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Boundary Layer Treatment of the Porous Channel Flow with Wall Regression

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This work extends a sequence of studies devoted to the analysis of the laminar flow in porous channels with retracting walls. This problem was originally used to model slab propellant grain regression by Zhou and Majdalani (Zhou, C., and Majdalani, J., "Improved Mean Flow Solution for Slab Rocket Motors with Regressing Walls," ***Journal of Propulsion and Power,*** **Vol. 18, No. 3, 2002, pp. 703-711. doi: 10.2514/2.5987). After identifying a subtle endpoint singularity that affects the former solution in its third derivative, a slow variable is introduced to capture the rapid variations in the channel's core. The core refers to the midsection plane where the shear layer is displaced due to hard blowing at the walls. Then using matched-asymptotic expansions with logarithmic corrections, a composite solution is developed following successive integrations that start with the third derivative. In the process, the inner correction is retrieved from the fourth-order equation governing the symmetric injection-driven flow near the core. The resulting approximation is expressed in terms of generalized hypergeometric functions and is confirmed using numerics and limiting process verifications. The composite solution is shown to outperform the former, outer solution, as the core is approached or as the injection Reynolds number is increased. Without undermining the practicality of the former solution outside the thin core region, the development of a matched-asymptotic approximation enables us to suppress singular terms, thus ensuring a uniformly valid outcome down to the fourth derivative.**

I. Introduction

THIS work seeks to provide a complete asymptotic solution to the steady two-dimensional flow of a viscous fluid in a porous channel with expanding or contracting walls. The channel is taken to be semi-infinite with uniformly porous sidewalls. An incompressible fluid is injected with constant relative velocity across its walls as shown in Fig. 1. Such an idealization serves to model a range of physical mechanisms including transpiration cooling, boundary layer control, jet mixing, surface ablation, propellant burning, and membrane separation. It has recently led to new exact solutions to the porous channel flow problem in the form of homotopy-based series that are not limited by the size of the crossflow Reynolds number.[1]

In transpiration cooling applications, the injection of a lower temperature fluid across the walls creates, on the one hand, a thermal barrier that protects the walls of the channel carrying a higher temperature fluid.[2] Boundary layer control, on the other hand, can be accomplished by injecting, redirecting, or pulsating streams of fluid that can serve to reduce drag or acoustic resonance on an aircraft wing or upstream of an ammunition bay.[3-6] In jet mixing processes, pulsating the jet in a controllable

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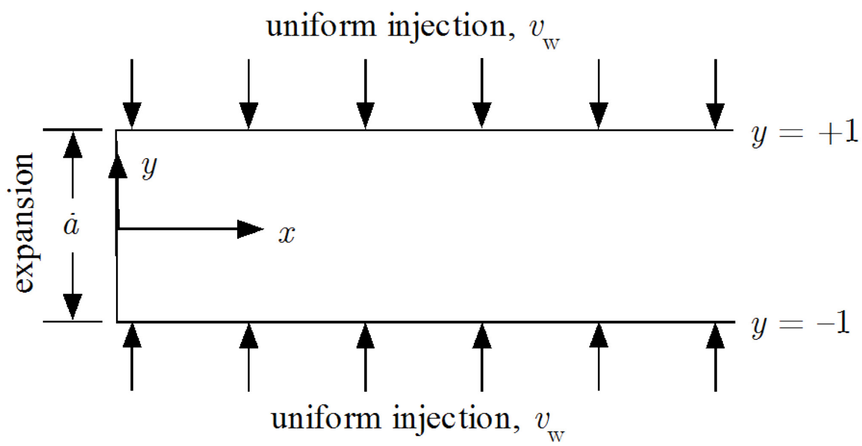

Figure 1. Flow geometry.

manner can substantially improve jet turbulence and penetration while reducing the length to achieve a given mixing state.[7,8] In combustion chambers and nozzles, injecting a cold layer of oxidizer can be instrumental in maintaining tolerable wall temperatures.[9] In chemical propulsion, the ejection of gases inside a thrust chamber can be simulated by the uniform injection of a fluid across porous and regressing walls.[10-16] Another application that has provided the original motivation for this class of studies is connected with the separation of uranium isotopes ^{235}U and ^{238}U by differential gaseous diffusion.[17]

In a prequel by Dauenhauer and Majdalani,[18] both injection and suction-driven flowfields are characterized inside a uniformly porous viscous channel with expanding and contracting walls. This model is also employed in the context of a slab burning solid propellant rocket motor.[19] By introducing a similarity solution for the streamfunction of the form $\psi = xF$, the Navier–Stokes equations are reduced to a single ODE for the characteristic function F. An asymptotic solution is then obtained, valid for a large crossflow Reynolds number $R = v_w a / \nu$, where R is based on the wall injection (relative) speed v_w and the channel half-spacing a.[20] This is followed by several related investigations,[21-26] including a recent study by Xu *et al*.[1] in which the Homotopy Analysis Method is used to obtain a series solution.

The motivation for this article and, hence, the driving factor behind the forthcoming analysis are connected to the following paradigm. The outer solution presented by Majdalani and Zhou[20] is marred by an essential singularity in its midsection plane that once appeared in Yuan's model[27] for porous channels with stationary walls. This endpoint singularity stems from a logarithmic term that appears in the solution's third derivative. The derivative in question controls the axial pressure gradient in the porous chamber and must be rectified lest its magnitude becomes suddenly unbounded at the core. Physically, the onset of irregular behavior signals the presence of a viscous layer along the channel's midsection plane, as once shown by Terrill[28] in similar context. In the present work, we follow Terrill's approach to the extent of not only uncovering the size and shape of the viscous layer, but also showing how this singularity may be removed through the use of matched-asymptotic expansions. Before closing, we present a uniformly valid approximation that leads to holomorphic vorticity, shear, and pressure fields across the fluid domain.

