^{(2)}}{\partial \theta \partial r} + \frac{1}{r^{2}} \frac{\partial u_{r}^{(2)}}{\partial \theta} + \frac{1}{r^{2}} \frac{\partial^{2} u_{\theta}^{(2)}}{\partial \theta^{2}} + \frac{1}{r \delta \partial \theta \partial \zeta} \right] \mathbf{e}_{\theta} \\ + \left[ \frac{1}{\sigma^{2} u_{r}^{(2)}} + \frac{1}{r \delta \partial \zeta} \frac{\partial u_{r}^{(2)}}{\partial \zeta} + \frac{1}{r \delta \partial \zeta} \frac{\partial^{2} u_{\theta}^{(2)}}{\partial \theta^{2}} + \frac{1}{\sigma^{2} \partial \theta^{2}} \right] \mathbf{e}_{z} \\ + \left[ \frac{1}{\sigma^{2} u_{r}^{(2)}} + \frac{1}{r \delta \partial \zeta} \frac{\partial u_{r}^{(2)}}{\partial \zeta} + \frac{1}{r \delta \partial \zeta} \frac{\partial^{2} u_{\theta}^{(2)}}{\partial \theta^{2}} + \frac{1}{\sigma^{2} \partial \theta^{2}} \right] \mathbf{e}_{z} \\ - \delta^{2} \nabla \times \mathbf{u}^{(2)} = \delta^{2} \\ \left\{ \begin{bmatrix} -\frac{1}{r} \left( \frac{\partial^{2} u_{\theta}^{(2)}}{\partial \theta \partial r} + \frac{1}{r \partial \theta} - \frac{1}{r \partial \theta} - \frac{1}{r \partial \theta^{2}} \right) + \left( -\frac{\partial^{2} u_{\theta}^{(2)}}{\partial \theta^{2}} \right) - \frac{1}{\delta} \left( \frac{1}{\delta} \frac{\partial^{2} u_{r}^{(2)}}{\partial \zeta^{2}} - \frac{\partial^{2} u_{z}^{(2)}}{\partial \zeta \partial r} \right) \right] \mathbf{e}_{r} \\ - \left[ \frac{1}{\delta} \left( \frac{1}{r \partial u_{z}^{(2)}} - \frac{1}{\delta \partial \theta} - \frac{1}{\delta \partial \theta^{2}} \right) + \left( -\frac{\partial^{2} u_{\theta}^{(2)}}{\partial r^{2}} - \frac{1}{r \partial \theta} - \frac{1}{r \partial \theta} + \frac{u_{\theta}^{(2)}}{r^{2}} + \frac{1}{r \partial \theta} - \frac{\partial^{2} u_{r}^{(2)}}{r^{2}} \right] \mathbf{e}_{z} \\ - \frac{1}{r} \left[ \left( \frac{1}{\delta u_{r}^{(2)}} + \frac{r}{\delta \partial \zeta} - \frac{1}{\delta \partial \zeta} - \frac{\partial^{2} u_{\theta}^{(2)}}{r^{2}} - \frac{1}{\sigma^{2}} \frac{\partial^{2} u_{\theta}^{(2)}}{r^{2}} \right) - \left( \frac{1}{r \partial \theta} - \frac{1}{r \partial \theta} - \frac{1}{\sigma^{2}} \frac{\partial^{2} u_{\theta}^{(2)}}{r^{2} \partial \zeta} - \frac{1}{\sigma^{2}} \frac{\partial^{2} u_{\theta}^{(2)}}$$

$$-\frac{1}{2}\nabla\left(\mathbf{u}^{(1)}\cdot\mathbf{u}^{(1)}\right) = \begin{cases} -\left[u_{r}^{(1)}\frac{\partial u_{r}^{(1)}}{\partial r} + u_{\theta}^{(1)}\frac{\partial u_{\theta}^{(1)}}{\partial r} + u_{z}^{(1)}\frac{\partial u_{z}^{(1)}}{\partial r}\right]\mathbf{e}_{r} \\ -\frac{1}{r}\left[u_{r}^{(1)}\frac{\partial u_{r}^{(1)}}{\partial \theta} + u_{\theta}^{(1)}\frac{\partial u_{\theta}^{(1)}}{\partial \theta} + u_{z}^{(1)}\frac{\partial u_{z}^{(1)}}{\partial \theta}\right]\mathbf{e}_{\theta} \\ -\frac{1}{\delta}\left[u_{r}^{(1)}\frac{\partial u_{r}^{(1)}}{\partial \zeta} + u_{\theta}^{(1)}\frac{\partial u_{\theta}^{(1)}}{\partial \zeta} + u_{z}^{(1)}\frac{\partial u_{z}^{(1)}}{\partial \zeta}\right]\mathbf{e}_{z} \end{cases} \end{cases},$$
(97)

