40th Fluid Dynamics Conference and Exhibit
28 June - 1 July 2010, Chicago, Illinois

AIAA 2010-4288

Pressure Integration Rules and Restrictions for the Navier-Stokes Equations

Tony Saad* and Joseph Majdalani†
University of Tennessee Space Institute, Tullahoma, TN 37388, USA

The Navier-Stokes formulation represents a uniquely challenging system of partial differential equations that continues to influence modern applied science and engineering. In its simplest form, the system can be used to prescribe the motion of a viscous incompressible fluid with constant properties. It consists of four equations in three-dimensional space that account for both the kinematic and dynamic conditions that a fluid element senses. In this work, we investigate the pressure integration rules and restrictions that affect the resolution of the scalar pressure field in these equations. We begin the analysis by exploring the integration properties of Euler's equations in two dimensions while making use of Clairaut's theorem on the commutativity of mixed partial derivatives. We then extend our findings to three-dimensional space. This effort gives rise to a theorem and three corollaries that help to clarify the conditions needed to obtain exact or asymptotic solutions for the pressure distribution. We thusly identify the fundamental conditions under which the Navier-Stokes equations can be successfully integrated to arrive at an analytic expression for the pressure field, namely, one that is continuous and twice differentiable. In closing, several configurations are used to test the theorem and showcase its connection with the pressure formulation. These include potential flows for which the pressure can be obtained unconditionally, and inviscid rotational motions of the Taylor-Culick type with and without headwall injection.

Nomenclature

$\boldsymbol{B}$	=	body force per unit volume
C	=	differentiability class of functions
C^k	=	class of functions with continuous derivatives up to order k
$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$	=	unit base vectors for curvilinear coordinates
$\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$	=	unit base vectors for Cartesian coordinates
$\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z$	=	unit base vectors for cylindrical coordinates
$\mathcal{F}, \mathcal{G}$	=	auxiliary functions
p	=	pressure
r, θ, z	=	cylindrical coordinates
t	=	temporal coordinate
$\boldsymbol{u}$	=	velocity vector
u, v, w	=	x-y-z or r-θ-z velocity components
x, y, z	=	Cartesian coordinates
x_1, x_2, x_3	=	generalized curvilinear coordinates

Greek Symbols

ε	=	small perturbation parameter
μ	=	dynamic viscosity
$\boldsymbol{\Omega}$	=	vorticity vector, $\nabla \times \boldsymbol{u}$

*Graduate Research Assistant, Mechanical, Aerospace and Biomedical Engineering Department. Currently Post Doctoral Fellow, Institute for Clean and Secure Energy, University of Utah. Member AIAA.
†H. H. Arnold Chair of Excellence in Advanced Propulsion, Mechanical, Aerospace and Biomedical Engineering Department. Senior Member AIAA. Fellow ASME.

ρ	=	density
ψ	=	fluid streamfunction

I. Introduction

The mathematical relations describing fluid motions form such fascinating systems of partial differential equations (PDEs) that, for decades, they have intrigued and challenged mathematicians and scientists alike. Yet the inherent nonlinearity and tight coupling embedded in these equations make it nearly impossible to obtain closed form solutions except within the edifices of simplified models. To partially alleviate the emerging difficulties in these strongly nonlinear PDEs, a multitude of flow regimes have been identified wherein the flow equations could be systematically reduced to more manageable levels. Instances include classifying the field as irrotational, isentropic, inviscid-rotational, compressible, creeping, viscous with a prescribed pressure gradient, or of the boundary layer type. To be worthwhile, each approximation thus conceived had to capture the essential physics of the flow under examination despite the set of deficiencies that it inevitably carried. Over the course of the years, appropriate solution methodologies and mathematical tools have henceforth been developed, some for the specific purpose of complementing the resulting idealizations. These have entailed concepts such as velocity potentials and conformal mapping for irrotational flows, vorticity-streamfunction formulations for inviscid-rotational motions, similarity transforms and triple-deck theory for boundary layers, Prandtl-Glauert and hodograph methods for compressible flows, and so on.[1,2]

