40th Fluid Dynamics Conference and Exhibit
28 June - 1 July 2010, Chicago, Illinois

AIAA 2010-4861

Inversion of the Fundamental Isentropic Expansion Equations in Variable Area Duct Flow

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The isentropic flow equations relating the thermodynamic pressures, temperatures, and densities to their stagnation properties are solved in terms of the area ratio and Mach number. These relationships are inverted asymptotically and presented to arbitrary order. Both subsonic and supersonic branches of the solution are systematically identified and produced separately. Two types of recursive formulations are provided, with one exhibiting a universal character by virtue of its applicability to all three properties under consideration. In the case of the subsonic branch, the asymptotic series expansion is shown to be recoverable from Bürmann's theorem of classical analysis. Bosley's technique is then applied to produce a graphical confirmation of the theoretical truncation order in each approximation. The final expressions permit the pressure, temperature, and density to be estimated for any chosen area ratio and gas constant with no intermediate Mach number calculation or tabulation. The techniques are shown in detail so as to facilitate future explorations of transcendental problems where numerical solutions may be difficult to achieve.

Nomenclature

A	=	local cross sectional area
A_t	=	nozzle throat area
E_n	=	absolute error between s_N and s_n, or S_N and S_n
s, S	=	subsonic or supersonic property
s_n, S_n	=	asymptotic property at iteration order n
s_N, S_N	=	numeric property reflecting true value
p, T, ρ	=	normalized pressure, temperature, and density
α, β	=	arbitrary exponents in $s^{\alpha} - s^{\beta} = \varepsilon\xi$
ε	=	perturbation parameter, $(A_t / A)^2$
γ	=	ratio of specific heats
ξ	=	constant related to γ through Eq. (9)

Subscripts and Symbols

0 ,1	=	leading and first order
c	=	chamber stagnation condition
e	=	exit plane condition
n	=	asymptotic level
N	=	numerically calculated value
t	=	nozzle throat condition
$\bar{\ }$	=	dimensional property

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I. Introduction

ONE-DIMENSIONAL nozzle theory employs a set of isentropic flow equations that have proven so useful over the years that they continue to receive attention in a variety of technological applications. Their surprising simplicity is perhaps responsible for granting them wide acceptance in both academic and industrial circles, particularly, in the communities that are concerned with propulsion and power generation equipment. From gas turbines to rockets, several fundamental thermodynamic relations stand at the foundation of standard performance measures and design criteria.[1] Gas turbine efficiencies, ideal thrust coefficients, and rocket specific impulses are some of the examples that may be cited in this context. The utility of such relations extends beyond the mundane performance prediction of a propulsive system to encompass sizing, shape selection, loss estimation, and product optimization. Applications abound and one may list chemical rockets, ramjets, scramjets, afterburners, and a variety of gas turbine engine components in which gases are expanded.

By assuming a chemically non-reactive, calorically perfect ideal gas, the basic thermodynamic principles give rise to a well established set of mathematical equations that relate pressures, temperatures, densities, local Mach numbers, ratios of specific heat, and the critically important area expansion ratio. These relations are often transcendental to the extent of requiring numerical root finding. For this reason, compressible flow tables have for decades graced the appendices of textbooks on the subject.[2-4] These tables often present discrete numerical solutions over a finite range of operating parameters. Interpolation is subsequently required to obtain the desired outcome.

In this study, an asymptotic alternative is pursued. In lieu of numerical root solving, closed form expressions are derived for each of the principal variables. The work complements a recent study in which Stodola's area-Mach number relation[5] was tacitly inverted under both subsonic and supersonic conditions.[6] The corresponding expression has since been integrated into Rocflu,[7] a compressible Navier-Stokes solver module that complements a massively parallel program coined Rocstar.[7] The latter is a rocket simulation program developed at the University of Illinois by the Center for Simulation of Advanced Rockets (CSAR) (e.g. Najjar *et al.*[8]).

In practical applications, Thakre and Yang[9] and Zhang *et al.*[10] have used the relations in question to verify their codes on nozzle erosion. Similarly, Haselbacher *et al.*[11] have used quasi-one-dimensional models in formulating test cases for their slow-time acceleration study and to estimate relevant time scales. In this vein, it may be useful to remark that the main usefulness of the analytical formulation that we seek is not so much in expediting root solving as it is in securing a reliable expression for each specific solution. This in turn could be used in future codes to ensure the proper and swift convergence to the desired supersonic or subsonic branches of solution. Additionally, the technique itself may prove valuable in seeking analytical approximations to multi-valued functions, especially to those that may prove intractable when pursued with classical methods.

In setting the stage for this study, we note that the present article comprises correctly reconstructed and re-rendered forms of solutions to a problem that originated in previous work by the first author. In its new and improved version, the analysis is organized as follows. First, the pressure, temperature and density relations are derived as function of the nozzle area and gas compression ratios. A uniquely developed universal form is also advanced for the purpose of providing a direct representation of all three properties using a single expression. Then one-by-one, the resulting correlations are expanded and solved asymptotically. The resulting series expansions are generalized to arbitrary order before undergoing a strict numerical verification. This step is accompanied by a rigorous order validation process that leads to an explicit representation of the error in each approximation. The theoretical error is further confirmed by means of Bosley's graphical error assessment technique.[12] For the particular case of the subsonic branch, the solution is also reproduced using Bürmann's theorem of classical analysis.[13]

II. Fundamental Isentropic Flow Relations

In a variety of compressible fluid applications, one-dimensional nozzle flow equations have been reported to predict performance within a few percent of actual values. The conditions leading to their inception assume a homogeneous perfect gas with inviscid, adiabatic flow conditions and uniform properties at all cross-sections. Under these auspices, one may write Stodola's area-Mach number relation[5] as

$$\left(\frac{A}{A_{\rm t}}\right)^2 = \frac{1}{M^2}\left\{\frac{2}{\gamma+1}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]\right\}^{\frac{\gamma+1}{\gamma-1}} \tag{1}$$

As usual, pressures, temperatures, and densities may be calculated at any cross section by first solving for the appropriate Mach number from Eq. (1) and then substituting the result into any of the isentropic relations

$$\frac{\bar{p}_{\rm c}}{\bar{p}} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}, \qquad \frac{\bar{T}_{\rm c}}{\bar{T}} = 1+\frac{\gamma-1}{2}M^2, \qquad \frac{\bar{\rho}_{\rm c}}{\bar{\rho}} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} \tag{2}$$

where the overbar denotes a dimensional thermodynamic quantity and c denotes a reference value that stands for chamber conditions. Because the velocities are relatively low in this region, chamber conditions may be exchanged for stagnation values. To circumvent the last step, one may extract the Mach number directly from Eq. (2) and write

$$M = \sqrt{\frac{2}{\gamma-1}\left[\left(\bar{p}_{\rm c}/\bar{p}\right)^{\frac{\gamma-1}{\gamma}}-1\right]} = \sqrt{\frac{2}{\gamma-1}\left(\bar{T}_{\rm c}/\bar{T}-1\right)} = \sqrt{\frac{2}{\gamma-1}\left[\left(\bar{\rho}_{\rm c}/\bar{\rho}\right)^{\gamma-1}-1\right]} \tag{3}$$

