46th AIAA/ASME/SAE/ASEE Joint Propulsion Conference & Exhibit
25 - 28 July 2010, Nashville, TN

AIAA 2010-7074

Pressure Variations in Rocket Nozzles. Part 3: Direct Calculation of the Local Mach Number

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Foremost amongst rocket nozzle relations is the area-Mach number expression linking the local velocity normalized by the speed of sound to the area ratio A_t/A, and the ratio of specific heats. Known as Stodola's equation, the attendant expression is transcendental and requires iteration or numerical root finding in extracting the solution under subsonic or supersonic nozzle operation. In this work, a novel analytical inversion of the problem is pursued to the extent of providing the local Mach number directly at any given cross-section. The inversion process is carried out using two unique approaches. In the first, Bürmann's Theorem of classical analysis is employed to undertake a functional reversion from which the subsonic solution may be retrieved. In the second, the Successive Approximation Approach is repeatedly applied to arrive at a closed-form representation of the supersonic root. Both methods give rise to unique recursive formulations that permit the selective extraction of the desired solution to an arbitrary level of accuracy. Results are verified numerically and the precision associated with the supersonic approximation is shown to improve with successive increases in the ratio of specific heats.

Nomenclature

A	=	local cross-sectional area
A_t	=	nozzle throat area
M	=	local Mach number
α	=	constant exponent in Eq. (4), $\frac{1}{2}(\gamma+1)/(\gamma-1)$
ε	=	perturbation parameter, $(A_t/A)^2$
γ	=	ratio of specific heats, c_p/c_v
κ	=	constant exponent, $(\gamma-1)/(\gamma+1)$
ζ	=	constant related to γ in Eq. (4), $[\frac{1}{2}(\gamma+1)]^{\alpha}$

Subscripts and Symbols

$0,1,2$	=	leading, first and second order
c	=	chamber stagnation condition
e	=	exit plane condition
n	=	asymptotic level
$\mathcal{O}(\varphi)$	=	Landau's big-O symbol, of same order as φ
$o(\varphi)$	=	Landau's little-o symbol, of smaller order than φ
sub, sup	=	subsonic or supersonic
t	=	nozzle throat condition

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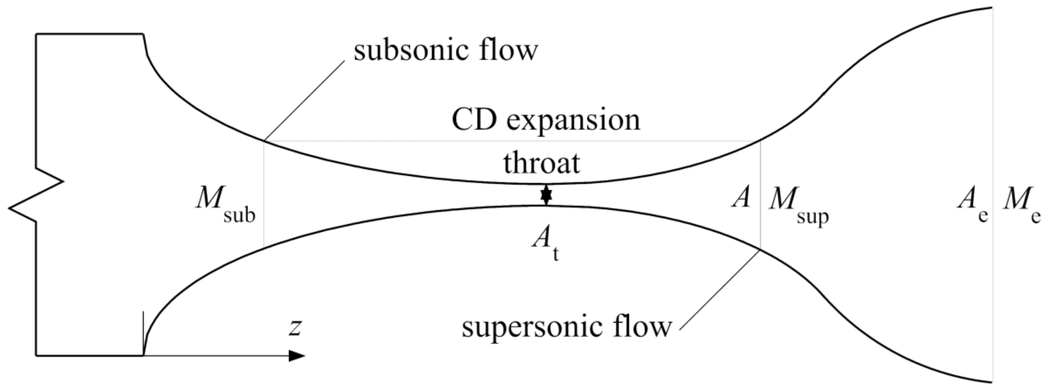

Figure 1. Converging-diverging nozzle schematic showing relevant properties and Mach numbers.

I. Introduction

For isentropic flow through a converging-diverging nozzle with throat area A_{t} (see Fig. 1), a transcendental equation named after Stodola[1] relates the area ratio A_{t} / A and the Mach number M at any cross-section of surface area A:

$$\left(\frac{A}{A_{\text{t}}}\right)^2 = \frac{1}{M^2}\left\{\frac{2}{\gamma+1}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]\right\}^{\frac{\gamma+1}{\gamma-1}} \tag{1}$$

