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Analytical Methodologies for Hypersonic Propulsion

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In this study, we review several analytical frameworks that may be used in the context of high speed propulsion. To this end, inviscid and boundary layer models are examined in the light of modern mathematical techniques. A modified Rayleigh-Janzen approach is offered as a possible improvement over previous inviscid techniques for handling hypersonic flow problems. The Successive Complementary Expansions Method (SCEM) is also examined as a general perturbation technique with application to the hypersonic boundary layer equations. Furthermore, the Homotopy Analysis Method (HAM) is introduced as a powerful yet seemingly overlooked approach for solving highly nonlinear problems. The HAM framework has been successfully employed over a wide range of topics ranging from fluid mechanics to finance. The strengths and weaknesses of these analytical methods are discussed to the extent of providing the reader with a roadmap directed toward their effective use.

I. Introduction

THOUGH frequently neglected in this day and age, analytical methods remain among the most powerful tools of modern engineering development. With the notable advancements in computational approaches, analytical methods are often abandoned in favor of more inclusive CFD models. While it is true that comprehensive CFD simulations can capture a level of detail and complexity not often matched by analytical models, they also require a considerable investment of time and talent development that can make them untenable options in preliminary design stages.

Instead of considering analytical methods and CFD models as rival techniques, it may be best to view them as mutually complementary tools in the development process. Analytical methods are best applied early in the design process where changing project requirements prohibit extensive computational treatments. Their (often) explicit closed forms provide valuable insight into the fundamental physical mechanisms that control a problem and this, in turn, guides the deployment of CFD codes to reduce computation time via an approximate, analytical starting point. The insight gained from a properly executed analytical framework also helps designers to avoid potential shortcomings early in the design process, when changes can be incorporated less expensively than in later testing stages.

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To apply such a framework to hypersonic engine development, it is beneficial to separate the scramjet engine into component parts and then evaluate each component separately. In a typical integrated hypersonic vehicle, the engine consists of the inlet, combustor, and nozzle. An isolator is often included between the inlet and the combustor to facilitate lower speed operation but in practice, can be lumped with the inlet or, more commonly, the combustor. This focus on component-level modeling allows each separate section to be studied independently, without affecting the performance of the other components. The result is an analytical framework that can be tuned both for accuracy where complicated models are needed and for performance where simpler analytical formulations may suffice.

The modular approach to hypersonic systems analysis is not a new one. In the 1960s, the work by Fred Billig[1] gave rise to the modular Ramjet Performance Analysis (RJPA) code developed by the Applied Physics Laboratory at Johns Hopkins University. As technology and modeling capabilities improved, so did the code. Many improvements were initiated by other research groups such as integration of the Naval Ordnance Testing Station's chemical modeling routines.[2] Continual revisions and improvements, as well as premier technical support, made the RJPA code a useful tool for hypersonic designers.[3] The RJPA code and its architecture have had a positive impact on the development of modern hypersonic performance analysis. The modular structure approach has allowed investigators to develop codes focusing on very specific problems or concepts, which then could be integrated into a larger, more comprehensive code.[4-7]

Boundary layer modeling is a natural starting point for the improvement of hypersonic engine analysis through asymptotic methods. The recent advances in the modeling of interactive boundary layer flows[8-10] provide a solid foundation to extend our analytical framework. Additionally, Majdalani and Halpenny,[11] Vyas and Majdalani,[12] and Batterson and Majdalani[13] have examined viscous effects inside the bidirectional vortex engine and their methodology could be adapted to the internal modeling of viscous effects in a combustion chamber as well as on the outer surfaces of a hypersonic vehicle. The quest for a closed form, analytical solution to the interactive boundary layer constitutes a challenging paradigm that will hopefully be addressed in future research.

Improved boundary layer stability and transition models constitute another research avenue that requires exploration. Because of the increased thermal loads at hypersonic speeds, accurate prediction of flow characteristics near all surfaces is of paramount importance. The triple deck theory provides the basis for much of the analytical research into hypersonic boundary layer analysis.[14] Taking advantage of asymptotic methods[15] and advances in perturbation theory, it stands to reason that our understanding of hypersonic boundary layer stability can be improved. Current research in nonlinear stability theory may also prove helpful in furthering our understanding of the highly unpredictable transition problem.

In the inviscid modeling of hypersonic flows, there is potential for improvement through an inverse form of the Rayleigh-Janzen perturbation approach. The traditional form is successfully applied to the study of compressibility effects in a chamber. Though the extension to the hypersonic regime is not a straightforward adaptation, an improved model for the inviscid flowfield can prove useful by allowing researchers to employ one robust model to resolve most of their hypersonic modeling needs, rather than selecting between three or four approximate models that depend on fitting parameters.

Another promising topic for asymptotic exploration stands in the resolution of pressure effects. There are a number of methods that could be employed to determine pressure effects analytically. These can range from basic oblique shock theory to a more complete accounting using piston theory[16] to determine the unsteady pressure effects. This particular area shows great potential for improvement using the asymptotic techniques applied by Majdalani and Flandro[17] to internal flowfields. This study captures unsteady pressure effects in a combustion chamber and, if adapted to an external configuration, could provide a more complete pressure model. In addition, there is a possible application to the stability analysis of a hypersonic combustion chamber (see Flandro, Fischbach and Majdalani[18]). However, the supersonic speeds in a scramjet combustor would require a modification of the technique.