$$\mathbf{u}^{(1)} \times \mathbf{u}^{(1)} = \begin{cases} \left[ u_{\theta}^{(1)} \left( \frac{\partial u_{\theta}^{(1)}}{\partial r} + \frac{u_{\theta}^{(1)}}{r} - \frac{1}{r} \frac{\partial u_{r}^{(1)}}{\partial \theta} \right) - u_{z}^{(1)} \left( \frac{1}{\delta} \frac{\partial u_{r}^{(1)}}{\partial \zeta} - \frac{\partial u_{z}^{(1)}}{\partial r} \right) \right] \mathbf{e}_{r} \\ + \left[ u_{z}^{(1)} \left( \frac{1}{r} \frac{\partial u_{z}^{(1)}}{\partial \theta} - \frac{1}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} \right) - u_{r}^{(1)} \left( \frac{\partial u_{\theta}^{(1)}}{\partial r} + \frac{u_{\theta}^{(1)}}{r} - \frac{1}{r} \frac{\partial u_{r}^{(1)}}{\partial \theta} \right) \right] \mathbf{e}_{\theta} \\ + \left[ u_{r}^{(1)} \left( \frac{1}{\delta} \frac{\partial u_{r}^{(1)}}{\partial \zeta} - \frac{\partial u_{z}^{(1)}}{\partial r} \right) - u_{\theta}^{(1)} \left( \frac{1}{r} \frac{\partial u_{z}^{(1)}}{\partial \theta} - \frac{1}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} \right) \right] \mathbf{e}_{z} \end{cases}$$
(98)

and

$$\frac{-M_b}{\gamma} p^{(1)} \nabla \left( \mathbf{U} \cdot \mathbf{u}^{(1)} \right) = \frac{-M_b}{\gamma} \left( p^{(1)} \frac{\partial u_z^{(1)}}{\partial r} \mathbf{e}_r + \frac{p^{(1)}}{r} \frac{\partial u_z^{(1)}}{\partial \theta} \mathbf{e}_\theta + \frac{p^{(1)}}{\delta} \frac{\partial u_z^{(1)}}{\partial \zeta} \mathbf{e}_z \right), \tag{99}$$

$$\frac{M_b}{\gamma} p^{(1)}(\mathbf{U} \times \mathbf{u}^{(1)}) = \frac{-M_b}{\gamma} p^{(1)} \left( \frac{1}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} - \frac{\partial u_z^{(1)}}{\partial r} \right) \mathbf{e}_r + \frac{M_b}{\gamma} p^{(1)} \left( \frac{1}{r} \frac{\partial u_z^{(1)}}{\partial \theta} - \frac{1}{\delta} \frac{\partial u_\theta^{(1)}}{\partial \zeta} \right) \mathbf{e}_\theta + (0) \mathbf{e}_z, \tag{100}$$

$$-M_b \nabla \left(\mathbf{U} \cdot \mathbf{u}^{(2)}\right) = -M_b \left(\frac{\partial u_z^{(2)}}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial u_z^{(2)}}{\partial \theta} \mathbf{e}_\theta + \frac{1}{\delta} \frac{\partial u_z^{(2)}}{\partial \zeta} \mathbf{e}_z\right),\tag{101}$$

$$M_{b}[\mathbf{U} \times \mathbf{u}^{(2)}] = -M_{b}\left(\frac{1}{\delta}\frac{\partial u_{r}^{(2)}}{\partial \zeta} - \frac{\partial u_{z}^{(2)}}{\partial r}\right)\mathbf{e}_{r} + M_{b}\left(\frac{1}{r}\frac{\partial u_{z}^{(2)}}{\partial \theta} - \frac{1}{\delta}\frac{\partial u_{\theta}^{(2)}}{\partial \zeta}\right)\mathbf{e}_{\theta} + (0)\mathbf{e}_{z}.$$
(102)

Equation (94) can be projected into three vector directions to obtain

$$\frac{\partial u_r^{(2)}}{\partial t} = -\frac{M_b}{\delta} \frac{\partial u_r^{(2)}}{\partial \zeta} - \frac{1}{\gamma} \frac{\partial p^{(2)}}{\partial r} - \frac{p^{(1)}}{\gamma} \left[ \frac{M_b}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} + \frac{\partial u_r^{(1)}}{\partial t} \right] - \left\{ \frac{u_z^{(1)}}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} + u_r^{(1)} \frac{\partial u_r^{(1)}}{\partial r} + \frac{u_\theta^{(1)}}{r} \left[ \frac{\partial u_r^{(1)}}{\partial \theta} - u_\theta^{(1)} \right] \right\} - \left\{ \delta^2 \left\{ \frac{1}{r} \left[ \frac{\partial^2 u_\theta^{(2)}}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial u_\theta^{(2)}}{\partial \theta} - \frac{1}{r} \frac{\partial^2 u_r^{(2)}}{\partial \theta^2} \right] - \frac{1}{\delta} \left[ \frac{1}{\delta} \frac{\partial^2 u_r^{(2)}}{\partial \zeta^2} - \frac{\partial^2 u_z^{(2)}}{\partial \zeta \partial r} \right] \right\} + \delta_d^2 \left[ \frac{\partial^2 u_r^{(2)}}{\partial r^2} + \frac{1}{r} \frac{\partial u_r^{(2)}}{\partial r} - \frac{u_r^{(2)}}{r^2} - \frac{1}{r^2} \frac{\partial u_\theta^{(2)}}{\partial \theta} + \frac{1}{r} \frac{\partial^2 u_\theta^{(2)}}{\partial r \partial \theta} + \frac{1}{\delta} \frac{\partial^2 u_z^{(2)}}{\partial r \partial \zeta} \right],$$
(103)