At this point, substitution of Eq. (3) into Eq. (1) yields a set of three ideal correlations:

$$\left(\frac{A}{A_{\rm t}}\right)^2 = \frac{\gamma-1}{2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[\left(\frac{\bar{p}_{\rm c}}{\bar{p}}\right)^{\frac{\gamma-1}{\gamma}}-1\right]^{-1}\left(\frac{\bar{p}_{\rm c}}{\bar{p}}\right)^{\frac{\gamma+1}{\gamma}} \tag{4}$$

$$\left(\frac{A}{A_{\rm t}}\right)^2 = \frac{\gamma-1}{2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{\bar{T}_{\rm c}}{\bar{T}}-1\right)^{-1}\left(\frac{\bar{T}_{\rm c}}{\bar{T}}\right)^{\frac{\gamma+1}{\gamma-1}} \tag{5}$$

$$\left(\frac{A}{A_{\rm t}}\right)^2 = \frac{\gamma-1}{2}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[\left(\frac{\bar{\rho}_{\rm c}}{\bar{\rho}}\right)^{\gamma-1}-1\right]^{-1}\left(\frac{\bar{\rho}_{\rm c}}{\bar{\rho}}\right)^{\gamma+1} \tag{6}$$

In this work, these expressions will be considered one-by-one and inverted asymptotically. The objective is to recast each thermodynamic property as a function of the local area ratio, thus averting the Mach number calculation and leading, instead, to closed-form approximations. Furthermore, the asymptotic inversion will be carried out under both subsonic and supersonic expansion states to provide a description of the behavior of a supersonic nozzle, such as the one depicted in Fig. 1.

III. Solutions

As a basis for solving Eqs. (4)–(6) we consider the size of the area expansion ratio and realize that it remains small. This prompts the introduction of $\varepsilon(z) \equiv A_{\rm t}^2 / A^2(z)$, the reciprocal of the left-hand-side term, where z represents the local axial coordinate. As usual, the ratio of specific heats γ is taken to be a constant that varies between 1 and 1.67. In rocket motors, γ varies between 1.1 and 1.4, although a value of 1.12 may be representative of metalized propellant mixtures. For example, the Reusable Shuttle Rocket Motor (RSRM) exhibits a mean chamber pressure of $\bar{p}_{\rm c} = 6.28$ MPa, a ratio of specific heats of

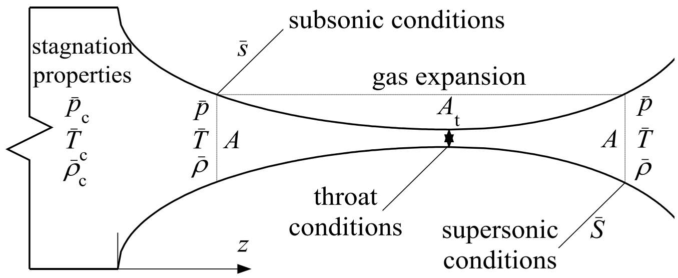

Figure 1. Variable area duct showing relevant thermodynamic properties and physical stations.

$\gamma = 1.17$, and a nozzle expansion ratio yielding an exit value of $\varepsilon_e = 0.017$. In many propulsive applications, the squared area ratio ε_e varies between 0.1 and 0.001 although values as low as 6×10^6 have been reported for high altitude nozzle applications (cf. Sutton[1]). Throughout a converging-diverging area duct, ε may hence alternate from a value near unity in the proximity of the throat section to a small value in highly expanded nozzle sections.

Our principal variables consist of the three dimensionless ratios that are extensively described and tabulated in textbooks on the subject. These are

$$p(z) = \frac{\bar{p}(z)}{\bar{p}_c} \qquad T(z) = \frac{\bar{T}(z)}{\bar{T}_c} \qquad \rho(z) = \frac{\bar{\rho}(z)}{\bar{\rho}_c} \tag{7}$$

where $\bar{p}_c$, $\bar{T}_c$ and $\bar{\rho}_c$ represent the dimensional stagnation properties. It should be noted that in standard tables and charts, the reciprocals of Eq. (7) are rather furnished. The present use of fractions to represent local over stagnation properties stems from a strictly perturbative perspective as it leads to faster converging asymptotic series expansions for transonic flows. This may be verified in the upcoming analysis which is pursued to obtain the appropriate subsonic (s) or supersonic (S) solutions.

A. Property Specific Subsonic Solution

For the subsonic roots, a regular perturbation approximation may be applied with ε as the baseline parameter. Each quantity is then expanded into

$$\begin{cases} p(\varepsilon,\gamma,n) = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots + \varepsilon^n p_n + O(\varepsilon^{n+1}) \\ T(\varepsilon,\gamma,n) = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots + \varepsilon^n T_n + O(\varepsilon^{n+1}) \\ \rho(\varepsilon,\gamma,n) = \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \ldots + \varepsilon^n \rho_n + O(\varepsilon^{n+1}) \end{cases} \tag{8}$$

Before linearization, one may introduce the gas compression related constant

$$\xi \equiv \frac{\gamma-1}{\gamma+1}\left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} \tag{9}$$

At the outset, Eqs. (4) through (6) may be rearranged and expressed as

$$\varepsilon\xi = \begin{cases} p^{2/\gamma} - p^{(\gamma+1)/\gamma} \\ T^{2/(\gamma-1)} - T^{(\gamma+1)/(\gamma-1)} = s^\alpha - s^\beta; \\ \rho^2 - \rho^{\gamma+1} \end{cases} \quad \alpha = \begin{cases} 2/\gamma \\ 2/(\gamma-1), \\ 2 \end{cases} \quad \beta = \begin{cases} (\gamma+1)/\gamma \\ (\gamma+1)/(\gamma-1) \\ (\gamma+1) \end{cases} \tag{10}$$

where $s = (p, T, \rho)$ represents any of the three thermodynamic quantities. By inserting the three-pronged system of Eq. (8) into Eq. (10), terms of the same order may expanded and collected. In the case of the dimensionless pressure ratio, one obtains,

$$p_0^\alpha - p_0^\beta + \left(\alpha p_1 p_0^{\alpha-1} - \beta p_1 p_0^{\beta-1} - \xi\right)\varepsilon + O(\varepsilon^2) = 0; \; \alpha = 2/\gamma; \quad \beta = (\gamma+1)/\gamma \tag{11}$$

and, more collectively, a generic relation of the type

$$s_0^{\alpha} - s_0^{\beta} + \left(\alpha s_1 s_0^{\alpha-1} - \beta s_1 s_0^{\beta-1} - \xi\right)\varepsilon + O(\varepsilon^2) = 0 \tag{12}$$

This expansion may be carried up to the fifth order where the asymptotic solution becomes accurate over a wide range of area ratios. By retaining additional terms, one can solve for the sequential corrections. At length, one finds

$$p = 1 - 2^{\frac{2}{\gamma-1}}\gamma(1+\gamma)^{\frac{1+\gamma}{1-\gamma}}\varepsilon - 3\gamma 2^{\frac{5-\gamma}{\gamma-1}}(1+\gamma)^{\frac{2(1+\gamma)}{1-\gamma}}\varepsilon^2 + O(\varepsilon^3) \tag{13}$$

$$T = 1 + 2^{\frac{2}{\gamma-1}}(1-\gamma)(1+\gamma)^{\frac{1+\gamma}{1-\gamma}}\varepsilon + 2^{\frac{3+\gamma}{\gamma-1}}(1-\gamma)(1+\gamma)^{\frac{2(\gamma+1)}{1-\gamma}}\varepsilon^2 + O(\varepsilon^3) \tag{14}$$

$$\rho = 1 - 2^{\frac{2}{\gamma-1}}(1+\gamma)^{\frac{1+\gamma}{1-\gamma}}\varepsilon - 2^{\frac{5-\gamma}{\gamma-1}}(2+\gamma)(1+\gamma)^{\frac{2(\gamma+1)}{1-\gamma}}\varepsilon^2 + O(\varepsilon^3) \tag{15}$$

Upon further scrutiny, a recursive relation may be obtained from which all subsonic roots may be retrieved. Further details on this relationship can be found in Appendix A.