where γ denotes the ratio of specific heats. Due to the nature of this expression, a perturbation parameter may be defined as $\varepsilon \equiv (A_{\text{t}} / A)^2$, where $\varepsilon \ll 1$ except for flow in the direct vicinity of the throat section. In practice, ε will reach its lowest value in the nozzle exit plane where $\varepsilon_{\text{e}} = (A_{\text{t}} / A_{\text{e}})^2$ represents the square of the so-called *area expansion ratio*. Naturally, the calculation of the corresponding local or exit Mach number M_{e} is of interest to the propulsion and power generation subdisciplines with particular areas of concentration in nozzle design and optimization. This can be attributed to the connection between exit Mach numbers and ideal thrust coefficients, rocket specific impulses, characteristic exhaust velocities, gas turbine efficiencies, and so on, and the relevance of Stodola's relation to the problem of sizing and shape selection in a variety of combustion devices in which gases are expanded, such as rockets, ramjets, scramjets, and afterburners.[2-4]

Despite its simplicity and one-dimensional form, Eq. (1) continues to find uses in modern propulsion-related studies. In this context, one may cite Najjar *et al.*[5] who employ this relation to set up the initial conditions in their compressible flow solver Rocflu.[6] Along similar lines, Haselbacher *et al.*[7] use this expression to establish a test case for their slow-time acceleration problem in which time scales are estimated. Thakre and Yang[8] and Zhang *et al.*[9] also make use of the relation in question for the purpose of verifying their codes on nozzle erosion.

In order to determine the local and/or exit Mach numbers at a given cross-section, one can solve for M straightforwardly using either numerical root finding or compressible flow tables. However, to solve the problem analytically, it is helpful to rearrange Eq. (1) into

$$\boxed{\varepsilon^{\frac{1}{2}}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}} - M\left[\tfrac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 0} \tag{2}$$

In this study an explicit solution for the Mach number will be presented in terms of the area ratio A/A_t and the ratio of specific heats γ.

II. Analysis

For a fixed nozzle area ratio, two mathematically and physically possible Mach number roots exist: one subsonic and the other supersonic. In what follows, the analysis leading to each of these roots will be separately described. Subsonic and supersonic solutions at order $n \in \mathbb{N}$ will be denoted by $M_{\text{sub}}^{(n)}$ and $M_{\text{sup}}^{(n)}$, respectively.

A. Subsonic Treatment Based on Bürmann's Theorem

In previous work by the authors,[10] the subsonic solution to Stodola's equation was pursued using regular perturbation theory. At present, we provide a simple alternative based on classical analysis, namely, by means of Bürmann's theorem. This form will lead to the establishment of a general recursive formulation from which the solution may be generated to any order. Bürmann's theorem forms an extension of a Taylor series reversion for arbitrary functions. In layman's words, if a function can be expanded about a particular point in terms of a second function, the converse is true, and one may express the second function in terms of the first. This theorem enables us to write an expression for the Mach number in terms of the local area ratio to arbitrary precision.

To start the analysis we revisit Stodola's equation and simplify it into

$$\varepsilon^{\frac{1}{2}}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{\alpha} = M\zeta \tag{3}$$

where

$$\alpha \equiv \frac{\gamma+1}{2(\gamma-1)} \quad \text{and} \quad \zeta \equiv \left(\frac{\gamma+1}{2}\right)^{\alpha} \tag{4}$$

Before we apply Bürmann's theorem, we rearrange Eq. (3) such that

$$\boxed{\varepsilon^{\frac{1}{2}} = \phi(M) = M\zeta\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{-\alpha}} \tag{5}$$

In the above, $\phi(M)$ constitutes an analytic function over the closed region associated with the subsonic branch of solution, $0 \le M \le 1$. To ensure that an expansion is valid in this region, we examine the behavior of the function and its first derivative as $\varepsilon \to 0$. We identify, in particular, the point $M_0 \equiv 0$ where

$$\phi(0) = 0 \quad \text{and} \quad \phi'(0) \neq 0 \tag{6}$$

Since $\phi(M_0)$ is finite and the derivative is non-zero, Bürmann's theorem can indeed be applied in this situation.[11] We subsequently define $f(M) = M$ and use M_0 as the anchor point to construct

$$\psi(M) = \frac{f(M)-f(M_0)}{\phi(M)-\phi(M_0)} = \frac{M}{\phi(M)} \tag{7}$$

Given $\psi(M)$, a straightforward application of the theorem leads to

$$f(M) = f(M_0) + \sum_{m=0}^{n} \frac{\phi^{m+1}(M)}{(m+1)!}\left[\frac{\mathrm{d}F}{\mathrm{d}M}\frac{\mathrm{d}^m\psi^{m+1}(M)}{\mathrm{d}M^m}\right]_{M=M_0} + R_{n+1} \tag{8}$$