$$\frac{\partial u_{\theta}^{(2)}}{\partial t} = -\frac{M_b}{\delta} \frac{\partial u_{\theta}^{(2)}}{\partial \zeta} - \frac{1}{\gamma r} \frac{\partial p^{(2)}}{\partial \theta} - \frac{p^{(1)}}{\gamma} \left[ \frac{M_b}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} + \frac{\partial u_{\theta}^{(1)}}{\partial t} \right] - \left[ \frac{u_z^{(1)}}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} + \frac{u_{\theta}^{(1)}}{r} \frac{\partial u_{\theta}^{(1)}}{\partial \theta} + u_r^{(1)} \left( \frac{u_{\theta}^{(1)}}{r} + \frac{\partial u_{\theta}^{(1)}}{\partial r} \right) \right] \\
- \delta^2 \left\{ \frac{1}{\delta} \left[ \frac{1}{r} \frac{\partial^2 u_z^{(2)}}{\partial \zeta \partial \theta} - \frac{1}{\delta} \frac{\partial^2 u_{\theta}^{(2)}}{\partial \zeta^2} \right] - \left[ \frac{\partial^2 u_{\theta}^{(2)}}{\partial r^2} - \frac{u_{\theta}^{(2)}}{r^2} + \frac{1}{r} \frac{\partial u_{\theta}^{(2)}}{\partial r} - \frac{1}{r} \frac{\partial^2 u_r^{(2)}}{\partial r \partial \theta} \right] \right\} \\
+ \delta^2_d \left[ \frac{1}{r} \frac{\partial^2 u_r^{(2)}}{\partial \theta \partial r} + \frac{1}{r^2} \frac{\partial^2 u_{\theta}^{(2)}}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 u_{\theta}^{(2)}}{\partial \theta^2} + \frac{1}{r \delta} \frac{\partial^2 u_z^{(2)}}{\partial \theta \partial \zeta} \right],$$
(104)

$$\frac{\partial u_{z}^{(2)}}{\partial t} = -\frac{M_{b}}{\delta} \frac{\partial u_{z}^{(2)}}{\partial \zeta} - \frac{1}{\gamma \delta} \frac{\partial p^{(2)}}{\partial \zeta} - \frac{p^{(1)}}{\gamma} \left[ \frac{M_{b}}{\delta} \frac{\partial u_{z}^{(1)}}{\partial \zeta} + \frac{\partial u_{z}^{(1)}}{\partial t} \right] - \left[ \frac{u_{z}^{(1)}}{\delta} \frac{\partial u_{z}^{(1)}}{\partial \zeta} + u_{r}^{(1)} \frac{\partial u_{z}^{(1)}}{\partial r} + \frac{u_{\theta}^{(1)}}{\rho} \frac{\partial u_{z}^{(1)}}{\partial \theta} \right] \\
- \frac{\delta^{2}}{r} \left\{ \left[ \frac{1}{\delta} \frac{\partial u_{r}^{(2)}}{\partial \zeta} + \frac{r}{\delta} \frac{\partial^{2} u_{r}^{(2)}}{\partial r \delta \zeta} - \frac{\partial u_{z}^{(2)}}{\partial r} - \frac{\partial^{2} u_{z}^{(2)}}{\partial r^{2}} \right] - \left[ \frac{1}{r} \frac{\partial^{2} u_{z}^{(2)}}{\partial \theta^{2}} - \frac{1}{\delta} \frac{\partial^{2} u_{\theta}^{(2)}}{\partial \theta \delta \zeta} \right] \right\} \\
+ \delta^{2}_{d} \left[ \frac{1}{\delta} \frac{\partial^{2} u_{r}^{(2)}}{\partial \zeta \delta r} + \frac{1}{r \delta} \frac{\partial u_{r}^{(2)}}{\partial \zeta} + \frac{1}{r \delta} \frac{\partial^{2} u_{\theta}^{(2)}}{\partial \zeta \delta \theta} + \frac{1}{\delta^{2}} \frac{\partial^{2} u_{z}^{(2)}}{\partial \zeta^{2}} \right].$$
(105)

Retaining terms at the zeroth order in  $\delta$  leaves us with

$$\begin{cases} \frac{\partial^2 u_r^{(2)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial u_r^{(2)}}{\partial \zeta} - \frac{\partial u_r^{(2)}}{\partial t} = \frac{1}{\gamma} \frac{\partial p^{(2)}}{\partial r} + \frac{p^{(1)}}{\gamma} \left[ \frac{M_b}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} + \frac{\partial u_r^{(1)}}{\partial t} \right] + \left[ \frac{u_z^{(1)}}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} + u_r^{(1)} \frac{\partial u_r^{(1)}}{\partial r} + \frac{u_\theta^{(1)}}{\rho} \left( \frac{\partial u_r^{(1)}}{\partial \theta} - u_\theta^{(1)} \right) \right] \\ \frac{\partial^2 u_{\theta}^{(2)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial u_{\theta}^{(2)}}{\partial \zeta} - \frac{\partial u_{\theta}^{(2)}}{\partial t} = \frac{1}{\gamma r} \frac{\partial p^{(2)}}{\partial \theta} + \frac{p^{(1)}}{\gamma} \left[ \frac{M_b}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} + \frac{\partial u_{\theta}^{(1)}}{\partial t} \right] + \left[ \frac{u_z^{(1)}}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} + \frac{u_r^{(1)}}{\theta} \frac{\partial u_{\theta}^{(1)}}{\partial \theta} + u_r^{(1)} \left( \frac{u_{\theta}^{(1)}}{r} + \frac{\partial u_{\theta}^{(1)}}{\partial r} \right) \right] \\ \frac{\partial^2 u_{\theta}^{(2)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial u_{\theta}^{(2)}}{\partial \zeta} - \frac{\partial u_{\theta}^{(2)}}{\partial t} = \frac{1}{\gamma \delta} \frac{\partial p^{(2)}}{\partial \zeta} + \frac{p^{(1)}}{\gamma} \left[ \frac{M_b}{\delta} \frac{\partial u_z^{(1)}}{\partial \zeta} + \frac{\partial u_{\theta}^{(1)}}{\partial t} \right] + \left[ \frac{u_z^{(1)}}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} + \frac{u_r^{(1)}}{\theta} \frac{\partial u_{\theta}^{(1)}}{\partial \theta} + u_r^{(1)} \left( \frac{u_{\theta}^{(1)}}{r} + \frac{\partial u_{\theta}^{(1)}}{\partial r} \right) \right] \quad , \quad (106)$$