B. Universal Subsonic Solution

An alternate, more portable expansion for the subsonic root may be obtained by recognizing that, by virtue of the constancy of $\beta/\alpha = (\gamma+1)/2$ for the three cases at hand, Eq. (10) may be collapsed into

$$\varepsilon\xi = \begin{cases} p^{2/\gamma} - p^{(\gamma+1)/\gamma} \\ T^{2/(\gamma-1)} - T^{(\gamma+1)/(\gamma-1)} = x - x^{(\gamma+1)/2} \\ \rho^2 - \rho^{\gamma+1} \end{cases} \quad \text{where} \quad x \equiv s^{\alpha} = \begin{cases} p^{2/\gamma} \\ T^{2/(\gamma-1)} \\ \rho^2 \end{cases} \tag{16}$$

Then using regular perturbations, a universal solution for x may be constructed with the added benefit of being simpler to solve, both numerically and asymptotically, while remaining equally applicable to all three thermodynamic quantities. This property independent expression collapses into

$$x(\varepsilon,\gamma,4) = \begin{cases} 1 - 2^{\frac{\gamma+1}{\gamma-1}}(1+\gamma)^{\frac{1+\gamma}{1-\gamma}}\varepsilon - 2^{\frac{4}{\gamma-1}}(1+\gamma)^{\frac{3+\gamma}{1-\gamma}}\varepsilon^2 - \frac{1}{3}2^{\frac{5+\gamma}{\gamma-1}}(3+\gamma)(1+\gamma)^{\frac{2(\gamma+2)}{1-\gamma}}\varepsilon^3 \\ -2^{\frac{9-\gamma}{\gamma-1}}(2+\gamma)(5+\gamma)(1+\gamma)^{\frac{5+3\gamma}{1-\gamma}}\varepsilon^4 + O(\varepsilon^5) \end{cases} \tag{17}$$

Furthermore, a recursive expression for x is found in Appendix A and may be offered to reproduce the solution to any desired level of precision.

C. Solution via Classical Analysis

It is interesting to note that the same solution can be arrived at via classical analysis. Accordingly, in seeking an expression for x in terms of ε, Bürmann's theorem may be employed to construct an expansion as $\varepsilon \to 0$. The first requirement for Bürmann's theorem to apply is the identification of an analytic function ϕ in a closed region.[13] In our case, we take

$$\varepsilon = \phi(x) = \frac{1}{\xi}\left[x - x^{(\gamma+1)/2}\right] \tag{18}$$

with the closed region for $\phi(x)$ being the subsonic branch in which the solution varies over the interval $0 \le x \le 1$. Because the function is analytic over this interval, a convenient anchor point may be chosen at $x = 1$ where

$$\phi(1) = 0. \tag{19}$$

Expansion about this point may be immediately carried out using a Taylor series expansion of the form

$$\phi(x) = \phi'(1)(x-1) + \frac{\phi''(1)}{2!}(x-1)^2 + \ldots \tag{20}$$

It is then possible to retrieve

$$(x-1)=\frac{1}{\phi'(1)}\phi(x)-\frac{\phi''(1)}{2\left[\phi'(1)\right]}\phi(x)^2+\ldots \tag{21}$$

Equation (21) is defined as the reversion of a standard Taylor series expansion about $x=1$.[13] The above extraction is sufficient, in this case, to obtain x in terms of ε. However, for a more general case we remark that, in Eq. (21), x appears as an analytic function of ϕ so long as $(x-1)$ remains small. It follows that if some arbitrary $f(x)$ is analytic near $x=1$, then it is also an analytic function of ϕ for sufficiently small values of $(x-1)$. We expect such an expansion to be reproducible from

$$f(x)=f(1)+a_1\phi(x)+a_2\phi(x)^2+\ldots \tag{22}$$

In practice, Bürmann's theorem provides the coefficients for Eq. (22). For the present study, based on Eq. (18), $f(x)$ may be taken simply as x. Furthermore, one may introduce $\psi(x)$ as a ratio of $f(x)$, $\phi(x)$, and their values about the expansion point, $x=1$. This quantity is defined as

$$\psi(x)=\frac{f(x)-f(1)}{\phi(x)-\phi(1)}=\frac{x-1}{\phi(x)} \tag{23}$$

The expansion of x may then be rendered directly from Bürmann's theorem using

$$x=1+\sum_{m=1}^{n-1}\frac{\varepsilon^m}{m!}\left\{\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\left[\psi\left(x\right)\right]^m\right\}+R_n \tag{24}$$

where R_n represents the remainder through which the truncation error may be inferred. According to Bürmann's theorem, the derivatives in Eq. (24) must be evaluated as $x\to 1$. When this operation is carried out, it reproduces identically the term-by-term expansion in Eq. (17) obtained using regular perturbation theory.

D. Subsonic Error Verification

To illustrate the accuracy entailed in these expressions, the asymptotic results derived from the universal approximation are compared to their numerical counterparts in Fig. 2 at $\gamma=1.2$. Graphically, it can be seen that the error remains tolerable up to an area ratio approaching unity. It can also be seen that the number of terms needed to achieve a desired level of accuracy is strongly dependent on the area ratio in question. To closely examine the behavior of the asymptotic error, the absolute and relative errors at order n may be computed viz.

$$E_n\left(\varepsilon,\gamma,s\right)=\left|s_N-s(\varepsilon,\gamma,n)\right| \quad \text{and} \quad e_n\left(\varepsilon,\gamma,s\right)=\left|\frac{s_N-s(\varepsilon,\gamma,n)}{s_N}\right| \tag{25}$$

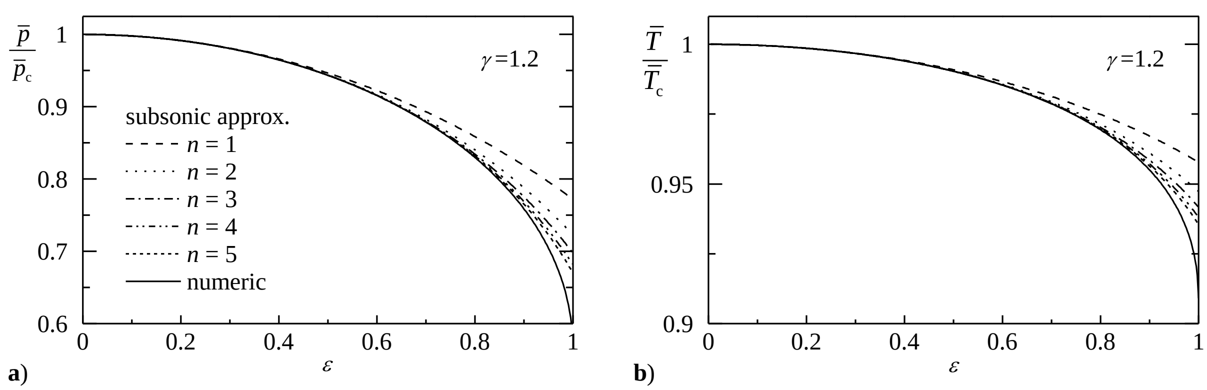

Figure 2. Numeric and asymptotic solutions for the subsonic a) pressure and b) temperature.