II. Outer Solution

We consider the injection-driven viscous flow inside a uniformly porous channel with expanding or contracting walls. As shown by Majdalani and Zhou,[20] one can apply similarity transformations in space and time to convert the Navier-Stokes equations into a well-posed fourth-order boundary value problem. This problem exhibits the form

$$R^{-1}F''' + \alpha R^{-1}\left(yF'' + 2F'\right) + FF'' - \left(F'\right)^2 = \lambda \tag{1}$$

where the boundary conditions are specified at the channel wall and midsection plane,

$$F''(0)=0\,,\;F(0)=0\,,\;F'(1)=0\,,\;F(1)=1. \tag{2}$$

As before, x and y represent the longitudinal and normal coordinates measured from the headwall and the midsection plane, respectively (see Fig. 1). They are made dimensionless by reference to the channel half height a. The wall expansion ratio $\alpha=\dot{a}a/\nu$ is the Reynolds number based on the speed of wall regression $\dot{a}$. For a large injection Reynolds number, the small parameter $\varepsilon=R^{-1}$ is used by Majdalani and Zhou[20] to solve Eq. (1) asymptotically. Following consecutive applications of the variation of parameters method, we obtain

$$F(\theta)=\sin\theta+\varepsilon\Big\{-(2\alpha/\pi)\theta+\left(\tfrac{1}{4}\pi-4\alpha/\pi\right)\left[(\theta\cos\theta-\sin\theta)\ln\tan\tfrac{1}{2}\theta-\cos\theta S(\theta)\right]+\alpha\sin\theta$$
$$+\left[\left(\tfrac{1}{2}-8\alpha\pi^{-2}\right)S\left(\tfrac{1}{2}\pi\right)+4\alpha\pi^{-2}-\tfrac{1}{2}\right]\theta\cos\theta\Big\}\,;\theta\equiv\tfrac{1}{2}\pi y\,;S(\theta)\equiv\int_0^\theta\phi\csc\phi\,\mathrm{d}\phi \tag{3}$$

where the x and y components of velocity can be deduced using $u=xF'$, and $v=-F$. In like fashion, other flow attributes may be extrapolated from F and its derivatives. The axial pressure gradient, for example, can be evaluated from F''' via

$$\partial p/\partial x=x\left[\varepsilon F'''+FF''+\left(F'\right)^2+\alpha\varepsilon\left(2F'+yF''\right)\right] \tag{4}$$

where

$$F'''=\tfrac{1}{8}\pi^3\Big(-\cos\theta+\varepsilon\left\{(\tfrac{1}{4}\pi-4\alpha/\pi)\left[-\sin\theta S(\theta)-\left(2\cos\theta-\theta\sin\theta\right)\ln\tan\tfrac{1}{2}\theta-1\right]-\alpha\cos\theta\right.$$
$$\left.-\left[\left(\tfrac{1}{2}-8\alpha\pi^{-2}\right)S\left(\tfrac{1}{2}\pi\right)+4\alpha\pi^{-2}-\tfrac{1}{2}\right]\left(3\cos\theta-\theta\sin\theta\right)\right\}\Big) \tag{5}$$

The singularity at hand is caused by the ($\cos\theta\ln\tan\frac{1}{2}\theta$) term that appears, for the first time, in F'''. Because this term becomes suddenly unbounded as $\theta\to 0$, it marks the presence of a core boundary layer that requires special treatment. This spurious behavior may be attributed to the effects of blowing at each wall, a convection mechanism that drives the shear layer away from the wall and into the core region. Inside this layer the viscous term $R^{-1}F'''$ in Eq. (1) becomes comparable to the inertial term $FF''-(F')^2$. Consequently, the full equation will have to be revisited and carefully rescaled before seeking a suitable outcome.

By retaining the highest derivative, however, the order of the equation is increased by one. Formerly, only three of the four boundary conditions had to be satisfied by the outer solution. Under the premise of a large injection Reynolds number, it was not necessary to impose the first condition, $F''(0)=0$, in Eq. (2). At present, two boundary conditions at $y=\theta=0$ must be secured by the inner approximation while two others must be employed to determine the remaining constants of integration through matching with the outer solution.

In retrospect, it may instructive to note that the case of large blowing at two opposing but fixed walls displays similar features to ours. This case is treated by Yuan[27] whose series solution also appears to exhibit a core anomaly: It continues to agree with numerical predictions until undergoing a threefold differentiation. Much like ours, Yuan's third derivative becomes infinite in its midsection plane. To suppress this unphysical behavior, a follow-up investigation by Terrill[28] may be consulted in which a mathematical strategy is introduced for the purpose of characterizing the subtle inner layer that is not considered in Yuan's analysis. In what follows, the existence of a comparable singularity will be shown to exist in the problem under investigation.

III. Inner Expansion of the Outer Solution

To eliminate the singularity in F''', an inner solution is attempted in the thin core region. This is accomplished by applying the concept of matched asymptotic expansions. Starting with

$$F=\sin\theta+\varepsilon\left\{-(2\alpha/\pi)\theta+B\left[(\theta\cos\theta-\sin\theta)\ln\tan\tfrac{1}{2}\theta-\cos\theta S(\theta)\right]+\alpha\sin\theta+A\theta\cos\theta\right\} \tag{6}$$

$$\begin{cases} A \equiv \left(\tfrac{1}{2} - 8\alpha / \pi^2\right) S\left(\tfrac{1}{2}\pi\right) + 4\alpha / \pi^2 - \tfrac{1}{2} \\ B \equiv \tfrac{1}{4}\pi - 4\alpha / \pi \end{cases} \tag{7}$$

a term-by-term expansion yields

$$\begin{aligned} F(\theta) = {} & \left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) + \varepsilon\Big\{-2(\alpha / \pi)\theta + B\Big[\theta\left(1 - \tfrac{1}{2!}\theta^2\right) - \left(\theta - \tfrac{1}{3!}\theta^3\right)\Big]\ln\tfrac{1}{2}\theta \\ & -B\left(1 - \tfrac{1}{2!}\theta^2\right)\left(\theta + \tfrac{1}{18}\theta^3\right) + \alpha\left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) + A\theta\left(1 - \tfrac{1}{2!}\theta^2\right)\Big\} \\ = {} & \left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) + \varepsilon\Big[\theta\left(\alpha + A - 2\alpha / \pi - B\right) \\ & - \tfrac{1}{3}B\theta^3 \ln\tfrac{1}{2}\theta + \theta^3\left(\tfrac{1}{3}B\ln 2 + \tfrac{4}{9}B - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\Big]. \end{aligned} \tag{8}$$