where R_{n+1} represents the truncation error remaining at that order. Substituting Eq. (7) into Eq. (8) yields

$$f(M) = \sum_{m=0}^{n} \frac{\varepsilon^{\frac{m+1}{2}}}{(m+1)!}\left[\frac{\mathrm{d}^m\psi^{m+1}(M)}{\mathrm{d}M^m}\right]_{M=0} + \mathcal{O}(\varepsilon^{\frac{n}{2}+1}) = \sum_{m=0}^{n-1} \frac{\varepsilon^{\frac{m+1}{2}}}{(m+1)!}\left\{\frac{\mathrm{d}^m}{\mathrm{d}M^m}\left[\frac{M^{m+1}}{\phi^{m+1}(M)}\right]\right\}_{M=0} + \mathcal{O}(\varepsilon^{\frac{n}{2}+1}) \tag{9}$$

whence

$$f(M) = \sum_{m=0}^{n} \frac{\varepsilon^{\frac{m+1}{2}}}{(m+1)!\,\zeta^{m+1}}\left(\frac{\mathrm{d}^m}{\mathrm{d}M^m}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{(m+1)\alpha}\right)_{M=0} + \mathcal{O}(\varepsilon^{\frac{n}{2}+1}) \tag{10}$$

Evaluating the first three terms of the summation renders

$$f = \frac{\varepsilon^{\frac{1}{2}}}{1!\zeta}\left\{\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{\alpha}\right\}_{M=0} + \frac{\varepsilon^{\frac{2}{2}}}{2!\zeta^2}\left\{\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{2\alpha}\right\}'_{M=0} + \frac{\varepsilon^{\frac{3}{2}}}{3!\zeta^3}\left\{\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{3\alpha}\right\}''_{M=0} + \ldots \quad (11)$$

Each member of this series may be separately determined from the limit of the corresponding derivative at M_0. We get, for the first, zeroth-order member,

$$\lim_{M\to 0}\frac{\varepsilon^{\frac{1}{2}}}{\zeta}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{\alpha} = \frac{\varepsilon^{\frac{1}{2}}}{\zeta} = \varepsilon^{\frac{1}{2}}\left(\frac{2}{\gamma+1}\right)^{\alpha} \quad (12)$$

Similarly, the second member gives

$$\lim_{M\to 0}\frac{\varepsilon^{\frac{2}{2}}}{2!\zeta^2}\left\{\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{2\alpha}\right\}' = \lim_{M\to 0}\frac{\varepsilon}{2\zeta^2}\left\{2\alpha\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{2\alpha-1}(\gamma-1)M\right\} = 0 \quad (13)$$

It may be easily proved that all terms containing whole powers of ε, $\{1,\varepsilon,\varepsilon^2,\varepsilon^3,\ldots\}$ vanish identically. Finally, the cubic term may be calculated from

$$\lim_{M\to 0}\frac{\varepsilon^{\frac{3}{2}}}{3!\zeta^3}\left\{\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{3\alpha}\right\}'' = \lim_{M\to 0}\frac{\varepsilon^{\frac{3}{2}}}{6\zeta^3}\left\{3\alpha\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{3\alpha-1}(\gamma-1)M\right\}'$$

$$= \lim_{M\to 0}\frac{\alpha(\gamma-1)\varepsilon^{\frac{3}{2}}}{2\zeta^3}\left\{(3\alpha-1)\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{3\alpha-2}(\gamma-1)M^2 + \left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{3\alpha-1}\right\} = \frac{\alpha(\gamma-1)}{2\zeta^3}\varepsilon^{\frac{3}{2}}$$

$$= \varepsilon^{\frac{3}{2}}2^{\frac{7-\gamma}{2(\gamma-1)}}(\gamma+1)^{-\frac{\gamma+5}{2(\gamma-1)}} = \frac{\varepsilon^{\frac{3}{2}}}{4}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+5}{2(\gamma-1)}} \quad (14)$$