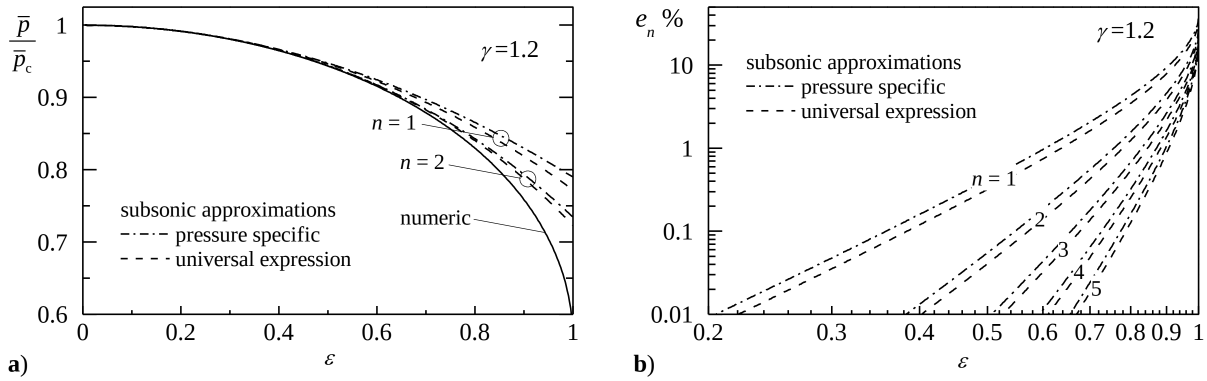

Figure 3. Performance comparison between a) pressure specific and universal approximations for the first two values of *n*, and b) the percent relative error in the subsonic solution at increasing asymptotic orders.

where s_N denotes the numerical solution for a given property $s(\varepsilon,\gamma,n)$. To characterize the error behavior, the universal and property specific expressions given by Eqs. (49) and (47) are compared side-by-side in Fig. 3a using $n=1$ and $n=2$, respectively. This is performed for the pressure variation with the area ratio as a representative of the group. Using the same number of asymptotic terms, the universal expression (broken line) is seen to fall closer to the numerical solution (solid line) than the property specific approximation (chained line). For further confirmation, the relative errors entailed in each of the asymptotic approaches are computed and displayed in Fig. 3b. We find that the $n=1$ universal solution entails a relative error e_1 of less than 1% for ratios up to $A_t/A=0.637$. Furthermore, e_1 remains bounded by 5% up to $A_t/A=0.846$. From an engineering perspective, a second-order solution will be sufficiently adequate although a fourth-order expansion may be needed to cover an appreciable range of practical interest with a minimum of four-digit accuracy. Note that γ has no effect on the error due to the use of a regular perturbation sequence that solely depends on the area expansion ratio. From a precision standpoint, the universal expression is found to be slightly more accurate than the property specific relation, albeit of the same asymptotic order. This is especially true for low order approximations where only a few asymptotic terms are retained. As confirmed in Fig. 3b, the relative errors incurred in either solution tend to merge with successive increases in *n*.

E. Property Specific Supersonic Solution

The regular perturbation approach is only effective in returning the subsonic root. This behavior can be connected to the properties of the isentropic equations at the origin. Unlike the subsonic branch which, Bürmann's theorem shows, can be written as a series expansion about $p=1$, the supersonic solution cannot be obtained using classical analysis. This may be attributed to the vanishing supersonic root as $\varepsilon\to 0$, a condition that prevents us from expanding the solution as a Taylor series about $p=0$. While the supersonic branch appears to be more elusive to track, it succumbs, after some effort, to the use of successive approximations. To this end, a systematic strategy is required as delineated below.

First, the terms in each of the original equations are scrutinized for the purpose of identifying the most dominant member in each. The ensuing selection is performed while assuming conditions appropriate of supersonic behavior. The dominant term is coined the leading order contributor. Other members of the series are then rescaled in reference to their largest contributor. To solve for the subsequent candidate, the procedure is repeated by searching for the next dominant term that may be extracted from the original equation after expansion. This process may be continued until a certain degree of precision is reached. Unlike the regular perturbation approach in which the truncation error is determined beforehand, the successive approximation technique does not yield a plain roadmap for estimating the error. Another exacerbating factor emerges due to the error becoming dependent on both ε and γ in a non-intuitive way. For this reason, the error will be evaluated analytically and then verified numerically using Bosley's order assessment technique.[12]

Letting $S=(p,T,\rho)$ be the placeholder for supersonic pressure, temperature and density ratios, we take $S=S_0+S_1+S_2+.....$ to be a series expansion of diminishing terms. In order to ensure a uniformly valid outcome, we insist on the solvability condition being, as usual, $S_{n+1}=o(S_n)$. Accordingly, successive corrections must observe

$$\lim_{\varepsilon\to 0}\frac{S_{n+1}}{S_n}=0 \tag{26}$$

We start the analysis by manually calculating the order of each term in Eq. (10). Taking a cue from the largest, we then collect the leading order quantities and discount the trailing elements as per Eq. (26). We hence identify $S_0=\left(\varepsilon\xi\right)^{1/\alpha}$ and write

$$S_0=\left[\left(\varepsilon\xi\right)^{\frac{1}{2}\gamma},\left(\varepsilon\xi\right)^{\frac{1}{2}(\gamma-1)},\left(\varepsilon\xi\right)^{\frac{1}{2}}\right] \tag{27}$$

The purpose of the first correction is to capture those secondary terms that are not retained at leading order. In what follows, details of the perturbative expansion are illustrated for the pressure ratio. A similar technique may be followed to obtain the remaining quantities.

After substituting $p=p_0+p_1$ into the pressure equation, we may factor out p_0 and subject the remaining part to a binomial series expansion in p_1/p_0. This operation yields

$$p_0^{2/\gamma}\left[1+(2/\gamma)(p_1/p_0)+O\left(p_1/p_0\right)^2\right]-p_0^{(\gamma+1)/\gamma}\left[1+(1+1/\gamma)p_1/p_0+O\left(p_1/p_0\right)^2\right]-\varepsilon\xi=0 \tag{28}$$

whence

$$p_1=\frac{p_0\left(p_0^{2/\gamma}-p_0^{(\gamma+1)/\gamma}-\varepsilon\xi\right)}{[(\gamma+1)/\gamma]p_0^{(\gamma+1)/\gamma}-(2/\gamma)p_0^{2/\gamma}} \tag{29}$$

The second order correction p_2 may be retrieved along similar lines. We find

$$p_2=\frac{\left(p_0+p_1\right)\left[\left(p_0+p_1\right)^{2/\gamma}-\left(p_0+p_1\right)^{(\gamma+1)/\gamma}-\varepsilon\xi\right]}{[(\gamma+1)/\gamma]\left(p_0+p_1\right)^{(\gamma+1)/\gamma}-(2/\gamma)\left(p_0+p_1\right)^{2/\gamma}} \tag{30}$$