where terms involving the ratio  $M_b/\delta$  are kept, being non-negligible. Solving the second-order equations requires greater algebraic detail. By scrutinizing the right-hand side of Eq. (106), it may be seen that several quadratic combinations of trigonometric functions appear. Such combinations give rise to both time-dependent and steady terms. An example would be the  $2 \cos^2(k_{mn}t)$  term which can be recast as  $1 + \cos(2k_{mn}t)$ . To briefly sketch the procedure followed, the solution in the radial direction will be outlined.

After substituting the first-order solution on the right-hand side of Eq. (106), one recovers, for the steady part,

$$\frac{1}{\gamma} \frac{\partial p^{(2)}}{\partial r} = \frac{1}{2k_{mn}^2 \gamma^2 (1 - M_b)^2} \left[ \frac{m^2}{r^3} J_m^2 + \left( k_{mn}^2 - \frac{m^2}{r^2} \right) J_m J_m' - J_m' J_m'' \right],\tag{107}$$

$$\frac{1}{\gamma} \frac{M_b}{\delta} p^{(1)} \frac{\partial u_r^{(1)}}{\partial \zeta} = \frac{-J_m J_m' e^{X_r \zeta}}{2\gamma^2 k_{mn} (1 - M_b)^2} \frac{M_b}{\delta} \{ X_r \sin(X_i \zeta) + X_i \cos(X_i \zeta) \},\tag{108}$$

$$\frac{p^{(1)}}{\gamma} \frac{\partial u_r^{(1)}}{\partial t} = \frac{1}{2\gamma^2 (1 - M_b)^2} J_m J_m' [e^{X_r \zeta} \cos(X_i \zeta) - 1],$$
(109)

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$$-u_{\theta}^{(1)}\frac{u_{\theta}^{(1)}}{r} = -\frac{1}{2(1-M_b)^2} \left(\frac{1}{\gamma k_{mn}}\right)^2 \frac{m^2}{r^3} J_m^2 [1+e^{2X_r\zeta}-2e^{X_r\zeta}\cos(X_i\zeta)],\tag{110}$$

where  $J_m$  stands for  $J_m(k_{mn}r)$  and its primes denote derivatives with respect to the radial coordinate. In this context, we have

$$\frac{u_{\theta}^{(1)}}{r}\frac{\partial u_{r}^{(1)}}{\partial \theta} = \frac{1}{2(1-M_{b})^{2}} \left(\frac{1}{\gamma k_{mn}}\right)^{2} \frac{m^{2}}{r^{2}} J_{m} J_{m}' [1 + e^{2X_{r}\zeta} - 2e^{X_{r}\zeta} \cos(X_{i}\zeta)],$$
(111)

$$\frac{u_z^{(1)}}{\delta} \frac{\partial u_r^{(1)}}{\partial \zeta} = \frac{J_m J_m' e^{X_r \zeta} \sin(X_i \zeta)}{2\gamma^2 (1 - M_b)^2 (X_i^2 + X_r^2)} \left[ (X_r^2 - X_i^2) \frac{e^{X_r \zeta}}{\sin(X_i \zeta)} + (X_i^2 - X_r^2) \frac{\cos(X_i \zeta)}{\sin(X_i \zeta)} + 2X_i X_r \right],$$
(112)

and

$$u_r^{(1)} \frac{\partial u_r^{(1)}}{\partial r} = \frac{1}{2k_{mn}^2 \gamma^2 (1 - M_b)^2} J_m' J_m'' [1 + e^{2X_r \zeta} - 2e^{X_r \zeta} \cos(X_i \zeta)].$$
(113)

Equation (106) can now be rewritten as

$$\frac{\partial^{2} u_{r}^{(2)}}{\partial \zeta^{2}} - \frac{M_{b}}{\delta} \frac{\partial u_{r}^{(2)}}{\partial \zeta} = \frac{1}{2k_{mn}^{2} \gamma^{2}} \Biggl\{ \Biggl[ \Biggl( \frac{m^{2}}{r^{2}} + k_{mn}^{2} - \frac{k_{mn}^{2} X_{r}^{2}}{X_{i}^{2} + X_{r}^{2}} \Biggr) J_{m} J_{m}' - \frac{m^{2}}{r^{3}} J_{m}^{2} + J_{m}' J_{m}'' \Biggr] e^{2X_{r}\zeta} + \Biggl[ \frac{2m^{2}}{r^{3}} J_{m}^{2} - \Biggl( \frac{2m^{2}}{r^{2}} + \frac{M_{b} k_{mn}}{\delta} X_{i} - 2k_{mn}^{2} X_{i}^{2} \Biggr) J_{m} J_{m}' - 2J_{m}' J_{m}'' \Biggr] e^{X_{r}\zeta} \\ \times \cos(X_{i}\zeta) + k_{mn} \Biggl( \frac{2k_{mn} X_{i} X_{r}}{X_{i}^{2} + X_{r}^{2}} - \frac{M_{b} X_{r}}{\delta} \Biggr) J_{m} J_{m}' e^{X_{r}\zeta} \sin(X_{i}\zeta) \Biggr\}.$$
(114)