Letting the slow inner variable take the form $\xi = \theta / \varepsilon^n$, substitution into Eq. (1) reveals a balance between inertial and viscous forces for $n = \tfrac{1}{2}$. Recognizing that the core layer has a thickness of $O(\varepsilon^{1/2})$, the appropriate coordinate transformation becomes $\xi = \theta / \varepsilon^{1/2}$; forthwith, the outer solution given by Eq. (8) may be expanded in the inner variable using

$$\begin{aligned} (F^{(o)})^i = F^{(o)}(\xi) = {} & \varepsilon^{\frac{1}{2}}\xi + \varepsilon^{\frac{3}{2}}\left[-\tfrac{1}{6}\xi^3 + \left(\alpha + A - 2\alpha / \pi - B\right)\xi\right] - \tfrac{1}{6}\varepsilon^{\frac{5}{2}}\ln\varepsilon B\xi^3 \\ & + \varepsilon^{\frac{5}{2}}\left[\tfrac{1}{120}\xi^5 - \tfrac{1}{3}B\xi^3\ln\xi + \left(\tfrac{1}{3}B\ln 2 + \tfrac{4}{9}B - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\xi^3\right] \\ = {} & \varepsilon^{\frac{1}{2}}f_1 + \varepsilon^{\frac{3}{2}}f_2 + \varepsilon^{\frac{5}{2}}\ln\varepsilon f_3 + \varepsilon^{\frac{5}{2}}f_4. \end{aligned} \tag{9}$$

IV. Inner Solution

Next, the independent variable in Eq. (1) is changed from y to θ. This replacement returns

$$\tfrac{1}{4}\pi^2\varepsilon F_{\theta\theta\theta\theta} + \alpha\varepsilon\left(\theta F_{\theta\theta\theta} + 3F_{\theta\theta}\right) + \tfrac{1}{2}\pi F F_{\theta\theta\theta} - \tfrac{1}{2}\pi F_\theta F_{\theta\theta} = 0 \tag{10}$$

where the subscript denotes a derivative with respect to θ. Using the spatial distortion $\theta = \varepsilon^{\frac{1}{2}}\xi$, one obtains the inner equation that dominates near the core. One finds

$$\tfrac{1}{4}\pi^2\varepsilon^{\frac{1}{2}}F_{\xi\xi\xi\xi} + \alpha\varepsilon^{\frac{3}{2}}\left(\xi F_{\xi\xi\xi} + 3F_{\xi\xi}\right) + \tfrac{1}{2}\pi F F_{\xi\xi\xi} - \tfrac{1}{2}\pi F_\xi F_{\xi\xi} = 0 \tag{11}$$

with local boundary conditions

$$F_{\xi\xi}(0) = 0 \quad \text{and} \quad F(0) = 0 \tag{12}$$

At this juncture, the inner solution can be written in a series of progressively diminishing terms, specifically

$$F^{(i)}(\xi) = \varepsilon^{\frac{1}{2}}g_1(\xi) + \varepsilon^{\frac{3}{2}}g_2(\xi) + \varepsilon^{\frac{5}{2}}\ln\varepsilon g_3(\xi) + \varepsilon^{\frac{5}{2}}g_4(\xi). \tag{13}$$

When substituted into Eq. (11), terms of the same order can be gathered. One gets

$$O(\varepsilon): \quad \tfrac{1}{2}\pi g_{1,\xi\xi\xi\xi} + g_1 g_{1,\xi\xi\xi} - g_{1,\xi}g_{1,\xi\xi} = 0 \tag{14}$$

$$O(\varepsilon^2): \quad \tfrac{1}{2}\pi g_{2,\xi\xi\xi\xi} + \xi g_{2,\xi\xi\xi} - g_{2,\xi\xi} = 0 \tag{15}$$

$$O(\varepsilon^3\ln\varepsilon): \quad \tfrac{1}{2}\pi g_{3,\xi\xi\xi\xi} + \xi g_{3,\xi\xi\xi} - g_{3,\xi\xi} = 0 \tag{16}$$

$$O(\varepsilon^3): \quad \tfrac{1}{2}\pi g_{4,\xi\xi\xi\xi} + \xi g_{4,\xi\xi\xi} - g_{4,\xi\xi} = \tfrac{1}{3}\xi^3 + (2\alpha / \pi)\xi \tag{17}$$

According to Van Dyke's matching principle, Eqs. (14)–(17) must be integrated and then matched with Eq. (9). This operation yields

$$g_1 = \xi \tag{18}$$

$$g_2 = -\tfrac{1}{6}\xi^3 + \left(\alpha + A - 2\alpha / \pi - B\right)\xi \tag{19}$$

$$g_3 = -\tfrac{1}{6} B\xi^3 \tag{20}$$

However, applying $F_{\xi\xi}(0) = 0$ from Eq. (12), one deduces that $g_{4,\xi\xi}(0) = 0$. Equation (17) can then be integrated twice and put in the form

$$g_{4,\xi\xi} = Z\left(\pi^2 - 4\alpha\right)\xi + \tfrac{1}{6}\xi^3 - \tfrac{1}{2}\pi\xi\left(1 - 4\alpha / \pi^2\right)\int_0^\xi e^{-\frac{\xi^2}{\pi}} \mathrm{erfi}\left(\xi / \sqrt{\pi}\right)\mathrm{d}\xi$$
$$-\left(\tfrac{1}{4}\pi^2 - \alpha\right)e^{-\frac{\xi^2}{\pi}} \mathrm{erfi}\left(\xi / \sqrt{\pi}\right) \tag{21}$$

where $\mathrm{erfi}(x) = \tfrac{1}{2}\pi\int_0^x e^{t^3}\mathrm{d}t$ refers to the error function and Z alludes to a constant that must be determined through matching with $(F^{(o)})^i$.