In most large scale propulsion devices, the modeling of acoustic stability remains a challenge. From liquid and solid rocket motors to gas turbines, nearly every major propulsive program has had some experience with instability. An analytical framework is necessary to achieve an understanding of the problems inherent in the release of large quantities of energy in confined spaces and to eliminate any harmful effects that such problems may cause. Left untreated, motor instabilities can result in severe oscillations, increased localized heat transfer, and increased combustor pressure. Recent advances in the modeling of liquid rocket engine stability have shown promise in accurately predicting key parameters in nonlinear stability such as mean pressure increases, oscillatory frequencies, and thrust variations. Continuing research in this area is vital to the stability of hypersonic combustors as well as traditional propulsion engines. It would surely be detrimental to the development of hypersonic engines to forgo this area of research until instabilities suddenly appear, as they often do in large scale engine tests.

What follows is a description of available analytical methods for hypersonic analysis and a discussion of future research applications.

II. Inviscid Methods

Most of the inviscid models currently in use were developed during the 1950s and 1960s. The most well known of these are based on small perturbation theory, an approach that relies on the slender body assumption to linearize the attendant hypersonic equations. Because the analytical solutions are dependent on the slender body constraint, the usefulness of the model for more realistic problems is limited. Even a slightly blunted slender body can produce strong local shocks, which would then limit the predictive capabilities of the model. Newtonian theory[19] is also advanced as a possible simplification, using a limiting assumption of $(\gamma - 1)/(\gamma + 1) \ll 1$. However, this assumption may not be physically reasonable for the cases of hypersonic flow as it implies a thin shock layer that may not exist for certain cases. In addition to these two general models, several modifications are needed to account for various geometries or flow conditions. For example, methodologies exist that reduce the problem to an analogous wedge or cone flow configuration.

The dearth in recent inviscid hypersonic research may be connected to the prevalent perception that available models are adequate for most hypersonic modeling needs (some of these methods are described in gas dynamics textbooks). When trying to develop a mathematical model that can be reconciled with experimental or computational results, researchers are often forced to choose an approximate method that best describes their results. While problem-specific approximations may work well for a given case, an analytical method that is valid for a wide range of shapes and configurations is desirable. It appears that current advances in asymptotic theory and available computational tools can lead to useful solutions that were previously discarded due to mathematical complexity.

In previous work, Maicke and Majdalani[20] employed a perturbation approach to produce a compressible extension to the Taylor solution in a porous channel. The solution is important to the present problem of analytical hypersonic modeling in two ways. The first, and most direct, application pertains to a combined cycle engine. Because of the very specific operating range of an air-breathing hypersonic engine, most vehicle design concepts are paired with more traditional propulsion systems to bring the vehicle to the hypersonic operating range. In this case, the insight gained from the analytical modeling can be directly applied to the subsonic flow in a combined cycle engine. The second application consists of adapting the methodology to hypersonic combustion via an inverse approach. In order to fully explain this, we outline the compressible solution employed in the rectangular channel with sidewall injection.

To model a rectangular enclosure with an injecting wall, a chamber of length L_0 and height of h is used. The origin of the coordinate system describing the domain is located at the bottom of the inlet section. The spatial variables $\overline{x}$ and $\overline{y}$ are defined as the directions parallel and normal to the bottom wall, respectively. A solution can be obtained for the chamber, namely $0 \le \overline{y} \le h$ and $0 \le \overline{x} \le L_0$. The bottom surface at $y = 0$ can be used to represent a hard wall (see Figure 1).

Along the top wall, a uniform injection velocity of U_w is imposed. While there are a number of factors that can affect the local velocity at the injecting surface, including density fluctuations, localized non-homogeneity of the injectant, and unstable burning, this constraint gives a reasonable approximation of the mechanism at the injecting surface. The resulting model corresponds to the steady, inviscid, compressible, and non-heat-conducting flow of an ideal gas.

Employing a compressible stream function approach, we use the vorticity definition and set

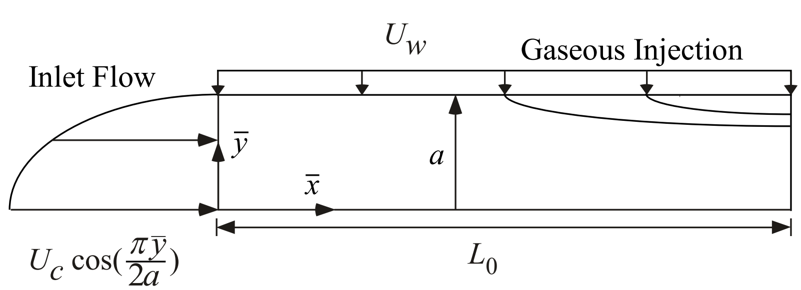

Figure 1. Chamber geometry including inlet injection profile.