Note that the term involving the partial derivative with respect to time has been removed because of our focus being on the steady flow. Equation (114) is a second-order linear PDE that is subject to

$$u_r^{(2)}(r,\theta,0) = 0; \quad u_r^{(2)}(r,\theta,\infty) = \text{finite.}$$
 (115)

A straightforward solution may be obtained using the method of undetermined coefficients. The steady part reads

$$u_{r}^{(2)} = \frac{1}{2k_{mn}^{2}\gamma^{2}(1-M_{b})^{2}} \begin{cases} \frac{(e^{2X_{r}\zeta}-1)}{2X_{r}\left(2X_{r}-\frac{M_{b}}{\delta}\right)} \left[ \left(\frac{m^{2}}{r^{2}}+k_{mn}^{2}\frac{X_{r}^{2}-X_{i}^{2}}{X_{r}^{2}+X_{i}^{2}}\right)J_{m}J_{m}'-\frac{m^{2}}{r^{3}}J_{m}^{2}+J_{m}'J_{m}''\right] \\ +\Lambda_{r,1}[e^{X_{r}\zeta}\sin(X_{i}\zeta)] +\Lambda_{r,2}[e^{X_{r}\zeta}\cos(X_{i}\zeta)-1] \end{cases} \right\},$$
(116)

where

$$\Lambda_{r,1} = \frac{1}{\beta_r} \begin{cases} [X_i(2\,\delta X_r - M_b)] \frac{2m^2}{r^3} J_m^2 + [2\,\delta X_i(M_b - 2\,\delta X_r)] \left(\frac{m^2}{r^2} J_m J_m' + J_m' J_m''\right) \\ + k_{mn}(M_b - \delta X_r) [M_b(X_i^2 + X_r^2) - k_{mn}] J_m J_m' \end{cases}$$

$$(117)$$

$$\Lambda_{r,2} = \frac{1}{\beta_r} \begin{cases} 2\,\delta [\delta X_i^2 + X_r(M_b - \delta X_r)] \left[J_m' J_m'' - \frac{m^2}{r^3} J_m^2\right] \\ + \left[2\frac{m^2}{r^2} \delta^2 \left(X_i^2 - X_r^2 + \frac{M_b}{\delta} X_r\right) + k_{mn} M_b \delta X_i(X_i^2 + X_r^2) \right] J_m J_m' \end{cases}$$



FIG. 3. Steady second-order approximations for (a) radial and (b) tangential velocities. The scale on the left axis corresponds to an injection Mach number of 0.3 while that on the right is set to display the  $M_b$ =0.03 and 0.003 cases.

(122)

and

$$\beta_r = (X_r^2 + X_i^2) [\delta^2 X_i^2 + (M_b - \delta X_r)^2].$$
(118)

In like manner, the steady streaming velocity in the tangential direction may be extracted from Eq. (106). Collecting the steady terms from the right-hand side of Eq. (106), one gets

$$\frac{M_b}{\gamma\delta}p^{(1)}\frac{\partial u_{\theta}^{(1)}}{\partial \zeta} = \frac{M_b}{2k_{mn}\gamma^2\delta(1-M_b)^2}\frac{m}{r}J_m^2e^{X_r\zeta} \times \{X_i\sin(X_i\zeta) - X_r\cos(X_i\zeta)\},$$
(119)

$$\frac{p^{(1)}}{\gamma} \frac{\partial u_{\theta}^{(1)}}{\partial t} = \frac{-J_m^2 e^{X_r \zeta}}{2\gamma^2 (1 - M_b)^2} \frac{m}{r} \sin(X_i \zeta),$$
(120)

and

$$u_{r}^{(1)} \frac{u_{\theta}^{(1)}}{r} = 0; \quad u_{r}^{(1)} \frac{\partial u_{\theta}^{(1)}}{\partial r} = 0;$$

$$\frac{1}{\gamma r} \frac{\partial p^{(2)}}{\partial \theta} = 0; \quad \frac{u_{\theta}^{(1)}}{r} \frac{\partial u_{\theta}^{(1)}}{\partial \theta} = 0,$$

$$\frac{u_{z}^{(1)}}{\delta} \frac{\partial u_{\theta}^{(1)}}{\partial \zeta} = \frac{J_{m}^{2} e^{X_{r}\zeta}}{2\gamma^{2}(1 - M_{b})^{2}} \left(\frac{m}{r}\right) \frac{(X_{i}^{2} - X_{r}^{2})}{X_{i}^{2} + X_{r}^{2}}$$

$$\times \left\{ \sin(X_{i}\zeta) + \frac{2X_{i}X_{r}}{(X_{i}^{2} - X_{r}^{2})} [e^{X_{r}\zeta} - \cos(X_{i}\zeta)] \right\}.$$
(121)