Retrospectively, our present understanding of fluid mechanics eclipses quite a turbulent yet engaging history. On the upshot, most of the hydrostatic principles were already known by the early eighteenth century owing, in large part, to the cumulative works of various scientists. To name a few, we enumerate: Evangelista Torricelli, Blaise Pascal, Christiaan Huygens, Edme Mariotte, Isaac Newton, and Daniel Bernoulli. On the downside, theoretically enticing formulations were timidly pursued during that era until somewhere in the middle of the eighteenth century. At that point, four key figures could be cited as having played a central role in advancing the theory of fluid mechanics. They are: Alexis Clairaut, for his work on differential forms; Jean le Rond d'Alembert, for his propositions on the differential treatment of fluids; Leonhard Euler, for his synthesis, notation, and rigorous generalizations; and Joseph Louis Lagrange, for introducing useful techniques, such as perturbation methods, to tackle the equations of motion.[3]

We start with Clairaut who, in his attempts at predicting the curvature of the earth and the flattening of the poles,[4,5] formulated a theorem on exact differentials that proved to be essential to many subsequent studies that were prompted by his work. In short, he derived the constraint needed for a total differential to be exact, which is still viewed today as a milestone achievement in differential calculus. About a decade later, circa 1749, d'Alembert presented his findings to the Academy of Sciences of Berlin in a manuscript that would officially appear later in 1755. In the interim, Euler as usual continued his groundbreaking inquiries, and may have been truly the first to approach the discipline of fluid mechanics from a purely generalistic and problem independent viewpoint.[3] In this regard, he is most famous for his four manuscripts,[6–9] published in the early 1750s, that lay down the foundations of fluid dynamics. It may be further argued that most of the fundamental theory that is taught today was handed down to us by Euler. Inspired no doubt by Euler's discourses and discoveries, Lagrange later presented in 1788 his treatise on analytical mechanics. His work was profound and provided the most comprehensive coverage of classical mechanics since Newton to the extent of precipitating the development of mathematical physics in the nineteenth century. He is credited today for being the first to have used asymptotic theory in the treatment of fluids-related problems.[3]

In their most general form, the equations of fluid motion cover the effects of inertial accumulation, convection, diffusion, compressibility, gravity, etc. They consist of four equations: the continuity relation that expresses the kinematic conditions on the fluid, and three momentum equations that capture the co-existing dynamical effects while asserting the balance of internal and external forces on the evolution of the flow field. The most remarkable feature of these equations stands, perhaps, in the wealth of physics that they embody. For example, they can be used for modeling both laminar and turbulent flows, although the physical processes that accompany these flow regimes are quite dissimilar.

In this study, we consider the incompressible Navier-Stokes equations and investigate a fundamental aspect of their solution. Our objective may be stated as follows: *Given a velocity field that satisfies the continuity equation, we seek to identify the conditions under which the Navier-Stokes equations can be integrated to the extent of returning an analytic expression for the pressure.* Our discussion will also address the resulting nonlinear constraints that will be shown to constitute necessary and sufficient conditions for obtaining the pressure distribution from the direct integration of the momentum equation.

The paper is organized as follows. First, we illustrate the process of integrating the two-dimensional Euler equations from which we are able to develop the pressure integrability rules. Then guided by Clairaut's theorem, the pressure integrability rules will be applied to the three-dimensional Navier-Stokes equations from which three constraints will be derived. A theorem and three corollaries will follow, and these will be extended to the treatment of small perturbation problems. Finally, a list of implications and pertinent examples will be presented and discussed.

II. Integration of Euler's Equations

Using standard nomenclature and a Cartesian reference frame, the normalized form of the two-dimensional, steady, Euler equations may be presented as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial p}{\partial x} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = -\boldsymbol{u}\cdot\nabla u \tag{2}$$

$$\frac{\partial p}{\partial y} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} = -\boldsymbol{u}\cdot\nabla v \tag{3}$$

Then given a velocity field $\boldsymbol{u}$ that satisfies continuity, our objective here is to determine the additional conditions that must be met by $\boldsymbol{u}$ to ensure that the corresponding pressure field is solvable using Eqs. (2–3). To this end, we first integrate Eq. (2) in the axial direction and write

$$p(x,y) = \int \frac{\partial p}{\partial x}\,\mathrm{d}x = -\int u\frac{\partial u}{\partial x}\,\mathrm{d}x - \int v\frac{\partial u}{\partial y}\,\mathrm{d}x + \mathcal{G}(y) \tag{4}$$

where $\mathcal{G}(y)$ is an arbitrary function of y. Forthwith, the substitution of $p(x,y)$ into Eq. (3) yields the differential equation for $\mathcal{G}(y)$:

$$\frac{\partial \mathcal{G}(y)}{\partial y} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\int u\frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial}{\partial y}\int v\frac{\partial u}{\partial y}\,\mathrm{d}x \tag{5}$$