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{1}{\rho}\left(\nabla\rho\cdot\nabla\psi\right)-\Omega\rho \tag{1}$$

To solve Eq. (1), the vorticity transport equation may be used, namely,

$$\nabla\times\left(U\times\Omega\right)=\frac{1}{\gamma M_w^2\rho^2}\nabla\rho\times\nabla p \tag{2}$$

where M_w is the wall Mach number based on U_w and the speed of sound. The momentum equation may then be solved to determine the pressure from

$$-\frac{\nabla p}{\gamma M_w^2}=\rho\nabla\left[\frac{1}{2\rho^2}\left(\nabla\psi\cdot\nabla\psi\right)\right]+\Omega\nabla\psi \tag{3}$$

Finally, the isentropic relations may be invoked to bring closure to the thermodynamic variables via

$$\rho=p^{1/\gamma},\quad T=p^{(\gamma-1)/\gamma} \tag{4}$$

The boundary conditions here stem from the physical determination of the system. Since the injection at the top wall is normal to the surface, there is no axial flow at the wall. The headwall of the chamber is an inlet, providing a similarity-conforming half-sinusoidal profile (see Figure 1). A uniform injection velocity is imposed at the top wall. Mathematically, these boundary conditions translate into

$$u(x,1)=0,\quad u(0,y)=U_0\sin\left(\tfrac{1}{2}\pi y\right),\quad v(x,1)=-1,\quad v(x,0)=0 \tag{5}$$

In order to solve (1)–(4), a Rayleigh–Janzen perturbation may be applied. This requires expanding:

$$\begin{aligned}
&u(r,z)=u_0+M_w^2u_1+O(M_w^4) && \rho(r,z)=1+M_w^2\rho_1+M_w^4\rho_2+O(M_w^6)\\
&v(r,z)=v_0+M_w^2v_1+O(M_w^4) && p(r,z)=1+M_w^2p_1+M_w^4p_2+O(M_w^6)\\
&\psi(r,z)=\psi_0+M_w^2\psi_1+O(M_w^4) && T(r,z)=1+M_w^2T_1+M_w^4T_2+O(M_w^6)\\
&\Omega(r,z)=\Omega_0+M_w^2\Omega_1+O(M_w^4) &&
\end{aligned} \tag{6}$$

Substitution of the expanded variables gives rise to the following leading order equations

$$\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial y^2}=-\Omega_0 \tag{7}$$

$$\nabla\times(\boldsymbol{U}_0\times\boldsymbol{\Omega}_0)=0 \tag{8}$$

$$-\frac{\nabla p_1}{\gamma}=\nabla\left(\frac{\nabla\psi_0\cdot\nabla\psi_0}{2}\right)+\Omega_0\nabla\psi_0 \tag{9}$$

$$\rho_1=\frac{p_1}{\gamma},\quad T_1=\frac{\gamma-1}{\gamma}p_1 \tag{10}$$

Similarly, the leading-order expansion of the boundary conditions, when expressed in terms of the streamfunction, becomes

$$\frac{\partial\psi_0(x,1)}{\partial y}=0,\quad \frac{\partial\psi_0(0,y)}{\partial y}=0,\quad \frac{\partial\psi_0(x,1)}{\partial x}=-1,\quad \frac{\partial\psi_0(x,0)}{\partial x}=0 \tag{11}$$

The first order governing equations, containing the compressible corrections, are similarly obtained; one extracts

$$\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_1}{\partial y^2}=\nabla\rho_1\cdot\nabla\psi_0-\Omega_0\rho_1-\Omega_1 \tag{12}$$

$$\nabla\times(\boldsymbol{U}_0\times\boldsymbol{\Omega}_1)+\nabla\times(\boldsymbol{U}_1\times\boldsymbol{\Omega}_0)=\nabla\rho_1\times\nabla p_1 \tag{13}$$

$$-\frac{\nabla p_2}{\gamma}=\nabla\Big[\Big(\nabla\psi_0\cdot\nabla\psi_1\Big)-\rho_1\Big(\nabla\psi_0\cdot\nabla\psi_0\Big)\Big]+\rho_1\nabla\left(\frac{\nabla\psi_0\cdot\nabla\psi_0}{2}\right)+\Omega_0\nabla\psi_1+\Omega_1\nabla\psi_0 \tag{14}$$

and

$$\rho_2=\frac{p_2}{\gamma}+\frac{1-\gamma}{\gamma}p_1^2,\quad T_2=\frac{\gamma-1}{\gamma}p_2+\frac{1-\gamma}{2\gamma^2}p_1^2 \tag{15}$$

Since the boundary conditions must be satisfied by the leading-order equation, a set of homogeneous boundary conditions must be imposed from this point forward. The compressible corrections for the stream function and pressure can thus be obtained. One finds

$$\psi_1=-\frac{1}{48}x\sin\left(\frac{1}{2}\pi y\right)\Big\{\pi^2x^2\Big[3+\cos\Big(\pi y\Big)\Big]+3\Big[7-\cos\Big(\pi y\Big)\Big]\Big\} \tag{16}$$

and

$$p_2=-\frac{1}{384}\gamma\pi^4x^4\Big(2-3\gamma\Big)+\frac{1}{64}\gamma\pi^2x^2\Big[7-\cos^2\Big(\pi y\Big)\Big]-\frac{1}{16}\gamma\Big[\cos\Big(\pi y\Big)+\cos^2\Big(\pi y\Big)\Big] \tag{17}$$