Higher-order terms may be readily extracted using symbolic programming of Eq. (10). At length, the subsonic solution to order $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ may be arrive at, specifically

$$M_{\text{sub}}^{(3)} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\varepsilon^{\frac{1}{2}} + \frac{1}{4}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+5}{2(\gamma-1)}}\varepsilon^{\frac{3}{2}} + \left[2^{\frac{15-5\gamma}{2(\gamma-1)}}(\gamma+1)^{\frac{3\gamma+7}{2-2\gamma}}(3\gamma+7)\right]\varepsilon^{\frac{5}{2}} + \mathcal{O}(\varepsilon^{\frac{7}{2}}) \quad (15)$$

Equation (15) compares quite well with the numerical solution. For most propulsive applications, the three terms retained above are quite sufficient for engineering accuracy. The attending error remains smaller than 5% up to $\varepsilon = 0.77$ or an area ratio of 0.88. Nonetheless, should further supplementary terms be required, Eq. (10) may be slightly modified to generate a solution to arbitrary order of precision viz.

$$\boxed{M_{\text{sub}}^{(n)} = \sum_{i=0}^{n}\frac{\varepsilon^{i+\frac{1}{2}}\zeta^{(2i+1)\alpha}}{(2i+1)!}\left\{\frac{\mathrm{d}^{2i}}{\mathrm{d}M^{2i}}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]^{(2i+1)\alpha}\right\}_{M=0} + \mathcal{O}(\varepsilon^{n+\frac{3}{2}})} \quad (16)$$

After some effort, a recursive relation may be seen to exist, thus permitting the direct extraction of the subsonic solution to any desired order of accuracy:

$$\boxed{M_{\text{sub}}^{(n)} = \sum_{i=0}^{n}(\gamma-1)^i\varepsilon^{i+\frac{1}{2}}\frac{(2i)!}{i!2^i}\frac{\prod_{j=0}^{i-1}\left[(2i+1)\alpha-j\right]}{(2i+1)!\zeta^{2i+1}} + \mathcal{O}(\varepsilon^{n+\frac{3}{2}})} \quad (17)$$

As shown in Fig. 2, this expression is virtually indiscernible from the numerical solution of the problem. A three-term approximation is more than adequate for the subsonic Mach number, even for large ε. For operational area ratios up to $A_t/A = 0.47$ and an extreme case of $\gamma = 1.7$, only *one term* needs to be retained in order to obtain a practical approximation that accrues a less than 5% error. The range of ε increases as more terms are retained or as γ is lowered.