The same procedure may be repeated by integrating Eq. (3) and then substituting the outcome into Eq. (2). A symmetrical result is obtained, namely,

$$\frac{\partial \mathcal{F}(x)}{\partial x} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{\partial}{\partial x}\int u\frac{\partial v}{\partial x}\,\mathrm{d}y + \frac{\partial}{\partial x}\int v\frac{\partial v}{\partial y}\,\mathrm{d}y \tag{6}$$

Recognizing that Eq. (5) can only be a function of y, it is clear that its partial derivative with respect to x must vanish. This implies

$$\frac{\partial^2 \mathcal{G}}{\partial y \partial x} = -\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right) = 0 \tag{7}$$

In like fashion, Eq. (6) may be differentiated with respect to y and set equal to zero. One gets

$$\frac{\partial^2 \mathcal{F}}{\partial x \partial y} = -\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right) = 0 \tag{8}$$

Equations (7–8) are certainly identical and can be re-arranged into

$$\frac{\partial}{\partial x}\left(\boldsymbol{u}\cdot\nabla v\right) - \frac{\partial}{\partial y}\left(\boldsymbol{u}\cdot\nabla u\right) = 0 \tag{9}$$

When the above constraint is fulfilled, it guarantees an analytic expression for the pressure field, namely, one that is continuous, smooth-differentiable, and equal to its Taylor series expansion at any point in its domain. This will be so granted that the velocity vector is observant of the continuity equation. Then recalling the right-hand-side of Eqs. (2–3), we may re-write Eq. (9) as

$$\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x} \tag{10}$$

The familiar expression just arrived at represents, in actuality, a statement of Clairaut's theorem on the equality of mixed partials,[10] also known as Schwarz's theorem.[11–13] Accordingly, *'the mixed second derivatives of a continuous function f on a domain $\mathcal{D}$ are equal if, and only if, its mixed derivatives f_{xy} and f_{yx} are continuous on $\mathcal{D}$.'*[14] Thus, whenever the mixed derivatives p_{xy} and p_{yx} prove to be continuous on a domain of fluid $\mathcal{D}$, their equality is ascertained along with the existence of a continuous parent function p on the same domain of interest. The commutative character of continuous mixed partials and their symmetry is also confirmed through Young's theorem.[15] For a review of the history of mixed derivatives, the reader is referred to Higgins.[11] These properties ensure the existence of an exact total differential for the pressure which can be synthesized from[16]

$$\mathrm{d}p = \frac{\partial p}{\partial x}\,\mathrm{d}x + \frac{\partial p}{\partial y}\,\mathrm{d}y \tag{11}$$

This simple, yet powerful result may now be extended to three-dimensional viscous motions.

III. Integration of the Navier-Stokes Equations

The analysis presented above may be further generalized by applying it to the three-dimensional Navier-Stokes equations with constant properties. These are routinely given by[17]

$$\nabla\cdot\boldsymbol{u} = 0 \tag{12}$$

and

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{B} \tag{13}$$

where ρ, μ and $\boldsymbol{B}$ denote the fluid density, molecular viscosity, and body force per unit volume. In what follows, one theorem and three corollaries connected to Clairaut's fundamental theorem will be formulated and discussed.

Theorem 1. *Given a velocity field* $\boldsymbol{u}$ *of class* C^2 *in a fluid region* $\mathcal{V}$ *that satisfies the incompressible continuity equation,* $\nabla \cdot \boldsymbol{u} = 0$*, the corresponding viscous momentum equation can be integrated for the pressure to within a constant if the following constraints are satisfied:*

$$\frac{\partial^2 p}{\partial x_1 \partial x_2} = \frac{\partial^2 p}{\partial x_2 \partial x_1}; \quad \frac{\partial^2 p}{\partial x_2 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_2}; \quad \frac{\partial^2 p}{\partial x_1 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_1} \tag{14}$$