Equation (106) can now be rewritten as

$$\frac{\partial^2 \mathbf{u}_{\theta}^{(2)}}{\partial \zeta^2} - \frac{M_b}{\delta} \frac{\partial \mathbf{u}_{\theta}^{(2)}}{\partial \zeta} = \mathbf{T}(\theta), \qquad (123)$$

$$\Gamma(\theta) = \frac{J_m^2 e^{2X_r \zeta}}{\gamma^2 (1 - M_b)^2} \frac{m}{r} \left\{ \left( \frac{M_b}{2k_{mn}\delta} - \frac{X_i}{X_i^2 + X_r^2} \right) e^{-X_r \zeta} \right.$$
$$\times \left[ X_i \sin(X_i \zeta) - X_r \cos(X_i \zeta) \right] - \frac{X_i X_r}{X_i^2 + X_r^2} \left\}.$$
(124)

Equation (123) can be carefully solved to obtain

$$u_{\theta}^{(2)} = \frac{m}{r} \frac{J_{m}^{2}[(X_{i}^{2} + X_{r}^{2})M_{b} - 2k_{mn}\delta X_{i}]}{2k_{mn}^{2}\gamma^{2}\beta_{\theta}(1 - M_{b})^{2}} \\ \times \{(M_{b} - \delta X_{r})[e^{X_{r}\zeta}\cos(X_{i}\zeta) - 1] - X_{i}e^{X_{r}\zeta}\sin(X_{i}\zeta)\} \\ + \frac{J_{m}^{2}\delta X_{i}}{2\gamma^{2}(1 - M_{b})^{2}(X_{i}^{2} + X_{r}^{2})(2\delta X_{r} - M_{b})}\frac{m}{r} \\ \times (e^{2X_{r}\zeta} - 1);$$

$$\beta_{\theta} = (X_{r}^{2} + X_{i}^{2})[\delta^{2}X_{i}^{2} + (M_{b} - \delta X_{r})^{2}].$$
(125)

For the sake of illustration, Fig. 3 displays the secondorder radial and tangential velocities at r=0.4,  $\theta=(1/3)\pi$ , and  $\delta = 0.000$  647 versus the axial coordinate at three headwall injection Mach numbers. The radial velocity exhibits an interesting trend, displaying alternating spatial excursions that shift outwardly toward increasingly more positive values. This behavior is most apparent in the case of  $M_b = 0.03$ [dashed line in Fig. 3(a)] where the radial velocity starts vacillating around  $u_r \approx 0.25$  and then  $u_r \approx 0.75$  in the short span of z = [0, 1]. The same pattern is repeated in the cases of  $M_b = 0.3$  and 0.003, but the positively shifting excursions are masked in the corresponding graphs by the relative scales. These particular trends suggest that when fluid particles convect downstream, away from the injector face, the secondorder flow field becomes increasingly influenced by a steady radial velocity that pushes the fluctuations outwardly toward the sidewall.

In order to compare the first- and second-order boundary layer flows, it may be useful to consider the entire wave structure at one particular instant of time. Figures 4 and 5 are snapshots of vector fields representing the first tangential mode of oscillation for the first- and second-order solutions,



FIG. 4. First-order traveling wave vector plot at z=0.01 and three headwall injection Mach numbers of (a)  $M_b=0.3$ , (b) 0.03, and (c) 0.003.

respectively, taken at fixed z=0.01, t=1, and  $\delta=0.000$  647. In Fig. 5, only the steady portion of the second-order solution is shown. Note that the first-order solution in Fig. 4 spins in a counterclockwise fashion as a consequence of the convention assumed in the exponential time dependence. The vector traces shown here have comparable patterns that are merely reoriented in the polar plane with successive decreases in the headwall injection Mach number. Velocity vectors moving from one nodal point to the other are identified in all three plots. These patterns stand in sharp contrast with the second-order results shown in Fig. 5, where the velocity vectors display distinctly dissimilar motions. In the cases of  $M_b=0.3$  and 0.03, the flow pattern is dominated by an inward pointing radial velocity drawing mass toward the chamber's centerline with a slight clockwise swirl velocity that is noticeable in the  $M_b=0.03$  case. At first glance, this pattern would appear to retard the first-order motion whose wave structure rotates in a counterclockwise direction. Although a similar conclusion is reported by Maslen and Moore,<sup>22</sup> a closer examination of the flow behavior seems to suggest the contrary. Note that Fig. 5(c) displays a strong outward pointing radial velocity with a similar counterclockwise swirl velocity. The disparity between Figs. 5(a)-5(c)suggests a closer look at Fig. 3. In plotting the second-order

radial component, it is seen that the velocity near the headwall fluctuates between positive and negative quantities. At z=0.01, deep within the boundary layer, the two larger injection Mach number cases fall in a negative  $u_r$  swing, whereas the smallest Mach number case falls in a positive swing. The corresponding arrowheads are inward pointing in Figs. 5(a) and 5(b) but outward in Fig. 5(c). However, outside the boundary layer, the arrowheads are always outward pointing as corroborated by the outer limit of  $u^{(2)}$ , namely, the induced streaming solution.