A. Outer Expansion of the Inner Solution

In order to evaluate the inner solution in the outer domain, it is useful to introduce the large ξ expression

$$\mathrm{erfi}(\xi) \approx \frac{B}{\xi}\frac{4\sqrt{\pi}}{(\pi^2 - 4\alpha)} e^{\xi^2} \tag{22}$$

Based on Eq. (22), the third and fourth terms on the right hand side of Eq. (21) become

$$\xi\left(\tfrac{1}{2}\pi - 2\alpha / \pi\right)\int e^{-\frac{\xi^2}{\pi}} \mathrm{erfi}\left(\xi / \sqrt{\pi}\right)\mathrm{d}\xi \sim \xi\left(\tfrac{1}{2}\pi - 2\alpha / \pi\right)\int \frac{B}{\xi(\frac{1}{4}\pi^2 - \alpha / \pi)}\mathrm{d}\xi \approx 2B\xi\ln\xi \tag{23}$$

$$(\tfrac{1}{4}\pi^2 - \alpha)e^{-\frac{\xi^2}{\pi}} \mathrm{erfi}\left(\xi / \sqrt{\pi}\right) \approx (\tfrac{1}{4}\pi^2 - \alpha)e^{-\frac{\xi^2}{\pi}} \frac{B}{\xi(\frac{1}{4}\pi - \alpha / \pi)} e^{\frac{\xi^2}{\pi}} \approx \frac{B\pi}{\xi} \tag{24}$$

The outer expansion of Eq. (21) can now be written as

$$(g_{4,\xi\xi})^o = \tfrac{1}{6}\xi^3 + Z(\pi^2 - 4\alpha)\xi - 2B\xi\ln\xi - B\pi\xi^{-1} + O(\xi^{-3}). \tag{25}$$

Next, Z needs to be determined before initiating the twofold integration of Eq. (25).

B. Matching with the Outer Solution

According to Eq. (13), $g_4(\xi)$ represents the $O(\varepsilon^{5/2})$ correction to the inner solution $F^{(i)}$; $(g_{4,\xi\xi})^o$ is hence the second derivative of the $\varepsilon^{5/2}$ correction arising in $(F^{(i)})^o$. Pursuant to Van Dyke's matching principle, this term must be set equal to the second derivative $f_{4,\xi\xi}$ of the $\varepsilon^{5/2}$ correction arising in $(F^{(o)})^i$. Returning to Eq. (9), it is clear that

$$f_4 = \tfrac{1}{120}\xi^5 - \tfrac{1}{3}B\xi^3\ln\xi + \left(\tfrac{1}{3}B\ln 2 + \tfrac{4}{9}B - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\xi^3 \tag{26}$$

and so

$$f_{4,\xi\xi} = \tfrac{1}{6}\xi^3 - 2B\xi\ln\xi + \left(2B\ln 2 - \alpha - 3A\right)\xi \tag{27}$$

By setting $(g_{4,\xi\xi})^o = f_{4,\xi\xi}$, Eqs. (25) and (27) can be equated in a manner to provide

$$Z = (2B\ln 2 - \alpha - 3A) / (\pi^2 - 4\alpha) \tag{28}$$

At the outset, Eq. (25) becomes

$$(g_{4,\xi\xi})^o = \tfrac{1}{6}\xi^3 - 2B\xi\ln\xi + \left(2B\ln 2 - \alpha - 3A\right)\xi - B\pi\xi^{-1} + O\left(\xi^{-3}\right) \tag{29}$$

hence

$$(g_4)^o = \tfrac{1}{120}\xi^5 + \left(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A + \tfrac{5}{18}B\right)\xi^3 - \tfrac{1}{3}B\xi^3\ln\xi + B\pi\xi - B\pi\xi\ln\xi \tag{30}$$

The outer expansion of the inner solution is finally at hand. Substituting all g_i back into Eq. (13), one collects

$$(F^{(i)})^o = \varepsilon^{\frac{1}{2}}\xi + \varepsilon^{\frac{3}{2}}\Big[-\tfrac{1}{6}\xi^3 + \big(\alpha + A - 2\alpha / \pi - B\big)\xi\Big] - \tfrac{1}{6}\varepsilon^{\frac{5}{2}}\ln\varepsilon B\xi^3$$
$$+\varepsilon^{\frac{5}{2}}\Big[\tfrac{1}{120}\xi^5 + \big(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A + \tfrac{5}{18}B\big)\xi^3 - \tfrac{1}{3}B\xi^3\ln\xi + B\pi\xi - B\pi\xi\ln\xi\Big]. \quad (31)$$

This solution can be expressed in the original variable θ and put in the form

$$(F^{(i)})^o = \big(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\big) + \varepsilon\Big[\theta\big(\alpha + A - 2\alpha / \pi - B\big)$$
$$+\big(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A + \tfrac{5}{18}B\big)\theta^3 - \tfrac{1}{3}B\theta^3\ln\theta\Big] + \tfrac{1}{2}\varepsilon^2\ln\varepsilon B\pi\theta + O(\varepsilon^2). \quad (32)$$