The first benefit to this analytical approximation is clear. The closed form nature of the solution permits the rapid calculation of the variables of interest for a large matrix of test cases. The second benefit is to drive future analysis of the design, either in CFD modeling or engine testing. For example, Figure 2 illustrates the axial velocity at two different locations in the channel. From the figures, we can see compressibility effects for this specific case become more important as the flow develops and fluid compression causes the velocity profile to steepen. In Figure 3 we show the streamlines for two different injection cases. In the first case, there is a significant difference between the compressible and incompressible streamlines, while in the second the difference is minor. One can thus isolate the range for which compressibility effects dominate.

Saad *et al.*[21] have determined the incompressible solution to a tapered channel via a transformation of the boundary conditions. Through this transformation, the directional derivative of the stream function is defined as

$$\frac{\mathrm{d}\psi_s}{\mathrm{d}s}=\frac{\partial\psi_s}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial\psi_s}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}=\frac{\partial\psi_s}{\partial x}\cos\alpha+\frac{\partial\psi_s}{\partial y}\sin\alpha \tag{18}$$

where ψ_s is the stream function at the burning surface, s is the length along the taper from the head end, and α is the taper angle. This directional derivative is applied to the stream function equation and is perturbed by $\epsilon=\sin\alpha$, where the taper angle is sufficiently small to allow such a perturbation. In the case of the compressible channel, the solution entails a doubly perturbed problem in the Mach number and the taper

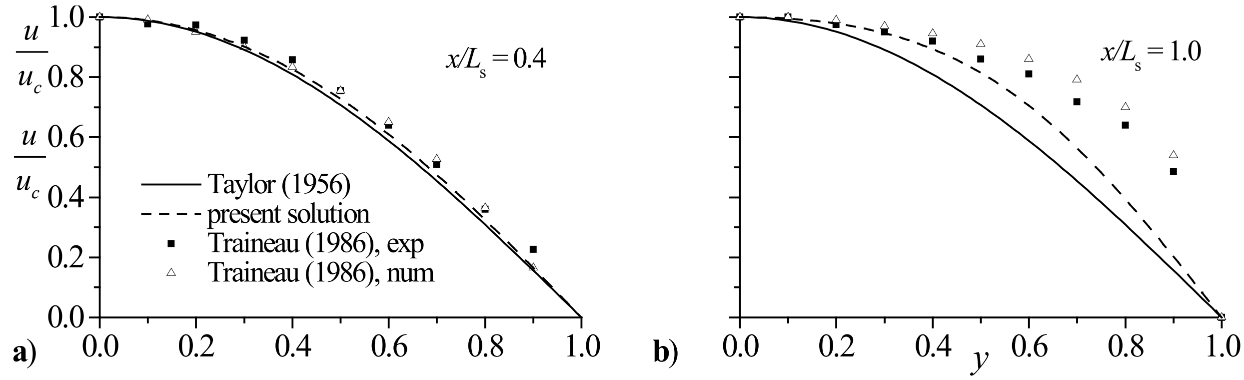

Figure 2. Axial velocity at two channel locations.

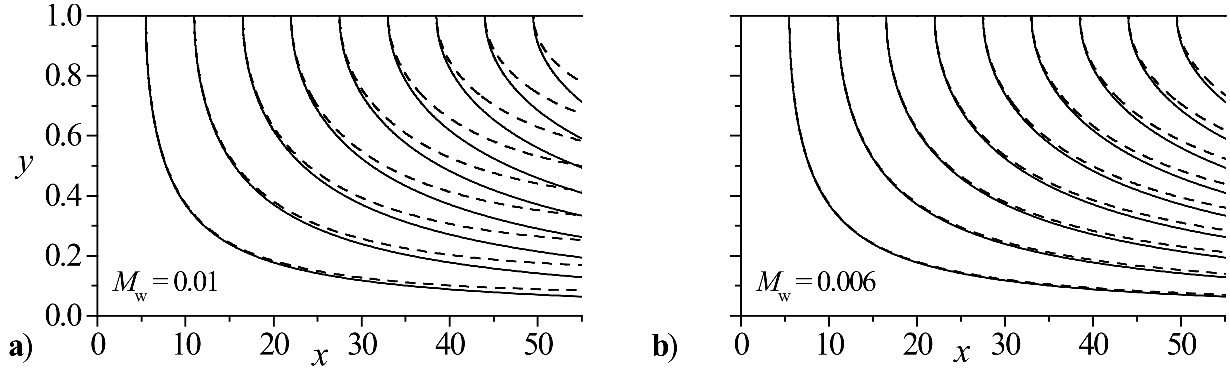

Figure 3. Streamlines for two injection Mach numbers.

angle. A similar approach can hence be used to solve the compressible problem described above assuming a tapered geometry.