It should be recalled that streaming flows and the focus of this investigation are normally associated with a secondorder steady rotational flow that is independent of viscosity. To extract these terms from the second-order flow solution, the limit is taken as the boundary layer coordinate approaches infinity. One obtains

$$\overline{u}_{r}^{(2)} = \lim_{\zeta \to \infty} u_{r}^{(2)}$$

$$= \frac{-\left[\left(\frac{m^{2}}{r^{2}} + k_{mn}^{2} \frac{X_{r}^{2} - X_{i}^{2}}{X_{r}^{2} + X_{i}^{2}}\right) J_{m} J_{m}' - \frac{m^{2}}{r^{3}} J_{m}^{2} + J_{m}' J_{m}''\right]}{4X_{r} (2X_{r} - M_{b} / \delta) \gamma^{2} k_{mn}^{2} (1 - M_{b})^{2}} - \frac{\Lambda_{r,2}}{2k_{mn}^{2} \gamma^{2} (1 - M_{b})^{2}},$$
(126)



FIG. 5. Steady second-order velocity vector plot at z=0.01 and three headwall injection Mach numbers of (a)  $M_b=0.3$ , (b) 0.03, and (c) 0.003.



FIG. 6. Total vector plot in the outer region illustrating the behavior of (a) the purely inviscid potential approximation up to the second order and [(b) and (c)] the same total potential solution augmented by the streaming contribution. Results are shown for t=1, n=1,  $\varepsilon=0.1$ ,  $M_b=0.3$ , and  $\delta=(a) 0$ , (b) 0.006 47, and (c) 0.0647.

$$\bar{u}_{\theta}^{(2)} = \lim_{\zeta \to \infty} u_{\theta}^{(2)}$$

$$= \frac{-J_m^2}{2\gamma^2 (1 - M_b)^2} \frac{m}{r} \left\{ \frac{(M_b - \delta X_r)}{2k_{mn}^2 \beta_{\theta}} \times [(X_i^2 + X_r^2)M_b - 2k_{mn}\delta X_i] - \frac{\delta X_i}{(X_i^2 + X_r^2)} \right\}.$$
(127)

Reflected in Eqs. (126) and (127) is a second-order steady flow that is deprived of viscous damping terms. Streaming flow investigators often refer to solutions similar to these limiting expressions as second-order "potential" solutions, although they are not totally independent of viscosity. Figures 6 and 7 display the radial dependent streaming terms as a function of  $\delta$  and  $M_b$ . The plots in Fig. 6 correlate to the streaming term in the radial direction for  $\varepsilon = 0.01$ , m = 1, n=0, and a Mach number of (a)  $M_b=0.003$ , (b) 0.03, and (c) 0.3. Recalling the expansion defined through Eq. (8), the order of magnitude of the first order potential solution is set by  $\varepsilon = 0.01$ . With this in mind one identifies that the second order terms would need to be of  $O(10^{-3})$  before having an appreciable impact on the first order flow field. As displayed in Fig. 6(b), the necessary condition is met when  $M_b = 0.03$ and  $\delta = 0.000$  647. When  $M_b = 0.3$ , a common condition in liquid rocket engines, the magnitude of  $\bar{u}_r^{(2)}$  is significant enough to affect the first order potential solution for all  $\delta$ . Yet contrary to what was seen in Fig. 6, the azimuthal component of the streaming solution,  $\bar{u}_{\theta}^{(2)}$ , does not reach a magnitude that can appreciably influence the potential motion. This behavior is displayed in Fig. 7.

To illustrate the impact of the streaming solution restored in the outer limit,  $\bar{u}_r^{(2)}$  and  $\bar{u}_{\theta}^{(2)}$ , on the total potential flow solution, Fig. 8 is used to display vector plots of the second-order approximation first without streaming (a), and then with streaming and either (b)  $\delta$ =0.006 47 or (c)  $\delta$ =0.0647. All results are shown at  $\varepsilon$ =0.01,  $M_b$ =0.3, t=1, m=1, and n=0 (first order tangential and zeroth order radial). In Fig. 8(a), only the total potential flow is depicted to second order. The results are found to be nearly identical to the first-order potential solution and to the patterns in Fig. 4 where the first-order viscous solution is shown. This agreement reflects the diminutive nature of the second-order potential flow contribution. When streaming effects are ac-



FIG. 7. Streaming velocity in the radial direction shown at several injection Mach and acoustic Reynolds numbers. Results correspond to m=1,  $\varepsilon=0.01$ ,  $\gamma=1.4$ , and an injection Mach number of (a) 0.003, (b) 0.03, and (c) 0.3. The scale on the left axis is specified for the  $\delta=0.000$  647 case while that on the right corresponds to the  $\delta=0.006$  47 and 0.0647 cases.



FIG. 8. Streaming velocity in the tangential direction shown at several injection Mach and acoustic Reynolds numbers. Results correspond to m=1,  $\varepsilon=0.01$ ,  $\gamma=1.4$ , and an injection Mach number of (a) 0.003, (b) 0.03, and (c) 0.3.

counted for, Figs. 8(b) and 8(c) bring into perspective the behavior of the total potential flow (to second order) combined with the streaming velocities,  $\bar{u}_r^{(2)}$  and  $\bar{u}_{\theta}^{(2)}$ . It is quite evident in Fig. 6(b) that, when we use a typical value of  $\delta$ =0.006 47, the streaming motion can markedly alter the fundamental flow structure observed in Figs. 4 and 8(a). The velocity vectors are pushed outwardly in all directions with the effect being most pronounced in the area around the nodal line where the potential flow vectors are mostly radial. The flow patterns in Figs. 4 and 8(a) comprise two regions with respect to the axis of rotation, an upstream region where the flow is directed toward the core, and a downstream region where the flow is outward. With the superposition of the streaming correction in Fig. 8(b), a reversal in the direction of flow upstream of the centerline may be noted. This flow reversal may be attributed to the large streaming amplitude resulting from the use of a relatively sizable  $\varepsilon = 0.01$ . In stark contrast with Fig. 8(b), no streaming consequences may be linked to Fig. 8(c), where the viscous parameter is increased to  $\delta$ =0.0647. At first glance, the diminishment in streaming intensity with successive increases in  $\delta$  appears to be paradoxical, or perhaps counterintuitive, because secondary flows are rooted deep within the viscous boundary layer. Upon further scrutiny, however, one realizes that increasing  $\delta$  leads to a smaller penetration number as expressed through Eq. (90). Decreasing the penetration number  $S_p$  reduces the boundary layer thickness and, thereby, the depth of penetration of the rotational segment through which streaming is generated.