C. Inner Correction

Having determined the constant in Eq. (21), one can integrate twice and write

$$g_4 = \tfrac{1}{120}\xi^5 + \big(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\big)\xi^3 - \tfrac{1}{120}\big(1 - 4\alpha / \pi^2\big)\xi^3\Big\{10\pi + \xi^2\,{}_pF_q\big[(1,1);(3,\tfrac{7}{2});-\xi^2 / \pi\big]\Big\}$$
$$+C_1\xi + C_0 \quad (33)$$

where C_1 and C_0 are pure constants and ${}_pF_q$ is the generalized hypergeometric function. Due to the boundary condition $F(0) = 0$, it can be readily seen that $g_4(0) = 0$ and $C_0 = 0$. The remaining constant C_1 becomes immaterial as it appears at $O(\varepsilon^2)$. This can be ascertained by first substituting Eq. (33) into Eq. (13) to retrieve

$$F^{(i)}(\xi) = \varepsilon^{\frac{1}{2}}\xi + \varepsilon^{\frac{3}{2}}\Big[-\tfrac{1}{6}\xi^3 + \big(\alpha + A - 2\alpha / \pi - B\big)\xi\Big] - \tfrac{1}{6}\varepsilon^{\frac{5}{2}}\ln\varepsilon B\xi^3$$
$$+\varepsilon^{\frac{5}{2}}\Big(\tfrac{1}{120}\xi^5 + \big(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\big)\xi^3$$
$$-\tfrac{1}{120}\big(1 - 4\alpha / \pi^2\big)\xi^3\Big\{10\pi + \xi^2\,{}_pF_q\big[(1,1);(3,\tfrac{7}{2});-\xi^2 / \pi\big]\Big\} + C_1\xi\Big). \quad (34)$$

With $\xi = \theta / \varepsilon^{1/2}$, the last term involving $\varepsilon^{5/2}C_1\xi = \varepsilon^2 C_1\theta$ can be dismissed when returning to the original coordinate representation. The inner solution becomes

$$F^{(i)}(\theta) = \big(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\big) - \tfrac{1}{120}\big(1 - 4\alpha / \pi^2\big)\theta^5\,{}_pF_q\big[(1,1),(3,\tfrac{7}{2}),-\theta^2 / (\varepsilon\pi)\big] - \tfrac{1}{6}\varepsilon\ln\varepsilon B\,\theta^3$$
$$+\varepsilon\Big[\big(\alpha + A - 2\alpha / \pi - B\big)\theta + \big(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\big)\theta^3 - \tfrac{1}{12}\big(\pi - 4\alpha / \pi\big)\theta^3\Big] + O(\varepsilon^2) \quad (35)$$

where terms at order ε^2 and higher are ignored. Having determined $F^{(o)}(\theta)$, $F^{(i)}(\theta)$ and $(F^{(i)})^o$, a uniformly valid composite solution $F^{(c)}$ can be reached by combining

$$F^{(c)} = F^{(o)} + F^{(i)} - (F^{(i)})^o. \quad (36)$$

One can evaluate the net correction $\overline{F}^{(i)}$ which, when added to the formerly reported outer solution,[20] makes it singularity-free down to the fourth derivative. Defining $\overline{F}^{(i)} \equiv F^{(i)} - (F^{(i)})^o$, one can put

$$\overline{F}^{(i)} = \tfrac{1}{120}\big(4\alpha / \pi^2 - 1\big)\theta^5\,{}_pF_q\big[(1,1),(3,\tfrac{7}{2}),-\theta^2 / (\varepsilon\pi)\big] - \tfrac{1}{6}\varepsilon\ln\varepsilon B\theta^3$$
$$+\varepsilon\Big[\tfrac{1}{3}B\theta^3\ln\theta - \tfrac{5}{18}B\theta^3 + \tfrac{1}{12}\big(4\alpha / \pi - \pi\big)\theta^3\Big] - \tfrac{1}{2}\pi\varepsilon^2\ln\varepsilon B\theta \quad (37)$$

Based on this correction, the third derivative in $F^{(c)}$ is no longer unbounded at the core. The net corrections that must be added to the derivatives obtained by Majdalani and Zhou[20] are, therefore,

$$\overline{F}_\theta^{(i)} = \tfrac{1}{40}\big(4\alpha / \pi^2 - 1\big)\theta^4\,{}_pF_q\big[(1,1),(3,\tfrac{7}{2}),-\theta^2 / (\varepsilon\pi)\big] - \tfrac{1}{2}\varepsilon\ln\varepsilon B\theta^2$$
$$+\varepsilon\theta^2\Big(B\ln\theta - \tfrac{1}{2}B + \tfrac{1}{12}(4\alpha / \pi - \pi)\Big\{\tfrac{13}{10} - {}_pF_q\big[(1,1),(2,\tfrac{5}{2}),-\theta^2 / (\varepsilon\pi)\big]\Big\}\Big) - \tfrac{1}{2}\pi\varepsilon^2\ln\varepsilon B \quad (38)$$

$$\overline{F}_{\theta\theta}^{(i)} = -\tfrac{1}{20}\big(1 + 4\alpha / \pi^2\big)\theta^3\,{}_pF_q\big[(1,1),(3,\tfrac{7}{2}),-\theta^2 / (\varepsilon\pi)\big] - \varepsilon\ln\varepsilon B\theta$$

$$+\varepsilon\theta\left(2B\ln\theta+(4\alpha/\pi-\pi)\left\{\tfrac{7}{15}-\tfrac{1}{4}\,{}_pF_q\left[(1,1),(2,\tfrac{5}{2}),-\theta^2/(\varepsilon\pi)\right]\right\}\right)+\varepsilon^2\left(\tfrac{1}{4}\pi^2-\alpha\right)/\theta$$
$$+\tfrac{1}{8}\pi\left(4\alpha-\pi^2\right)\varepsilon^{5/2}\operatorname{erfi}(\theta/\sqrt{\varepsilon\pi})\exp[-\theta^2/(\varepsilon\pi)]/\theta^2 \tag{39}$$