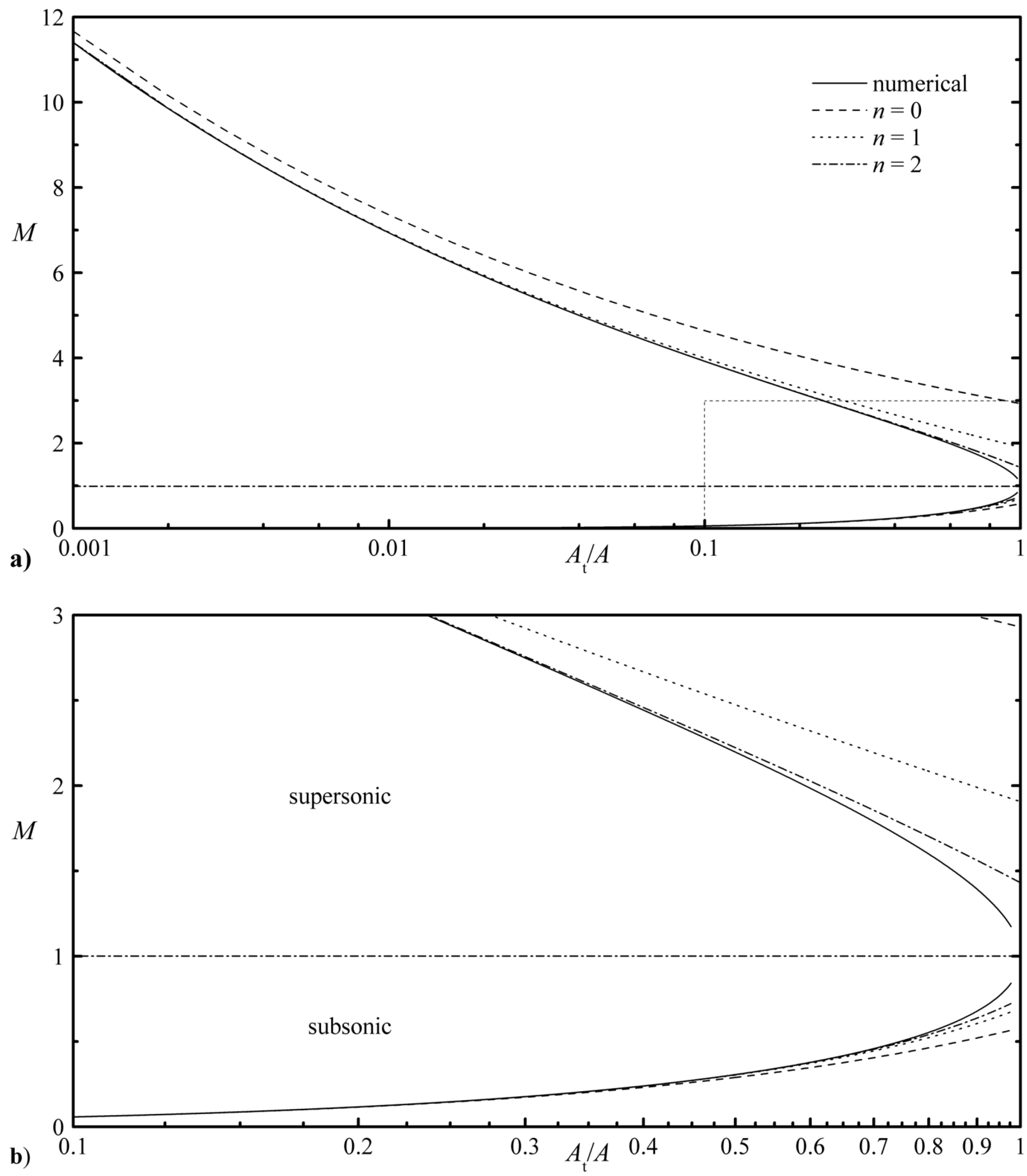

Figure 2. Comparison between numerical and asymptotic solutions for $\gamma = 1.4$.

B. Supersonic Treatment Based on Successive Approximations

It seems that all regular and non-regular perturbation attempts will fail in extracting the supersonic root directly from Eq. (2). Instead, we find it necessary to elevate Stodola's equation to the power of $\alpha^{-1} = 2(\gamma-1)/(\gamma+1)$. The exponent-inverted form becomes

$$\varepsilon^{\frac{\gamma-1}{\gamma+1}}\left[1+\tfrac{1}{2}(\gamma-1)M^2\right]-M^{\frac{2(\gamma-1)}{\gamma+1}}\left[\tfrac{1}{2}(\gamma+1)\right]=0 \tag{18}$$

The resulting expression may then be multiplied by 2 and rearranged into

$$(\gamma+1)M^{\frac{2(\gamma-1)}{\gamma+1}}-(\gamma-1)\varepsilon^{\frac{\gamma-1}{\gamma+1}}M^2-2\varepsilon^{\frac{\gamma-1}{\gamma+1}}=0 \tag{19}$$

or

$$\boxed{(\gamma+1)M^{2\kappa}-(\gamma-1)\varepsilon^{\kappa}M^2-2\varepsilon^{\kappa}=0;\quad \kappa\equiv(\gamma-1)/(\gamma+1)} \tag{20}$$

Equation (20) is a keystone relation that can be managed to produce the supersonic root, albeit equivalent to Eq. (2). In what follows, we apply the Successive Approximation Method to obtain the first three terms of the solution from which a recursion formula may be deduced. We also recognize that $1 \le \gamma \le \frac{5}{3}$ and so $0 \le \kappa \le \frac{1}{4}$. Then for $M > 1$, the members in Eq. (20) may be presented in descending order, with the largest and smallest terms corresponding to $(\gamma+1)M^{2\kappa}$ and $2\varepsilon^{\kappa}$, respectively. Assuming $M = M_0 + \mathcal{O}(M_0)$ one gets, at leading order:

$$(\gamma+1)M_0^{2\kappa} - (\gamma-1)\varepsilon^{\kappa}M_0^2 - 2\varepsilon^{\kappa} = 0 \tag{21}$$