In Fig. 9, three diagrams are provided to help visualize the key regions of interest. In Fig. 9(a), we seek to isolate the coupling between streaming and radial waves. Being radially outward in all directions, streaming opposes the radial velocity waves in the right-hand side sector of the domain, thus leading to a decreased local wave speed. By the same token, it enhances the radial wave in the left-hand side sector, where it promotes further growth in the radial velocity. In Fig. 9(b), the coupling with the tangential wave is examined. Given that streaming in the outer region is accompanied by small clockwise rotation [see Fig. 5(c)], its superposition on the counterclockwise motion of the tangential waves gives rise to regions with tangential velocity defect or excess, in the top



FIG. 9. (Color online) Sectors in which oscillatory waves are enhanced or weakened by virtue of streaming. These illustrate the outcome of interactions between (a) radial and (b) tangential velocities with the streaming motion. For example, in part (a) the radially outward streaming contributions act to decelerate the radial wave in the right half while accelerating it in the left half. In part (c) the main regions of interest are delineated along with their pertinent equations.

and bottom halves of the domain, respectively. In practice, the coupling configurations shown in Figs. 9(a) and 9(b) occur simultaneously, thus leading to the patterns shown in Fig. 8(b). Finally, to summarize the results obtained heretofore, Fig. 9(c) is used to delineate the main regions of interest and their pertinent solutions. For example, within the boundary layer region, the viscous treatment is most relevant. Applicable solutions include Eq. (87) for the first-order traveling wave solution and Eqs. (116)–(125) for the steady, secondorder transverse velocities. In the outer region, the complete potential flow solution is depicted as the sum of the inviscid, irrotational, time-dependent field, given by Eqs. (37) and (50), and the viscous, rotational, steady streaming field given by Eqs. (126) and (127).

#### V. CONCLUSIONS

In this study, closed-form analytical solutions are derived to describe the behavior of secondary flows generated by parallel wave incidence over a uniformly injecting headwall. Of particular interest is how the streaming motion affects the oscillating field, especially in the tangential and radial directions. From the flow patterns depicted in Figs. 8 and 9, some interesting results may be inferred. Along the nodal pressure line [equator line in Fig. 8(a)], the flow field is heavily dominated by radial velocities. Specifically, it is shown that along the nodal lines the flow is directed toward the center of the chamber on one side and out from the center on the other. Assuming that the velocity is proportional to the gradient of the pressure, a conclusion about the corresponding wave form may be inferred. In Fig. 8, the region where the velocity vectors are counterclockwise corresponds to a positive pressure region with the peak amplitude occurring along the outer circumference. Conversely, in the region where the flow is clockwise (down below the nodal line), a negative pressure region is formed with the troughs occurring along the outer circumference as well. Along the nodal line, where the velocity vectors converge or diverge, a transition from a positive to a negative pressure region is realized. We note that the second-order streaming flow for a traveling wave is axisymmetric, with a strong outward pointing radial component. Therefore, in the case where the secondary flow is large enough to influence the first-order oscillations, the radial coupling along the nodal line is affected the most. In the absence of streaming, an observer situated at the north or south poles [Fig. 8(a)] will witness the largest tangential velocities sweeping by. In the presence of streaming, the flow will no longer be tangential as it gains an outward pointing radial component near the poles [Fig. 8(b)]. Along the equator line, the potential flow that is originally radial will be either enhanced or weakened downstream and upstream of the core, respectively. The result is a steepened wave form similar to that described by Pierce<sup>36</sup> in the case of a plane wave. It should be noted that as per Fig. 3, the secondary flow is one order of magnitude smaller than  $\mathbf{u}^{(1)}$ . Recalling that the problem is linearized by the ratio of the pressure fluctuations to the mean pressure,  $\varepsilon$ , terms at the second order in  $\varepsilon$  are small. This will remain true until the peak-to-peak amplitudes of the pressure oscillations become comparable to the chamber pressure, as reported in Clayton's data<sup>18</sup> and other experimental measurements taken in liquid rockets.

Our work clearly demonstrates the origination of streaming flows induced by tangential oscillations near a liquid rocket engine faceplate. Solutions to the first and secondorder boundary layers are presented and discussed. The secondary flow patterns are found to increase the first-order pressure gradient in some areas and to decrease it in others. This process is associated with a steepening of the wave profile. Experimental evidence in the above discussion supports the development of a traveling wave form that displays a sharp pressure spike followed by a long shallow trough. Our study calls for further investigations to relax some of the limiting assumptions used here. Since tangential wave structures can steepen when interacting with an injector faceplate, a more elaborate model may be required to obtain a complete solution for the fully steepened waves. A theoretical model that mirrors the analysis provided here can also be pursued to capture the motion of standing transverse waves in chambers with headwall mass addition. These models as well as others involving nonuniform injection are hoped to be discussed in forthcoming work.

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