$$\bar{F}^{(i)}_{\theta\theta\theta}=-\tfrac{1}{20}\left(1+4\alpha/\pi^2\right)\theta^2\,{}_pF_q\left[(1,1),(3,\tfrac{7}{2}),-\theta^2/(\varepsilon\pi)\right]-\varepsilon\ln\varepsilon B$$
$$+\varepsilon\left(2B-\tfrac{29}{30}(\pi-4\alpha/\pi)+\tfrac{1}{4}\left\{(\pi-4\alpha/\pi)\,{}_pF_q\left[(1,1),(2,\tfrac{5}{2}),-\theta^2/(\varepsilon\pi)\right]+8B\ln\theta\right\}\right)$$
$$+\tfrac{1}{4}\left(\pi^2-4\alpha\right)\varepsilon^{3/2}\operatorname{erfi}(\theta/\sqrt{\varepsilon\pi})\exp[-\theta^2/(\varepsilon\pi)]/\theta$$
$$+\tfrac{1}{4}\left(\pi^2-4\alpha\right)\varepsilon^{5/2}\left\{1-\tfrac{1}{2}\pi\operatorname{erfi}(\theta/\sqrt{\varepsilon\pi})\exp[-\theta^2/(\varepsilon\pi)]/\theta\right\}/\theta^2 \tag{40}$$

and, finally,

$$\bar{F}^{(i)}_{\theta\theta\theta\theta}=\tfrac{1}{2}\left(4\alpha/\pi-\pi\right)\varepsilon^{1/2}\operatorname{erfi}(\theta/\sqrt{\varepsilon\pi})\exp[-\theta^2/(\varepsilon\pi)]+2B\varepsilon/\theta \tag{41}$$

V. Corrected Core Values

Recalling that $\mathrm{d}^nF/\mathrm{d}y^n=(\tfrac{1}{2}\pi)^n\mathrm{d}^nF/\mathrm{d}\theta^n$, one can use either L'Hôpital's rule or Taylor series expansions to study the behavior of F and its derivatives as $y\to 0^+$. Without the inner correction, the formerly reported solution can be shown to exhibit

$$F(0)=F''(0)=0 \tag{42}$$

$$F'(0)=\tfrac{1}{2}\pi\left\{1+\varepsilon\left[\mathcal{C}-\tfrac{1}{2}-\tfrac{1}{4}\pi+\left(4-16\mathcal{C}+2\pi+\pi^2\right)\alpha/\pi^2\right]\right\} \tag{43}$$

$$F'''(y)\approx\tfrac{1}{8}\pi^3\varepsilon\left(8\alpha/\pi-\tfrac{1}{2}\pi\right)\ln\left(\tfrac{1}{2}\pi y\right)\xrightarrow[y\to 0^+]{}\left[\operatorname{sgn}(\pi^2-16\alpha)\right]\infty \tag{44}$$

and

$$F''''(y)\approx\tfrac{1}{16}\pi^4\varepsilon\left(16\alpha/\pi^2-1\right)/y\xrightarrow[y\to 0^+]{}-\left[\operatorname{sgn}(\pi^2-16\alpha)\right]\infty \tag{45}$$

where $\mathcal{C}=0.9159656$ is Catalan's constant. When the composite solution is constructed, singular terms take leave; one finds

Table 1. Spatial comparison in outer (*o*), composite (*c*), and numeric (*n*) predictions for F, F', F'' and F''' at $\alpha=10$ and $\varepsilon=0.01$ ($R=100$)

	F			F'			F''			F'''		
$y/(2\varepsilon)$	$F^{(o)}$	$F^{(c)}$	$F^{(n)}$	$F^{(o)\prime}$	$F^{(c)\prime}$	$F^{(n)\prime}$	$F^{(o)\prime\prime}$	$F^{(c)\prime\prime}$	$F^{(n)\prime\prime}$	$F^{(o)\prime\prime\prime}$	$F^{(c)\prime\prime\prime}$	$F^{(n)\prime\prime\prime}$
0.0	0	0	0	1.6525	1.6389	1.6476	0	0	0	-∞	-6.2752	-6.0567
0.1	0.0033	0.0033	0.0033	1.6525	1.6389	1.6476	-0.0190	-0.0126	-0.0121	-8.5722	-6.2750	-6.0562
0.2	0.0066	0.0066	0.0066	1.6524	1.6389	1.6475	-0.0354	-0.0251	-0.0242	-7.9301	-6.2746	-6.0547
0.3	0.0099	0.0098	0.0099	1.6523	1.6388	1.6475	-0.0509	-0.0376	-0.0363	-7.5543	-6.2740	-6.0523
0.4	0.0132	0.0131	0.0132	1.6522	1.6387	1.6474	-0.0657	-0.0502	-0.0484	-7.2875	-6.2732	-6.0489
0.5	0.0165	0.0164	0.0165	1.6521	1.6386	1.6473	-0.0801	-0.0627	-0.0605	-7.0803	-6.2722	-6.0446
0.6	0.0198	0.0197	0.0198	1.6519	1.6385	1.6471	-0.0941	-0.0753	-0.0726	-6.9109	-6.2709	-6.0393
0.7	0.0231	0.0229	0.0231	1.6517	1.6383	1.647	-0.1077	-0.0878	-0.0847	-6.7674	-6.2695	-6.0331
0.8	0.0264	0.0262	0.0264	1.6515	1.6381	1.6468	-0.1211	-0.1004	-0.0967	-6.6430	-6.2678	-6.0259
0.9	0.0297	0.0295	0.0296	1.6512	1.6379	1.6466	-0.1343	-0.1129	-0.1088	-6.5330	-6.2660	-6.0178
1.0	0.0330	0.0328	0.0329	1.6509	1.6377	1.6464	-0.1473	-0.1254	-0.1208	-6.4345	-6.2639	-6.0088