In view of $M_0 > 1$ and $\kappa < 1$, the last term may be ignored, being of higher order. This enables us to achieve balance between the first two terms by setting

$$(\gamma+1)M_0^{2\kappa} = (\gamma-1)\varepsilon^{\kappa}M_0^2 \;\text{ or }\; M_0 = \left(\kappa\varepsilon^{\kappa}\right)^{\frac{1}{2(\kappa-1)}} = \left[\frac{\gamma+1}{(\gamma-1)\varepsilon^{\kappa}}\right]^{\frac{\gamma+1}{4}} = \left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma+1}{4}} \frac{1}{\varepsilon^{\frac{1}{4}(\gamma-1)}} \tag{22}$$

Next, we let $M = M_0 + M_1 + \mathcal{O}(M_1)$ and expand Eq. (21) into

$$(\gamma+1)M_0^{2\kappa}\left(1 + 2\kappa\frac{M_1}{M_0}\right) - (\gamma-1)\varepsilon^{\kappa}M_0^2\left(1 + 2\frac{M_1}{M_0}\right) - 2\varepsilon^{\kappa} = 0 \tag{23}$$

where a binomial series is used. This enables us to extract the first order correction from

$$2\kappa(\gamma+1)M_0^{2\kappa-1}M_1 - 2(\gamma-1)\varepsilon^{\kappa}M_0M_1 - 2\varepsilon^{\kappa} = 0 \tag{24}$$

whence

$$M_1 = \frac{\varepsilon^{\kappa}M_0}{\kappa(\gamma+1)M_0^{2\kappa} - (\gamma-1)\varepsilon^{\kappa}M_0^2} = \frac{1}{(\gamma-1)(\varepsilon^{-\kappa}M_0^{2\kappa-1} - M_0)} \tag{25}$$

To determine $\mathcal{O}(M_1)$ and, with it, the order of leading order truncation error, we evaluate

$$M_1 \approx \frac{1}{(\gamma-1)}\varepsilon^{\kappa}M_0^{1-2\kappa} = \frac{\varepsilon^{\frac{\gamma-1}{\gamma+1}}}{(\gamma-1)}\left[\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma+1}{4}}\varepsilon^{\frac{1}{4}(1-\gamma)}\right]^{\frac{3-\gamma}{\gamma+1}} = \frac{\varepsilon^{\frac{\gamma^2-1}{4(\gamma+1)}}}{(\gamma-1)}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{3-\gamma}{4}} = \mathcal{O}(\varepsilon^{\frac{1}{4}(\gamma-1)}) \tag{26}$$

Repeating once more, we substitute $M = M_0 + M_1 + M_2 + \mathcal{O}(M_2)$ into Eq. (21) and collect

$$(\gamma+1)(M_0+M_1)^{2\kappa}\left(1 + 2\kappa\frac{M_2}{M_0+M_1}\right) - (\gamma-1)\varepsilon^{\kappa}(M_0+M_1)^2\left(1 + 2\frac{M_2}{M_0+M_1}\right) - 2\varepsilon^{\kappa} = 0 \tag{27}$$

The second order correction may thus be retrieved. We obtain

$$M_2 = \frac{2\varepsilon^{\kappa} - (\gamma+1)(M_0+M_1)^{2\kappa} + (\gamma-1)\varepsilon^{\kappa}(M_0+M_1)^2}{2\kappa(\gamma+1)(M_0+M_1)^{2\kappa-1} - 2(\gamma-1)\varepsilon^{\kappa}(M_0+M_1)} \tag{28}$$

An asymptotically expanded form of the above may be expressed as

$$M_2 \simeq -\frac{M_1^3}{2}\frac{(\gamma-1)\varepsilon^{\kappa} - (\gamma+1)\kappa(2\kappa-1)M_0^{2\kappa-2}}{\left[(\gamma-1)\varepsilon^{\kappa} - (\gamma+1)\kappa(2\kappa-1)M_0^{2\kappa-2}\right]M_1^2 - \varepsilon^{\kappa}} \tag{29}$$

The corresponding order may be retrieved using binomial expansions and simplifications leading to the identification of

$$M_2 \approx \tfrac{1}{2}(\gamma-1)M_1^3 = \mathcal{O}(\varepsilon^{\frac{3}{4}(\gamma-1)}) \tag{30}$$