In the previous discussion we illustrated how the compressible approximation could be of use in analyzing the flow in a combined cycle engine design. However, the method described here could also be modified and applied to the analysis of a purely hypersonic engine. In our analysis, we exploit the small injection Mach number present in large combustors in order to perturb the governing equations and arrive at an asymptotic solution that captures compressibility effects. In a hypersonic system, one could utilize an inverse approach that is based on the reciprocal of the average flow Mach number:

$$\epsilon = \frac{1}{M_\infty^2} \tag{19}$$

One can thus take advantage of the large Mach number and then apply a similar expansion to the variables of interest via

$$\begin{aligned}
u(r,z) &= u_0 + \epsilon u_1 + O(\epsilon^2) & \rho(r,z) &= 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + O(\epsilon^4) \\
v(r,z) &= v_0 + \epsilon v_1 + O(\epsilon^2) & p(r,z) &= 1 + \epsilon p_1 + \epsilon^2 p_2 + O(\epsilon^4) \\
\psi(r,z) &= \psi_0 + \epsilon\psi_1 + O(\epsilon^2) & T(r,z) &= 1 + \epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^4) \\
\Omega(r,z) &= \Omega_0 + \epsilon\Omega_1 + O(\epsilon^2) & &
\end{aligned} \tag{20}$$

For a similar two dimensional approach, one could solve the hypersonic equations using

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho \nabla \cdot \boldsymbol{u} = 0 \tag{21}$$

$$\rho\left(u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y}\right) + \nabla p = 0 \tag{22}$$

$$u\frac{\partial (p\,/\,\rho^{\gamma})}{\partial x} + v\frac{\partial (p\,/\,\rho^{\gamma})}{\partial y} = 0 \tag{23}$$

By applying an asymptotic expansion in the reciprocal of M_∞^2, a viable hypersonic flow expansion can thus be pursued.

III. Viscous Methods

The hypersonic boundary layer problem has piqued the interest of numerous researchers since the late 1960s. In traditional boundary layer theory, the inviscid mean flow is determined analytically and the boundary layer equations are calculated near the wall. The solutions are then matched asymptotically to provide a uniformly valid result. However, this methodology does not provide acceptable results for a hypersonic boundary layer. For example, shock / boundary layer interactions can cause severe flow separation which, in turn, would render traditional methods inadequate.[22] Additionally, during hypersonic reentry, low Reynolds numbers arise due to increased kinematic viscosity, thus creating a thicker boundary layer that requires special treatment.[23]

The primary challenge in the modeling of hypersonic boundary layers lies in the coupling of the mean flow and the boundary layer. This coupling results in a change in the mean flow outside of the boundary layer, which often lends itself to an iterative numerical procedure. Indeed, most of these analytical models for interactive boundary layers are eventually solved numerically. Though not as useful as a closed form solution in either understanding or calculation, the numerical treatment has several benefits. First, it is often easier to adapt to more complicated geometries. Fully analytical solutions are often necessarily limited by the geometry of the problem and extensions to other geometries are not always apparent. Secondly, the numerical solution of the analytical framework can provide a reasonable approximation of the complex interactive flowfield at a fraction of the computation time that a full solution of the Navier-Stokes equations would require. This is a key benefit of the analytical framework, especially in the evaluation stages of hypersonic engine design.

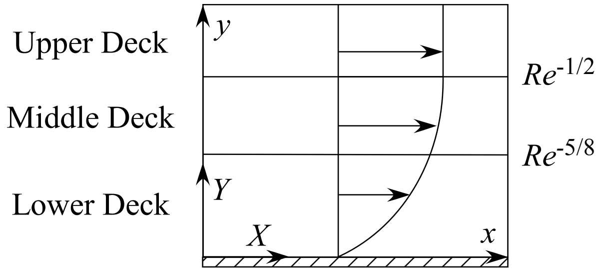

Figure 4. Description of the triple deck structure.

A. Triple Deck Theory

The first hypersonic boundary layer models are advanced using triple deck theory, generally attributed to Stewartson and Williams,[22] Neiland,[24] and Messiter[25] whose collective contributions to this approach have been invaluable. The triple deck theory posits that the boundary layer is structured as a thin, viscous-dominated layer near the wall. Outside this layer, a thicker middle deck appears and may be determined from the alterations made to the mean flow by the coupled boundary layer equations. Finally, the outer deck contains the traditional mean flow solution (cf. Figure 4). Stewartson's work is directly motivated by a high speed flow application, namely the separation of a laminar boundary layer interacting with a strong shock. The work is primarily directed towards determining a rational explanation for the laminar boundary layer that separates before the incident shock. The parabolic nature of the boundary layer equations prevents upstream propagation of disturbances. The supersonic mean flow does likewise. However, between the supersonic flow and the boundary layer at the wall there exists a subsonic region that allows for such a propagation. Though not a complete explanation of the phenomenon, it provides a starting point to formulate a boundary layer framework that can account for the type of separation observed experimentally.