$$F^{(c)}(0) = F^{(c)''}(0) = 0 \quad (46)$$

$$F^{(c)'}(0) = \tfrac{1}{2}\pi\left\{1 + \varepsilon\left[\mathcal{C} - \tfrac{1}{2} - \tfrac{1}{4}\pi + \left(4 - 16\mathcal{C} + 2\pi + \pi^2\right)\alpha / \pi^2\right] + \varepsilon^2 \ln\varepsilon\left(2\alpha - \tfrac{1}{8}\pi^2\right)\right\} \quad (47)$$

$$F^{(c)'''}(0) = \tfrac{1}{8}\pi^3\left(-1 + \tfrac{1}{4}\varepsilon\ln\varepsilon\left(16\alpha / \pi - \pi\right) + \varepsilon\left\{3\left(\tfrac{1}{2} - \mathcal{C}\right) + \pi\left(\tfrac{1}{5} + \tfrac{1}{2}\ln 2\right) - \left[\pi^2 + \left(\tfrac{19}{5} + 8\ln 2\right)\pi + 12 - 48\mathcal{C}\right]\alpha / \pi^2\right\}\right) \quad (48)$$

$$F^{(c)''''}(0) = 0 \quad (49)$$

Clearly, a distinct improvement can be observed near the core in the first, third, and fourth derivatives. The difference between $F'(0)$ and $F^{(c)'}(0)$ appears at a small logarithmic order in view of

$$\overline{F}^{(i)'}(0) = \tfrac{1}{2}\pi\left(2\alpha - \tfrac{1}{8}\pi^2\right)\varepsilon^2 \ln\varepsilon \quad (50)$$

Insofar as Eq. (50) enhances the accuracy of the solution in the vicinity of the chamber midsection, it leads to local ameliorations in axial velocity and both axial and normal pressure gradients.[20] In fact, when inner corrections are accounted for, a better agreement is achieved at all levels with the numerical solution of the problem. This can be seen in Table 1 where the outer, composite, and numerical predictions for F and its derivatives are reported. This comparison is focused on the limited core region associated with $0 \le y \le 2\varepsilon$ and representative values of $R = 100$ and $\alpha = 10$. In addition to the gradual refinement in the composite solution over the outer expressions for F, F', and F'', a significant

Table 2. Outer (*o*), composite (*c*), and numeric (*n*) predictions for $F'(0)$ and $F'''(0)$ at $\alpha = 10$ and an increasing range of R. The composite solutions at the core are given using the first term, the first two terms, and the first three terms in Eqs. (47) and (48)

	F' : Eq. (47)					F''' : Eq. (48)				
R	$F^{(o)'}$	$(F^{(c)'})_1$	$(F^{(c)'})_2$	$(F^{(c)'})_3$	$F^{(n)'}$	$F^{(0)'''}$	$(F^{(c)'''})_1$	$(F^{(c)'''})_2$	$(F^{(c)'''})_3$	$F^{(n)'''}$
10	2.3877	1.5708	2.3877	1.7089	2.1281	-∞	-3.8758	-14.538	-17.208	-18.165
20	1.9793	1.5708	1.9793	1.7585	1.8963	-∞	-3.8758	-10.812	-12.147	-12.294
50	1.7342	1.5708	1.7342	1.6881	1.7173	-∞	-3.8758	-7.4986	-8.0326	-7.8156
100	1.6525	1.5708	1.6525	1.6389	1.6476	-∞	-3.8758	-6.0082	-6.2752	-6.0567
200	1.6116	1.5708	1.6116	1.6077	1.6102	-∞	-3.8758	-5.1025	-5.2360	-5.0798
500	1.5871	1.5708	1.5871	1.5864	1.5869	-∞	-3.8758	-4.4513	-4.5047	-4.4242
1,000	1.5790	1.5708	1.5790	1.5788	1.5789	-∞	-3.8758	-4.1956	-4.2223	-4.1776
10,000	1.5716	1.5708	1.5716	1.5716	1.5716	-∞	-3.8758	-3.9184	-3.9211	-3.9160
100,0000	1.5709	1.5708	1.5709	1.5709	1.5709	-∞	-3.8758	-3.8811	-3.8814	-3.8809

Table 3. Outer (*o*), composite (*c*), and numeric (*n*) predictions for $F'(0)$ at $\alpha = \pm 5$ and an increasing range of R. The composite solution at the core is obtained using the first term, the first two terms, and the first three terms in Eq. (47)

	$\alpha = 5$ (expanding walls)					$\alpha = -5$ (contracting walls)				
R	$F^{(o)'}$	$(F^{(c)'})_1$	$(F^{(c)'})_2$	$(F^{(c)'})_3$	$F^{(n)'}$	$F^{(0)'}$	$(F^{(c)'})_1$	$(F^{(c)'})_2$	$(F^{(c)'})_3$	$F^{(n)'}$
10	1.9502	1.5708	1.9502	1.6332	1.8116	1.0753	1.5708	1.0753	1.4816	1.5659
20	1.7605	1.5708	1.7605	1.6574	1.7152	1.3230	1.5708	1.3230	1.4552	1.4061
50	1.6467	1.5708	1.6467	1.6251	1.6372	1.4717	1.5708	1.4717	1.4993	1.4888
100	1.6087	1.5708	1.6087	1.6024	1.6059	1.5212	1.5708	1.5212	1.5294	1.5263
200	1.5898	1.5708	1.5898	1.5879	1.5890	1.5460	1.5708	1.5460	1.5484	1.5475
500	1.5784	1.5708	1.5784	1.5780	1.5782	1.5609	1.5708	1.5609	1.5613	1.5612
1,000	1.5746	1.5708	1.5746	1.5745	1.5745	1.5658	1.5708	1.5658	1.5660	1.5659

Table 4. Outer (*o*), composite (*c*), and numeric (*n*) predictions for $F'(0)$ and $F'''(0)$ at $R = 1000$ and a range of α