By linking consecutive terms in a recursive fashion (see Appendix B), the correction at arbitrary order may be deduced. In the interest of clarity, the three-term expansions of these quantities are given below:

$$p=\overbrace{\left(\varepsilon\xi\right)^{\frac{1}{2}\gamma}}^{p_0}+\overbrace{\frac{\gamma\left(p_0^{2/\gamma}-p_0^{(\gamma+1)/\gamma}-\varepsilon\xi\right)}{(\gamma+1)p_0^{1/\gamma}-2p_0^{2/\gamma-1}}}^{p_1}+\overbrace{\frac{\gamma\left[\left(p_0+p_1\right)^{2/\gamma}-\left(p_0+p_1\right)^{(\gamma+1)/\gamma}-\varepsilon\xi\right]}{(\gamma+1)\left(p_0+p_1\right)^{1/\gamma}-2\left(p_0+p_1\right)^{2/\gamma-1}}}^{p_2}; \tag{31}$$

$$T=\left(\varepsilon\xi\right)^{\frac{1}{2}(\gamma-1)}+\frac{(\gamma-1)\left[T_0^{2/(\gamma-1)}-T_0^{(\gamma+1)/(\gamma-1)}-\varepsilon\xi\right]}{(\gamma+1)T_0^{2/(\gamma-1)}-2T_0^{(3-\gamma)/(\gamma-1)}}+\frac{(\gamma-1)\left[\left(T_0+T_1\right)^{2/(\gamma-1)}-\left(T_0+T_1\right)^{(\gamma+1)/(\gamma-1)}-\varepsilon\xi\right]}{(\gamma+1)\left(T_0+T_1\right)^{2/(\gamma-1)}-2\left(T_0+T_1\right)^{(3-\gamma)/(\gamma-1)}}; \tag{32}$$

$$\rho=\left(\varepsilon\xi\right)^{\frac{1}{2}}+\frac{\rho_0^2-\rho_0^{\gamma+1}-\varepsilon\xi}{(\gamma+1)\rho_0^{\gamma}-2\rho_0}+\frac{\left(\rho_0+\rho_1\right)^2-\left(\rho_0+\rho_1\right)^{\gamma+1}-\varepsilon\xi}{(\gamma+1)\left(\rho_0+\rho_1\right)^{\gamma}-2\left(\rho_0+\rho_1\right)}. \tag{33}$$

F. Universal Supersonic Solution

A more generic expansion may be obtained, specifically, one that simultaneously applies to the three thermodynamic quantities. Using $S\equiv X^{1/\alpha}$ and $X-X^{(\gamma+1)/2}=\varepsilon\xi$, we take $X=X_0+X_1+\ldots$ and solve

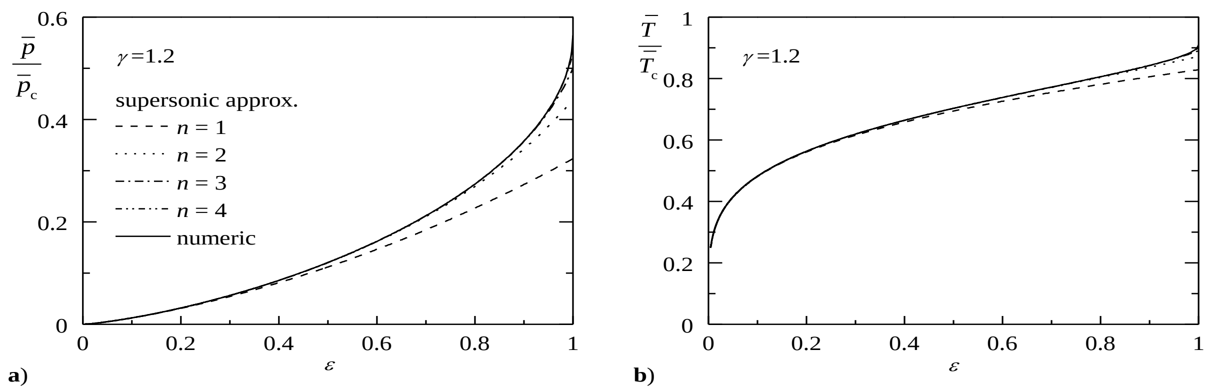

Figure 4. Numeric and asymptotic solutions for the subsonic a) pressure and b) temperature.

for the supersonic root. Recognizing that $X_0 = \varepsilon\xi$, we separate the higher order corrections and retrieve the common form

$$X(\varepsilon,\gamma,n) = X_0 + \sum_{m=1}^{n} X_m; \qquad X_m = \frac{\chi_m - \chi_m^{(\gamma+1)/2} - \varepsilon\xi}{\frac{1}{2}(\gamma+1)\chi_m^{(\gamma-1)/2} - 1}; \quad \chi_m \equiv \sum_{j=0}^{m-1} X_j; \qquad \begin{Bmatrix} p \\ T \\ \rho \end{Bmatrix} = \begin{cases} \sqrt{X^{\gamma}} \\ \sqrt{X^{\gamma-1}} \\ \sqrt{X} \end{cases} \tag{34}$$

Subsequently, the supersonic root to any of the thermodynamic properties may be calculated from

$$S(\varepsilon,\gamma,n) = \left[\varepsilon\xi + \sum_{m=1}^{n} \frac{\chi_m - \chi_m^{(\gamma+1)/2} - \varepsilon\xi}{\frac{1}{2}(\gamma+1)\chi_m^{(\gamma-1)/2} - 1} + O(\varepsilon^{n+1}) \right]^{1/\alpha} \tag{35}$$

where α represents the single parameter that varies between one property and another. Here too, the universal expression is seen to outperform the property specific parent relation given by Eq. (52). For example, given $\varepsilon = 0.1$, $\gamma = 1.2$, and an exact value of $T = 0.627051$, Eq. (35) predicts 0.623046 and 0.627024 for the temperature ratio using $n = 1$ and $n = 2$, respectively. For the same case, Eq. (52) in the Appendix returns 0.654231 and 0.630479. Both approximations, however, converge to 0.627051 at $n = 3$.

G. Supersonic Error Verification

For a typical $\gamma = 1.2$, a comparison between Eq. (35) and the numerical solution is showcased in Fig. 4 using $n = 1,2,3$ and 4. It is clear that the merging of asymptotic and numerical curves occurs so rapidly that the second and third order approximations become graphically indiscernible from the exact solution over an extended range of area ratios. This behavior is further confirmed in Fig. 5 where the orders of the relative errors in the pressure, temperature, and density estimates are captured on a log-log plot using two values of γ. Note that the increased accuracy of the third order expansion, which is observed in Fig. 4, may be attributed to the steep order jump preceding $n = 3$ in the relative error e_n. Furthermore, we find that increasing γ has a favorable influence on e_n and that the temperature approximation exhibits the lowest relative error. For this reason, it may be sufficient to use two corrections for the temperature and three for the pressure and density. Clearly, the use of $\gamma = 1.2$ for the purpose of illustration stands on the conservative side as the error only diminishes with further increases in γ.