Based on the work of Stewartson and Williams,[22] it is determined that the scale of the interactions evolves over $O(Re^{-3/8})$, which gives justification to the following transformations

$$\epsilon = Re^{-1/8}, \quad y = Y\epsilon^4, \quad x = x_0 + \epsilon^3 X \tag{24}$$

where X and Y are the boundary layer variables (see Figure 4). These are paired with the expansions of velocity and thermodynamic quantities, namely

$$\begin{aligned} u &= U_0(Y) + \epsilon u_1(X,Y) + \epsilon^2 u_2(X,Y) + \epsilon^3 u_3(X,Y) + \ldots \\ v &= \epsilon^2 v_1(X,Y) + \epsilon^3 v_2(X,Y) + \epsilon^4 v_3(X,Y) + \ldots \\ p &= p_\infty + \epsilon^2 p_2(X,Y) + \epsilon^3 p_3(X,Y) + \ldots \\ \rho &= \rho_0(Y) + \epsilon\rho_1(X,Y) + \epsilon^2\rho_2(X,Y) + \epsilon^3\rho_3(X,Y) \end{aligned} \tag{25}$$

Here u and v are the streamwise and normal velocities, p is the pressure and ρ is the density. For the two-dimensional problem, these expansions are applied to the governing equations to produce

$$\rho_0 U_0 \frac{\partial u_1}{\partial X} + \rho_0 v_1 \frac{\partial U_0}{\partial Y} = 0 \tag{26}$$

$$\rho_0 U_0 \frac{\partial u_2}{\partial X} + \rho_0 v_2 \frac{\partial U_0}{\partial Y} + \frac{\partial p_2}{\partial X} = -\rho_0 u_1 \frac{\partial u_1}{\partial X} + \rho_0 v_1 \frac{\partial u_1}{\partial Y} \tag{27}$$

$$0 = -\frac{\partial p_2}{\partial Y} \tag{28}$$

$$\rho_0 U_0 \frac{\partial v_1}{\partial X} = -\frac{\partial p_3}{\partial Y} \tag{29}$$

$$\gamma p_\infty \left(U_0 \frac{\partial \rho_2}{\partial X} + v_1 \frac{\partial \rho_0}{\partial Y} \right) = 0 \tag{30}$$

$$\rho_0 U_0 \frac{\partial p_2}{\partial X} - \gamma p_\infty \left(U_0 \frac{\partial \rho_2}{\partial X} + v_2 \frac{\partial \rho_0}{\partial Y} \right) = \gamma p_\infty \left(u_1 \frac{\partial \rho_1}{\partial X} + v_1 \frac{\partial \rho_1}{\partial Y} \right) \tag{31}$$

which are the equations that govern the middle deck. Stewartson and Williams[22] further manipulate these equations and match them to the inner boundary layer and the supersonic mean flow to find the equations for the fundamental boundary layer.

To arrive at a final answer, an ad hoc numerical method is employed to solve the fundamental boundary layer model. Qualitative agreement is found, but quantitative errors of up to 20% are present. This is expected, as the analysis here can be taken as a first order approximation of the hypersonic boundary layer interactions. Retaining more terms will invariably reduce the error residual. This is especially true for hypersonic boundary layers, which are often characterized by relatively low Reynolds numbers that reduce the accuracy of the perturbation solution. In the end, Stewartson and Williams[22] determine that it is not a subsonic area between the wall and the supersonic flow that is responsible for the early separation, but rather the interaction between the boundary layer and the supersonic stream. This example illustrates the subtle advantage of setting an analytical framework in place to study complex problems. The triple deck theory could be seen as having been inspired by the speculative idea of a subsonic middle region that can promote the upstream propagation of disturbances. In examining this hypothesis and using it to guide a structured approach to the problem, an unexpected result is found that provides insight into the actual mechanisms at work in self-induced separation.

Since the original studies in triple deck theory, a number of numerical techniques have been developed and refined to address the calculation of interacting boundary layers.[26,27] This often requires solving an inverse or semi-inverse problem in which the boundary layer displacement thickness or wall shear are specified as a boundary condition instead of the pressure. The triple deck theory is not limited only to numerical solutions of the analytical framework. Smith[28] shows that an analytical solution to the triple deck formulation can be obtained via Fourier transform methods. This methodology is later utilized by a number of researchers to effectively determine analytical solutions for interactive boundary layer problems.[29,30]

The triple deck theory is employed in the current literature as evidenced by the work of Inger and Gnoffo[30] in investigating temperature jumps induced by sudden changes in thermal protection properties and for shock/boundary layer interactions.[31] Their application of the triple deck theory to wall temperature jumps is of particular interest to the design of hypersonic motors. Thermal Protection Systems (TPS) remain one of the key challenges facing sustained hypersonic speeds. In studying the behavior of wall temperature jumps at hypersonic speeds, Inger and Gnoffo[30] attempt to assist in the preliminary design of TPS, particularly where seams or changes in protection can occur, either from design or manufacturing concerns.