α	$F^{(o)\prime}$	$(F^{(c)\prime})_1$	$(F^{(c)\prime})_2$	$(F^{(c)\prime})_3$	$F^{(n)\prime}$	$F^{(o)\prime\prime\prime}$	$(F^{(c)\prime\prime\prime})_1$	$(F^{(c)\prime\prime\prime})_2$	$(F^{(c)\prime\prime\prime})_3$	$F^{(n)\prime\prime\prime}$
-100	1.4827	1.5708	1.4827	1.4849	1.4872	-∞	-3.8758	-0.4459	-0.1589	-1.5951
-10	1.5615	1.5708	1.5615	1.5617	1.5616	-∞	-3.8758	-3.5139	-3.4835	-3.5544
-1	1.5693	1.5708	1.5693	1.5694	1.5694	-∞	-3.8758	-3.8207	-3.8160	-3.8250
-0.1	1.5701	1.5708	1.5701	1.5701	1.5701	-∞	-3.8758	-3.8513	-3.8492	-3.8529
0	1.5702	1.5708	1.5702	1.5702	1.5702	-∞	-3.8758	-3.8548	-3.8529	-3.8560
0.1	1.5703	1.5708	1.5703	1.5703	1.5703	-∞	-3.8758	-3.8582	-3.8566	-3.8591
1	1.5711	1.5708	1.5711	1.5711	1.5711	-∞	-3.8758	-3.8888	-3.8899	-3.8873
10	1.5790	1.5708	1.5790	1.5788	1.5789	-∞	-3.8758	-4.1956	-4.2223	-4.1776
100	1.6577	1.5708	1.6577	1.6556	1.6598	-∞	-3.8758	-7.2636	-7.5470	-8.1171

improvement in the performance of the composite solution can be noted for F'''. Overall, the ability of the matched-asymptotic approximation to outperform the outer solution is consistent as $\varepsilon \to 0$. This can be inferred from Table 2 where estimates for both $F'(0)$ and $F'''(0)$ are produced at $\alpha = 10$ and a progressively increasing Reynolds number ranging from 10 to 10^5. These estimates are compared to the numerical values with and without inner corrections. In order to depict the improvement with each successive asymptotic correction, the matched-asymptotic expansions at the core are provided using the first term, the first two terms, and the first three terms that appear in Eqs. (47) and (48), respectively. In the case of $F'(0)$, the two-term composite solution that emerges is identical to the outer approximation. It is in fact the third correction of order $\varepsilon^2 \ln \varepsilon$ that marks the difference in Table 2. This result is further confirmed in Table 3 where the behavior of $F'(0)$ is illustrated for expanding and contracting wall expansions with $\alpha = \pm 5$. In both situations, the asymptotic behavior improves as $\varepsilon \to 0$.

The dependence of the core values on α is also captured in Table 4 where both $F'(0)$ and $F'''(0)$ are calculated over a range of expansion ratios ranging from -100 to 100 at fixed $R = 1000$. It is interesting to note that the error in the asymptotic predictions becomes more appreciable as $|\alpha|$ is increased at constant R. This can be attributed to our approximation being contingent on $|\alpha R^{-1}| \ll 1$ in Eq. (1). In reality, this constraint does not pose any physical limitations in view of the small reported $|\alpha|$ in practical applications. The present development of a composite solution with critical core corrections completes our laminar flow treatment of the porous channel with retractable walls.

VI. Conclusions

In this study, we first identify and then suppress the spurious logarithmic singularity that affects the mean flow of a porous channel with regressing or contracting walls. The singularity in question is conspicuous by its sudden appearance in the third derivative of the characteristic mean flow function. At the outset, unboundedness unexpectedly appears in the axial pressure gradient. Its emergence signals the presence of a viscous boundary layer that must be carefully resolved. In this vein, the quest for an inner scaling transformation is initiated to the extent of striking a quasi balance between viscous dissipation and inertia. The ensuing scaling analysis unravels a slow-varying stretched coordinate of the form $\xi = \theta / \varepsilon^{1/2}$. Next, when the inner expansion of the outer solution is carried out in terms of the inner variable ξ, fractional powers of ε emerge along with the dreaded logarithmic order at irregular intervals. The gauge functions in the resulting series follow a sequence that starts with $\{\varepsilon^{1/2}, \varepsilon^{3/2}, \varepsilon^{5/2} \ln \varepsilon, \varepsilon^{5/2}, \ldots\}$. This particular establishment of a suitable asymptotic sequence proves to be essential to the construction of a meaningful inner approximation. The latter is pursued by first rescaling the governing equation into a form that is appropriate of the boundary layer region. Then, by assuming a parallel inner expansion of the form $F^{(i)} = \varepsilon^{1/2} g_1 + \varepsilon^{3/2} g_2 + \varepsilon^{5/2} \ln \varepsilon g_3 + \varepsilon^{5/2} g_4$, g_1, g_2 and g_3 are readily obtained, contrary to g_4; mathematical obstructions in the last term warrant a separate analysis. To determine g_4, special error function approximations are used to simplify its second derivative under farfield conditions. This is followed by matching with the second derivative of the outer solution expressed in the inner variable. This step allows us to deduce the constants in g_4''. Subsequently, pursuant to a twofold integration of g_4'', a complete representation of the inner approximation is arrived at. At length, the remaining constants are specified through matching with the outer solution. Finally, using Erdélyi's principle of composite expansions, the inner, outer, and common parts are algebraically combined into a matched-asymptotic solution that remains uniformly valid throughout the domain. In retiring, comparisons with numerics are used to show that the corrected formulation is no longer marred by singularity. The correction presented in this study is hence essential not only to the porous channel problem, but to other injection-driven flows that exhibit common attributes. In future work, it is hoped that a similar approach will be employed to rectify the outer approximations that have so far been developed for analogous injection-based flowfields with core singularities.

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