By comparison to the subsonic case, the asymptotic character of the supersonic approximation is more elusive to quantify. This behavior may be attributed to the subsonic approach being based on a simple perturbation expansion in which the truncation error can be directly estimated from the order of the largest unused term in the series. It can be easily seen, for example, that a three-term expansion in Eq. (8)

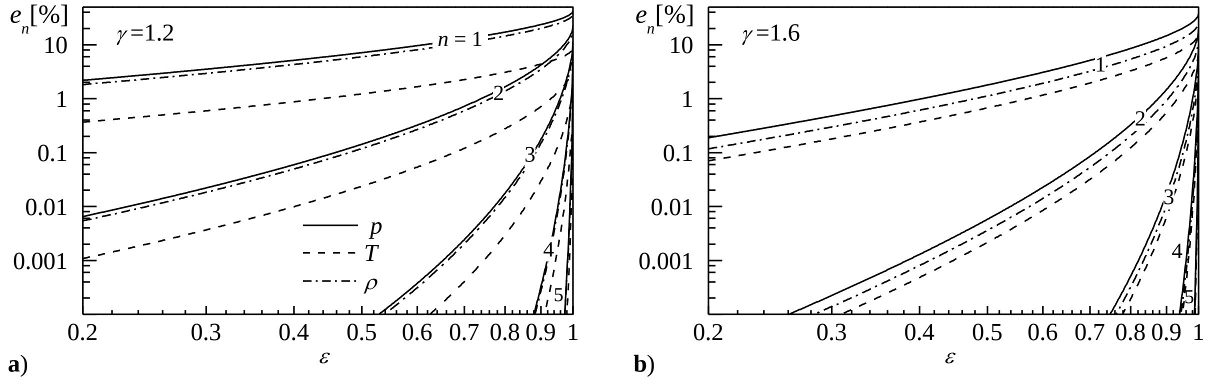

Figure 5. Percent relative error in the supersonic pressure, temperature, and density ratios based on the universal asymptotic representation. The error diminishes as γ is increased from a) 1.2 to b) 1.6.

accrues an error $E_2 = O(\varepsilon^3)$ or that, in general, $E_n = O(\varepsilon^{n+1})$. Deducing the truncation error for the supersonic solution may be seen to be considerably more elaborate, especially that the corresponding (successive) approximations do not employ an $\varepsilon-$based sequence of gauge functions. Instead, the dependence on ε entrenches itself in each successive correction term in a non-intuitive way.

To explore the implicit error dependence on ε, it is useful to re-examine the expanded form of $X - X^{(\gamma+1)/2} = \varepsilon\xi$. For the sake of illustration, we take the zeroth and first order corrections, namely,

$$\begin{cases} X_0 = \varepsilon\xi - X_0^{\frac{1}{2}(\gamma+1)} = \varepsilon\xi + O\left[\varepsilon^{\frac{1}{2}(\gamma+1)}\right]; \text{ since } \tfrac{1}{2}(\gamma+1) > 1 \\ X_0 + X_1 = \varepsilon\xi - X_0^{\frac{1}{2}(\gamma+1)}\left(1+\dfrac{X_1}{X_0}\right)^{\frac{1}{2}(\gamma+1)} \end{cases} \tag{36}$$

To extract the first order solution, the second member in Eq. (36) is subjected to a binomial expansion. This enables us to write

$$X_1 = -X_0^{\frac{1}{2}(\gamma+1)}\left(1+\frac{X_1}{X_0}\right)^{\frac{1}{2}(\gamma+1)} = -X_0^{\frac{1}{2}(\gamma+1)}\left[1+\tfrac{1}{2}(\gamma+1)\frac{X_1}{X_0}+\tfrac{1}{8}(\gamma+1)(\gamma-1)\left(\frac{X_1}{X_0}\right)^2+\ldots\right] \tag{37}$$

and retrieve

$$X_1 = -\frac{X_0^{\frac{1}{2}(\gamma+1)}}{1+\tfrac{1}{2}(\gamma+1)X_0^{\frac{1}{2}(\gamma-1)}}\left[1+O\left(\frac{X_1}{X_0}\right)^2\right] \tag{38}$$

Recalling that $X_0 \sim \varepsilon$, it follows that $X_0^{\frac{1}{2}(\gamma+1)} \sim \varepsilon^{\frac{1}{2}(\gamma+1)}$ and, according to Eq. (36), $X_1 \sim \varepsilon^{\frac{1}{2}(\gamma+1)}$. This leaves us with a truncation error of order

$$E_1 = O\left[X_0^{\frac{1}{2}(\gamma+1)}\frac{X_1^2}{X_0^2}\right] = O\left[X_0^{\frac{1}{2}(\gamma+1)}\frac{X_0^{\gamma+1}}{X_0^2}\right] = O\left[\varepsilon^{\frac{1}{2}(3\gamma-1)}\right] \tag{39}$$

The same procedure may be repeated to the extent of identifying the form of the error at arbitrary order. We find

$$E_n = O\left[X_0^{\frac{1}{2}(\gamma+1)}\left(1+\frac{X_1}{X_0}+\ldots+\frac{X_{n-1}}{X_0}\right)^{\frac{1}{2}(\gamma+1)}\frac{X_n^2}{\left(X_0+X_1+\ldots+X_{n-1}\right)^2}\right] \tag{40}$$

Then realizing that the leading X_0 term controls the order of the denominator, it can be readily factored out and simplified. This reduces Eq. (40) into

$$E_n = O\left[\varepsilon^{\frac{1}{2}(\gamma-3)} X_n^2\right] \tag{41}$$

At this point, a recursive relation for the order of X_n will be required before any further headway can be made. After some effort, we deduce

$$X_n = E_{n-1}\varepsilon^{\gamma-1}; \qquad n \ge 2 \tag{42}$$

and, consequently,

$$\begin{cases} E_0 = O\left[\varepsilon^{\frac{1}{2}(\gamma+1)}\right]; \quad E_1 = O\left[\varepsilon^{\frac{1}{2}(3\gamma-1)}\right] \\ E_n = O\left[E_{n-1}^2 \varepsilon^{\frac{1}{2}(5\gamma-7)}\right]; \qquad n \ge 2 \end{cases} \tag{43}$$

While Eq. (43) may be evaluated to render the error recursively, it requires an increasing number of algebraic operations before yielding the truncation order at successive values of n. Because our objective remains to seek a direct expression for the error, Eq. (43) is only used to determine the first ten members of the sequence of supersonic truncation orders. At the outset, the coefficients of γ in the various error exponents are judiciously collected and analyzed. An exact order relationship is subsequently constructed, namely,

$$E_n = O(\varepsilon^{\kappa_n}); \qquad \kappa_n = 2^{n+1}(\gamma-1) + \tfrac{1}{2}(7-5\gamma); \;\; n = 1, 2, ... \tag{44}$$

The correct supersonic error is thus at hand. In piecewise fashion, it may be expressed as

$$E_n = O(\varepsilon^{\kappa_n}); \quad \kappa_n = \begin{cases} \tfrac{1}{2}(\gamma+1); & n = 0 \\ \tfrac{1}{2}\left[(2^{n+2}-5)\gamma + 7 - 2^{n+2}\right]; & n \ge 1 \end{cases} \tag{45}$$

It should be emphasized that Eq. (45) applies equally uniformly to the absolute errors entailed in the specific thermodynamic properties. Their parallelism is depicted in Fig. 5 and confirmed asymptotically through Eq. (35). Their theoretical values are catalogued in Table 1 for the first 6 successive expansions and a wide spectrum of $\gamma = [1.1 - 1.6]$. Their rapidly escalating error exponents are gratifying to note as they suggest substantially improved accuracy in corresponding formulations. At $n = 4$, for example, the order increases from a conventional 3.95 at $\gamma = 1.1$ to a remarkable value of 9.85 at $\gamma = 1.4$, to a whopping 18.7 at $\gamma = 1.6$. As for the continual error reduction with upward increments in γ, it is attributable to the strictly positive coefficient of γ in Eq. (45) where it can be seen that $2^{n+2} - 5 > 0$ for all $n \ge 1$.