B. Successive Complementary Expansions

Recently, advances in perturbation theory have granted researchers freedom to generalize the triple deck theory and arrive at uniformly valid solutions for interactive boundary layers. Most notable is the work by Mauss and Cousteix[10] who, in a series of papers, formalized an approach to the interactive boundary layer problem using their

Successive Complementary Expansions Method (SCEM).[8-10,23] While not specifically focused on hypersonic analysis, the SCEM is a promising tool because it does not require the application of a matching principle to construct a uniformly valid approximation. The basis of this formulation is the use of non-regular expansions to generalize the problem, mathematically,

$$\Phi = \sum_{i=1}^{n} \delta_i(\epsilon)\Big[\phi_i(x,\epsilon) + \psi_i(X,\epsilon)\Big] \tag{32}$$

where δ_i is a gauge function and arguments ϕ_i and ψ_i are expansions of the outer and inner domains respectively. Equation (32) is a non-regular expansion because the arguments ϕ_i and ψ_i are functions of not only the variables of the problem (x and X) but also the perturbation parameter, ϵ. By contrast, the definition of a regular expansion stands as

$$\Phi = \sum_{i=1}^{n} \delta_i(\epsilon)\Big[\phi_i(x) + \psi_i(X)\Big] \tag{33}$$

Here, the arguments are only written in terms of the variables of the problem. Regular expansions are generally easier to handle analytically. The complexity of the problem is reduced by excluding the perturbation parameter dependence. With the generalized non-regular expansion, more information is captured at leading order, thus yielding a more accurate expansion at an earlier truncation order.

Cousteix and Mauss[8] show that by expressing the triple deck theory in terms of regular expansions, one can recover it as a subset of the Successive Complementary Expansion Method. The consistency of SCEM with triple deck theory is encouraging. It highlights the effectiveness of the triple deck theory in providing a judicious mathematical framework that can be continually improved. It also proves that for a subset of the SCEM, closed form analytical solutions are possible. Most of the SCEM endeavor, like much of the work in hypersonic boundary layer theory, applies numerical methods to solve a set of mathematical equations. There exists a great desire to apply this method to other problems that are amenable to closed form solutions. Knowledge gained from such a process can then be used to shape decisions in the preliminary design stages and to guide analysis of later stage designs with CFD tools.

IV. Homotopy Analysis Method

Though not strictly an inviscid method, the Homotopy Analysis Method (HAM) has been used to investigate viscous flows.[32-34] Liao[35] developed the technique in his 1991 dissertation to analyze strongly nonlinear problems. Its main difference from traditional perturbation methods is the following. Perturbation methods are limited by the necessity of a small (or large) parameter. It is through the exploitation of such a parameter that researchers are able to arrive at closed form series expansions. HAM has no such requirement. The homotopy analysis method arises from a topological concept that continuous transformations from one function to another are possible if they are homotopic. This basic notion is used in certain numerical methods which have been used in the design of control systems for hypersonic flight.[36] For a nonlinear problem, HAM is applied as follows; first define a nonlinear operator via

$$A[u(t)] = \dot{u} + u^2 - 1 = 0 \tag{34}$$

The nonlinear equation of interest is written in terms of the dependent variable u and the independent variable t. Next, define an arbitrary linear operator such that

$$L[f] = 0 \quad \text{when} \quad f = 0 \tag{35}$$

The homotopy equation can then be formed using these operators, namely,

$$H\left[\phi\left(t;q\right);q,\hbar\right] = \left(1-q\right)L\left[\phi\left(t;q\right) - u_0\left(t\right)\right] + q\hbar H(t)A\left[\phi\left(t;q\right)\right] \tag{36}$$

where q is called the embedding parameter, $\hbar$ is the convergence-control parameter, H is an auxiliary function, and ϕ is a temporary function. As the embedding parameter goes to zero, the homotopy equation reduces to the linear operator. As $q \to 1$, the original nonlinear operator is recovered. Taylor's theorem can be applied to ϕ, thus yielding

$$\phi\left(t;q\right) = u_0\left(t\right) + \sum_{m=1}^{\infty} q^m u_m\left(t\right) \tag{37}$$

The homotopy equation may be differentiated m times to obtain a system of linear ODEs. For example,

$$\begin{aligned} &m = 1 \qquad L\left[u_1\right] = \hbar H\left(t\right)\left(\dot{u}_0 + u_0^2 - 1\right) \\ &m = 2 \qquad 2L\left[u_2 - u_1\right] = \hbar H\left(t\right)\left(\dot{u}_1 + 2u_0 u_1\right) \\ &\vdots \end{aligned} \tag{38}$$

We then select a set of base functions and apply them to the set of ODEs to determine a solution, adjusting $\hbar$ as necessary for convergence. For a polynomial set of base functions, u can be expressed as

$$\begin{aligned} &u_1\left(t\right) = \frac{1}{3}\hbar\, t^3 \\ &u_2\left(t\right) = \frac{1}{3}\hbar\left(1+\hbar\right)t^3 + \frac{2}{15}\hbar^2 t^5 \\ &u_3\left(t\right) = \frac{1}{3}\hbar\left(1+\hbar\right)^2 t^3 + \frac{2}{15}\hbar^2\left(1+\hbar\right)t^5 + \frac{17}{315}\hbar^3 t^7 \\ &\vdots \end{aligned} \tag{39}$$

In summation notation, the complete solution reads

$$u_n\left(t\right) = \sum_{j=0}^{n} \alpha_{2j+1}\mu\left(\hbar\right)t^{2j+1} \tag{40}$$

To illustrate the power of the method, we compare results obtained using different techniques in Figure 5. In Figure 5a, we compare the exact solution, $\tanh t$, with solutions obtained via perturbation methods assuming small values of t, and to HAM results using polynomial and exponential base functions. For values of $t < 1$, all solutions show good agreement with the exact solution. After this point the perturbation solution diverges, as we no longer have a valid assumption regarding the size of t. The five term polynomial solution extends the range of validity to approximately 2.5, but then it also begins to diverge. However, polynomials are not particularly well-suited to approximate $\tanh t$. Because HAM allows for tremendous flexibility in choosing the base function, one can posit an exponential base function.