It is clear that for $i \ge 1$, a recursive relation exists for M_i in terms of M_{i-1} from which all terms beyond M_0 may be extrapolated. Higher order approximations may hence be realized by adding each successive correction to the sum:

$$M_{\sup}^{(n)} = M_0 + \sum_{i=1}^{n} M_i; \qquad M_i = \frac{2\varepsilon^{\kappa} - (\gamma+1)\left[M_{\sup}^{(i-1)}\right]^{2\kappa} + (\gamma-1)\varepsilon^{\kappa}\left[M_{\sup}^{(i-1)}\right]^{2}}{2\kappa(\gamma+1)\left[M_{\sup}^{(i-1)}\right]^{2\kappa-1} - 2(\gamma-1)\varepsilon^{\kappa} M_{\sup}^{(i-1)}}; \qquad M_{\sup}^{(i-1)} = \sum_{j=0}^{i-1} M_j \quad (31)$$

Results displayed in Fig. 2 are taken with expansions up to $n = 2$ so both the supersonic and subsonic solutions will contain three terms. The selected area in part a) is magnified in part b) to illustrate the degree of agreement with increasing asymptotic orders as $\varepsilon \to 1$. We note a substantial agreement with the use of three terms, even at large ε.

III. Concluding Remarks

In this study, two asymptotic formulations are presented as numerical equivalents to the traditional area-Mach number relationship that is ubiquitously used in rocket nozzle analysis. The first is derived from Bürmann's Theorem and the second, using the Method of Successive Approximations. Both techniques unravel the explicit dependence of the Mach number on the nozzle area ratio and the ratio of specific heats. The tacit relations that we arrive at allow for swift and robust computation of essential flowfield properties in a De Laval nozzle under either subsonic or supersonic operation. This is accomplished by granting the use of selectively controlled recursive formulations that produce the desired branch of solution to any degree of accuracy. Compared to other studies on the subject, the present work leads to simple and novel recursions with well prescribed truncation orders. These increase our repertoire of engineering approximations for compressible flows and enable us to compute the subsonic and supersonic Mach numbers at any cross-section and to an arbitrary degree of precision. In practice, a maximum of three non-zero terms in each approximation may be sufficient to yield a satisfactory level of precision.

Acknowledgments

This work was completed with the support of the National Science Foundation through Grant No. CMMI-0928762.

References

[1]Stodola, A., *Steam Turbines*, Springer-Verlag, Berlin, 1903.

[2]Anderson, J. D., *Modern Compressible Flow with Historical Perspective*, 3rd ed., McGraw-Hill, New York, 2003.

[3]Moran, M. J., and Shapiro, H. N., *Fundamentals of Engineering Thermodynamics*, 5th ed., John Wiley, New York, 2004.

[4]Çengel, Y. A., and Boles, M. A., *Thermodynamics: An Engineering Approach*, 4th ed., McGraw-Hill, New York, 2002.

[5]Najjar, F. M., Ferry, J. P., Haselbacher, A., and Balachandar, S., "Simulations of Solid-Propellant Rockets: Effects of Aluminum Droplet Size Distribution," *Journal of Spacecraft and Rockets,* Vol. 43, No. 6, 2006, pp. 1258-1270. doi: 10.2514/1.17326

[6]Haselbacher, A., "A WENO Reconstruction Algorithm for Unstructured Grids Based on Explicit Stencil Construction," AIAA Paper 2005-0879, January 2005.

[7]Haselbacher, A., Najjar, F. M., Massa, L., and Moser, R. D., "Slow-Time Acceleration for Modeling Multiple-Time-Scale Problems," *Journal of Computational Physics,* Vol. 229, No. 2, 2010, pp. 325-342. doi: 10.1016/j.jcp.2009.09.029

[8]Thakre, P., and Yang, V., "Chemical Erosion of Carbon-Carbon/Graphite Nozzles in Solid-Propellant Rocket Motors," *Journal of Propulsion and Power,* Vol. 24, No. 4, 2008, pp. 822-833. doi: 10.2514/1.34946

[9]Zhang, J., Jackson, T. L., Najjar, F. M., and Buckmaster, J., "High-Fidelity Multiphysics Simulations of Erosion in SRM Nozzles," AIAA Paper 2009-5499, August 2009.

[10]Majdalani, J., and Maicke, B. A., "Modeling Mach Number and Temperature Distributions in Supersonic Nozzle Flow," AIAA Paper 2010-6844, July 2010.

[11]Whittaker, E. T., and Watson, G. N., *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1927.