To verify Eq. (45) numerically, the absolute supersonic error is computed and plotted in Fig. 6 at select values of γ. Here we follow Bosley[12] in the use of a log-log scale so that κ_n can be graphically

Table 1. Theoretical asymptotic order in the universal supersonic solution

		γ					
n	Error order	1.1	1.2	1.3	1.4	1.5	1.6
0	$\frac{1}{2}(\gamma+1)$	1.05	1.1	1.15	1.2	1.25	1.3
1	$\frac{1}{2}(3\gamma-1)$	1.15	1.3	1.45	1.6	1.75	1.9
2	$\frac{1}{2}(11\gamma-9)$	1.55	2.1	2.65	3.2	3.75	4.3
3	$\frac{1}{2}(27\gamma-25)$	2.35	3.7	5.05	6.4	7.75	9.1
4	$\frac{1}{2}(59\gamma-57)$	3.95	6.9	9.85	12.8	15.75	18.7
5	$\frac{1}{2}(123\gamma-121)$	7.15	13.3	19.45	25.6	31.75	37.9
6	$\frac{1}{2}(251\gamma-249)$	13.55	26.1	38.65	51.2	63.75	76.3

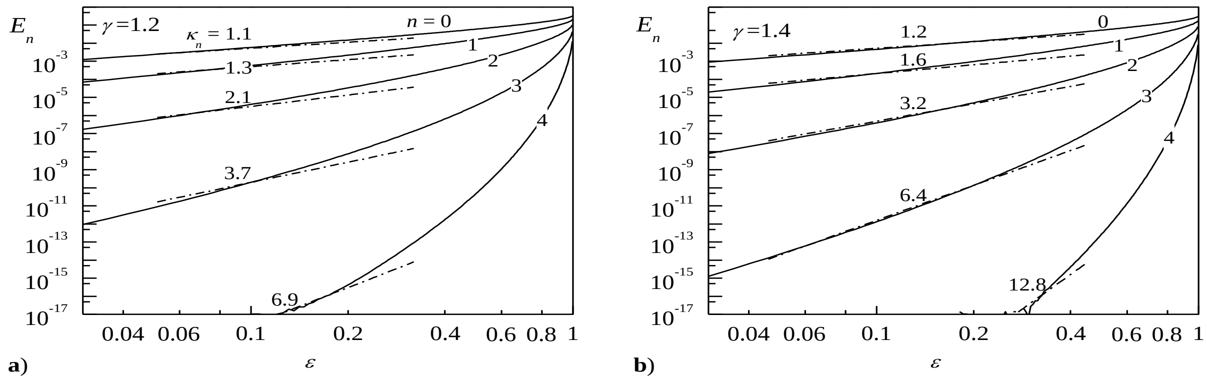

Figure 6. Absolute error in the supersonic solution using the universal representation for γ equaling a)1.2 and b) 1.4.

inferred from the slope of the error curves. While the actual error does not behave in a strictly linear fashion, it can be seen that the theoretical order expression captures quite favorably the fundamental character of the numerically computed error over a range of ε and γ. Furthermore, the magnitude of the absolute error can be seen to rapidly decrease as more terms are brought to bear. In fact, for sufficiently small values of ε, the error of the five-term expansion drops below the round-off error introduced by the numerical solution, thus introducing artificial noise in the $n = 4$ curves of Figs. 6b and 6c.

IV. Conclusions

In this study, we have considered three basic relations that arise in the context of isentropic flow analysis. Their numerical solutions have so far appeared in classic monographs on thermodynamics and compressible gas dynamics. Inasmuch as their transcendental nature has precluded an explicit inversion, we have managed to overcome their intractable character by means of asymptotic expansions. This process has required the implementation of two dissimilar asymptotic techniques, with one being original for the $M > 1$ case, to retrieve both subsonic and supersonic roots. The former branch of solution was also obtained using Bürmann’s theorem borrowed from classical analysis. At the outset, the analytical solutions have been presented using both a property specific expansion and a universal form that applies equally to all three properties. In addition to obtaining the solutions for the main thermodynamic properties to any desired order, recursive expressions have been produced for their generic formulations with arbitrary exponents (see Table 2). These have been verified both analytically and numerically. Although the details of the present derivation are shown for at least one representative property, the main results may be deduced directly from Eqs. (49) and (35) for the subsonic and supersonic branches, respectively. The strategies followed may hence find applicability in other physical settings where similar equations arise. What is most interesting, perhaps, concerns the techniques that are developed for the purpose of determining the supersonic solution and its unconventional truncation order. It is hoped that such an efficient procedure will be used in the treatment of other intransigent equations. It is also hoped that the compact relations presented here will be used to complement the collection of isentropic flow approximations that are often employed in the propulsion and power generation industries.

Acknowledgments

This work was completed with the support of the National Science Foundation through Grant No. CMMI-0928762.

Table 2. Summary of asymptotic solutions

Type	Subsonic	Supersonic
Universal	$-\sum_{m=0}^{n}\frac{\prod_{j=1}^{m-1}\left[m-1+\frac{1}{2}j(\gamma-1)\right]}{(-1)^{(2m)!}m!}\left(\frac{2\varepsilon\xi}{\gamma-1}\right)^m$	$X_0+\sum_{m=1}^{n}X_m;\ X_m=\frac{\chi_m-\chi_m^{\frac{\gamma+1}{2}}-\varepsilon\xi}{\frac{\gamma+1}{2}\chi_m^{\frac{\gamma-1}{2}}-1};\ \chi_m\equiv\sum_{j=0}^{m-1}X_j$
Pressure	$-\sum_{m=0}^{n}\frac{\prod_{j=1}^{m-1}\left\{\frac{2}{\gamma}\left[m+\frac{1}{2}j(\gamma-1)\right]-1\right\}}{(-1)^{(2m)!}m!\left[\frac{1}{\gamma}(\gamma-1)\right]^m}(\varepsilon\xi)^m$	$p_0+\sum_{m=1}^{n}p_m;\ p_m=\frac{\Pi_m^{\frac{2}{\gamma}}-\Pi_m^{\frac{\gamma+1}{\gamma}}-\varepsilon\xi}{\frac{\gamma+1}{\gamma}\Pi_m^{\frac{1}{\gamma}}-\frac{2}{\gamma}\Pi_m^{\frac{2}{\gamma}-1}};\ \Pi_m\equiv\sum_{j=0}^{m-1}p_j$
Density	$-\sum_{m=0}^{n}\frac{\prod_{j=1}^{m-1}\left\{2\left[m+\frac{1}{2}j(\gamma-1)\right]-1\right\}}{(-1)^{(2m)!}m!(\gamma-1)^m}(\varepsilon\xi)^m$	$\rho_0+\sum_{m=1}^{n}\rho_m;\ \rho_m=\frac{\mathrm{P}_m^2-\mathrm{P}_m^{\gamma+1}-\varepsilon\xi}{(\gamma+1)\mathrm{P}_m^{\gamma}-2\mathrm{P}_m};\ \mathrm{P}_m\equiv\sum_{j=0}^{m-1}\rho_j$
Temperature	$-\sum_{m=0}^{n}\frac{\prod_{j=1}^{m-1}\left\{\frac{2}{\gamma-1}\left[m+\frac{1}{2}j(\gamma-1)\right]-1\right\}}{(-1)^{(2m)!}m!}(\varepsilon\xi)^m$	$T_0+\sum_{m=1}^{n}T_m;\ T_m=\frac{\Theta_m^{\frac{2}{\gamma-1}}-\Theta_m^{\frac{\gamma+1}{\gamma-1}}-\varepsilon\xi}{\frac{\gamma+1}{\gamma-1}\Theta_m^{\frac{2}{\gamma-1}}-\frac{2}{\gamma-1}\Theta_m^{\frac{3-\gamma}{\gamma-1}}};\ \Theta_m\equiv\sum_{j=0}^{m-1}T_j$