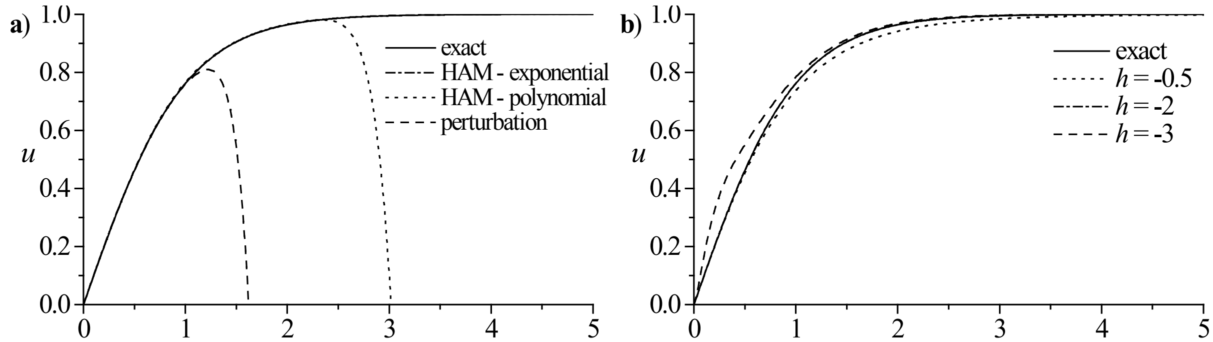

Figure 5. Comparison of solutions for a) different analytical techniques and b) different values of the convergence-control parameter $\hbar$.

When retaining five terms, the approximate expression from HAM is virtually indistinguishable from the exact solution.

In Figure 5b, we compare different values of the convergence-control parameter for the exponential solutions. For $h = -0.5$, there is acceptable agreement for the lower values of t, but the solution diverges slightly as t increases. When $h = -3$, there is a significant deviation for small values of t, but the difference vanishes as t increases. As expected, the ideal setting for the convergence-control parameter is between these values at approximately -2. For this case, the HAM solution is coincident with the exact expression.

This new ground-breaking method has been applied to a variety of nonlinear problems with remarkable results.[32,37,38] Owing to its generality, HAM has been successfully administered in a wide range of fields including biological systems,[39] soliton modeling,[40] shallow water waves,[40] boundary layer similarity equations,[41] heat transfer,[42] nonlinear oscillators,[43] finance,[44] and others.[45] Clearly, this method can be used to tackle many unsolved hypersonic problems featuring nonlinear interactions. A recent application by Xu *et al.*[46] illustrates the application of HAM to the regression porous channel flow problem.

V. Conclusions

In this work, the major analytical tools that are applicable to hypersonic systems are reviewed. Possible extensions to existing hypersonic theory are discussed and areas requiring additional research are highlighted. These include:

- An inverse Rayleigh-Janzen formulation of the inviscid hypersonic equations.
- Application of the Successive Complementary Expansion Method and other modern perturbation techniques to the hypersonic boundary layer equations.
- Extension of boundary layer stability models through a generalized SCEM representation.
- Acoustic wave modeling in a hypersonic combustor.
- Calculation of unsteady pressure effects in both the chamber and on control surfaces through a modified perturbation approach.

- Application of modern advances in nonlinear methods (such as the homotopy analysis method, HAM) to boundary layer stability calculations.

In addition to these problems for future consideration, a number of analytical techniques are discussed, along with their merits and limitations:

- Rayleigh-Janzen Perturbation Expansion
  - Applies to subsonic (or the in the inverse case, hypersonic) inviscid flowfields.
  - Provides closed form solution tied to physical mechanisms.
  - Requires small (or large) Mach numbers in the asymptotic expansions.
  - Solution deteriorates as parameter size increases.
- Triple Deck Theory
  - Applies to hypersonic boundary layer problems.
  - Based on perturbation theory and requires a small (or large) parameter.
  - Analytical solutions are possible using transform methods.
  - Framework is often solved numerically.
- Successive Complementary Expansions Method (SCEM)
  - Generalized technique for problems with multiple scales.
  - Requires small (or large) parameters for expansion.
  - Applicable to both traditional and hypersonic boundary layer problems.
  - Non-regular expansions are more difficult to solve analytically.
  - Solutions often obtained numerically.
- Homotopy Analysis Method (HAM)
  - Approximate analytical technique for solving nonlinear problems.
  - Presence of small or large parameters is not required.
  - Produces series expansion type analytical solutions in series form.
  - Solutions are exact in the limit of Taylor series expansions.

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