Appendix A. Subsonic Formulation

The equation for the subsonic root can be rewritten in a recursive formula to facilitate the solution of higher order terms. The recursive relation takes the general form of

$$
\begin{aligned}
s(\varepsilon,\gamma,n)&=-\sum_{m=0}^{n}\frac{\prod_{j=1}^{m-1}\left[(m-j)\alpha+j\beta-1\right]}{(-1)^{(2m)!}m!(\beta-\alpha)^m}(\varepsilon\xi)^m+O(\varepsilon^{n+1})\\
&=2+\sum_{m=0}^{n}\frac{(-1)^{-(2m)!}(\varepsilon\xi)^m}{m!(\alpha-\beta)}\text{Pochhammer}\left[\frac{\alpha(1-m)-\beta+1}{\alpha-\beta},m-1\right]+O(\varepsilon^{n+1})
\end{aligned}
\tag{46}
$$

where, according to Abramowitz and Stegun,[14] $\text{Pochhammer}(a,n)\equiv(a)_n=\Gamma(a+n)/\Gamma(a)$. Note that Eq. (46) represents the general asymptotic solution to any transcendental relation of the form $s^{\alpha}-s^{\beta}=\varepsilon\xi$. However, due to the interconnectivity of the properties given by Eq. (10), one may fully eliminate β and write the identical expression

$$
\begin{aligned}
s(\varepsilon,\gamma,n)&=-\sum_{m=0}^{n}\frac{\prod_{j=1}^{m-1}\left\{\alpha\left[m+\frac{1}{2}j(\gamma-1)\right]-1\right\}}{(-1)^{(2m)!}m!\left[\frac{1}{2}(\gamma-1)\alpha\right]^m}(\varepsilon\xi)^m+O(\varepsilon^{n+1})\\
&=2-\frac{2}{\alpha(\gamma-1)}\sum_{m=0}^{n}\frac{(\varepsilon\xi)^m}{(-1)^{(2m)!}m!}\frac{\Gamma\left[m+2(m-1/\alpha)/(\gamma-1)\right]}{\Gamma\left[1+2(m-1/\alpha)/(\gamma-1)\right]}+O(\varepsilon^{n+1})
\end{aligned}
\tag{47}
$$

The universal form has a similar recursive formula, simplified by the elimination of the α terms. To arbitrary order, this is

$$
x(\varepsilon,\gamma,n)=-\sum_{m=0}^{n}\frac{\prod_{j=1}^{m-1}\left[m-1+\frac{1}{2}j(\gamma-1)\right]}{(-1)^{(2m)!}m!}\left(\frac{2\varepsilon\xi}{\gamma-1}\right)^m+O(\varepsilon^{n+1})
$$

$$= 2 - \frac{2}{\gamma - 1}\sum_{m=0}^{n}\frac{(\varepsilon\xi)^m}{(-1)^{(2m)!}\, m!}\text{Pochhammer}\left[\frac{2m+\gamma-3}{\gamma-1}, m-1\right] + O(\varepsilon^{n+1}) \tag{48}$$

The actual thermodynamic properties may be deduced from Eq. (48) using $s = x^{1/\alpha}$ or

$$\begin{Bmatrix} p \\ T \\ \rho \end{Bmatrix} = \begin{cases} \sqrt{x^{\gamma}} \\ \sqrt{x^{\gamma-1}} \\ \sqrt{x} \end{cases} \quad \text{or} \quad s(\varepsilon,\gamma,n) = \left\{ -\sum_{m=0}^{n} \frac{\prod_{j=1}^{m-1}\left[m - 1 + \tfrac{1}{2} j(\gamma-1)\right]}{(-1)^{(2m)!}\, m!}\left(\frac{2\varepsilon\xi}{\gamma-1}\right)^{m} + O(\varepsilon^{n+1}) \right\}^{1/\alpha} \tag{49}$$

Appendix B. Supersonic Formulation

The supersonic solution may be expressed in terms of a recursive formula to arbitrary order. For the pressure equation, we find

$$p_m = \frac{\left(\sum_{j=0}^{m-1} p_j\right)^{2/\gamma} - \left(\sum_{j=0}^{m-1} p_j\right)^{(\gamma+1)/\gamma} - \varepsilon\xi}{\frac{\gamma+1}{\gamma}\left(\sum_{j=0}^{m-1} p_j\right)^{1/\gamma} - \frac{2}{\gamma}\left(\sum_{j=0}^{m-1} p_j\right)^{2/\gamma-1}}; \qquad m \ge 1 \tag{50}$$

Since the total pressure is the summation of its constituents, we simply collect

$$p(\varepsilon,\gamma,n) = p_0 + \sum_{m=1}^{n} p_m \tag{51}$$

To be more general, the governing equation may be recast in the form $S^{\alpha} - S^{\beta} = \varepsilon\xi$. The supersonic placeholder S may then be expanded and substituted as usual to the extent of retrieving

$$S(\varepsilon,\gamma,n) = S_0 + \sum_{m=1}^{n} S_m; \quad S_m = \frac{\left(\sum_{j=0}^{m-1} S_j\right)^{\alpha} - \left(\sum_{j=0}^{m-1} S_j\right)^{\beta} - \varepsilon\xi}{\beta\left(\sum_{j=0}^{m-1} S_j\right)^{\beta-1} - \alpha\left(\sum_{j=0}^{m-1} S_j\right)^{\alpha-1}} \tag{52}$$

The specific expansions for the temperature and density corrections follow suit. We find

$$T_m = \frac{\left(\sum_{j=0}^{m-1} T_j\right)^{\frac{2}{\gamma-1}} - \left(\sum_{j=0}^{m-1} T_j\right)^{\frac{\gamma+1}{\gamma-1}} - \varepsilon\xi}{\frac{\gamma+1}{\gamma-1}\left(\sum_{j=0}^{m-1} T_j\right)^{\frac{2}{\gamma-1}} - \frac{2}{\gamma-1}\left(\sum_{j=0}^{m-1} T_j\right)^{\frac{3-\gamma}{\gamma-1}}} \qquad \rho_m = \frac{\left(\sum_{j=0}^{m-1} \rho_j\right)^{2} - \left(\sum_{j=0}^{m-1} \rho_j\right)^{\gamma+1} - \varepsilon\xi}{(\gamma+1)\left(\sum_{j=0}^{m-1} \rho_j\right)^{\gamma} - 2\left(\sum_{j=0}^{m-1} \rho_j\right)} \tag{53}$$

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