Proof. For the sake of generality, we write the viscous momentum relation, Eq. (13), in orthogonal curvilinear coordinates using

$$\frac{\partial p}{\partial x_i} = \mathcal{F}_i(x_1, x_2, x_3, t); \quad i = 1, 2, 3 \tag{15}$$

where (x_1, x_2, x_3) denote the generalized coordinates in an orthogonal reference frame in which $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ represent the base unit vectors. Integration of Eq. (15) in the x_1 direction yields

$$p = \int \frac{\partial p}{\partial x_1} \mathrm{d}x_1 = \int \mathcal{F}_1 \, \mathrm{d}x_1 + \mathcal{G}_1(x_2, x_3, t) \tag{16}$$

As before, this expression may be substituted into the x_2 momentum equation to produce

$$\frac{\partial p}{\partial x_2} = \mathcal{F}_2 = \frac{\partial}{\partial x_2} \int \mathcal{F}_1 \, \mathrm{d}x_1 + \frac{\partial \mathcal{G}_1(x_2, x_3, t)}{\partial x_2} \tag{17}$$

At this point, we recall that $\mathcal{G}_1$ depends solely on x_2, x_3, and t; we may therefore impose

$$\frac{\partial^2 \mathcal{G}_1}{\partial x_2 \partial x_1} = 0 \tag{18}$$

Then, through the use of Eq. (17), we retrieve,

$$\frac{\partial^2 \mathcal{G}_1}{\partial x_2 \partial x_1} = \frac{\partial \mathcal{F}_2}{\partial x_1} - \frac{\partial \mathcal{F}_1}{\partial x_2} = 0 \tag{19}$$

With $\mathcal{F}_i$ being the partial derivative of the pressure through Eq. (15), we deduce

$$\frac{\partial^2 p}{\partial x_1 \partial x_2} = \frac{\partial^2 p}{\partial x_2 \partial x_1} \tag{20}$$

In a similar fashion, substitution of Eq. (16) into the x_3 momentum equation renders

$$\frac{\partial^2 p}{\partial x_1 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_1} \tag{21}$$

To retrieve the third constraint, Eq. (15) may be first integrated in the x_2 direction

$$p = \int \frac{\partial p}{\partial x_2} \mathrm{d}x_2 = \int \mathcal{F}_2 \, \mathrm{d}x_2 + \mathcal{G}_2(x_1, x_3, t) \tag{22}$$

and then inserted into the x_3 momentum equation to obtain

$$\frac{\partial p}{\partial x_3} = \mathcal{F}_3 = \frac{\partial}{\partial x_3} \int \mathcal{F}_2 \, \mathrm{d}x_2 + \frac{\partial \mathcal{G}_2(x_1, x_3, t)}{\partial x_3} \tag{23}$$

Since $\mathcal{G}_2 = \mathcal{G}_2(x_1, x_3, t)$, differentiation with respect to x_2 gives

$$\frac{\partial^2 \mathcal{G}_2}{\partial x_3 \partial x_2} = 0 \tag{24}$$

Finally, by taking the partial of Eq. (23) with respect to x_2, accounting for Eq. (24), and substituting the relations between $\mathcal{F}_2, \mathcal{F}_3$ and the pressure gradients that they represent, we get

$$\frac{\partial^2 p}{\partial x_2 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_2} \tag{25}$$

It is hence established that the three constraints given by Eqs. (20), (21), and (25) provide necessary and sufficient conditions for deriving an analytic expression for the pressure field directly from the momentum equation. □

Corollary 1. *The pressure is integrable if its total differential is exact.*

Proof. In general orthogonal coordinates, the total differential for the pressure may be written as

$$\mathrm{d}p = \frac{\partial p}{\partial x_1}\mathrm{d}x_1 + \frac{\partial p}{\partial x_2}\mathrm{d}x_2 + \frac{\partial p}{\partial x_3}\mathrm{d}x_3 + \frac{\partial p}{\partial t}\mathrm{d}t \equiv F\,\mathrm{d}x_1 + G\,\mathrm{d}x_2 + H\,\mathrm{d}x_3 + E\,\mathrm{d}t \tag{26}$$

Note that the time derivative of the pressure is excluded in Eq. (26) by virtue of its irrelevance to the steps that lie ahead.[3] The total differential in Eq. (26) will be exact only if[16]

$$\frac{\partial G}{\partial x_1} = \frac{\partial F}{\partial x_2}; \quad \frac{\partial F}{\partial x_3} = \frac{\partial H}{\partial x_1}; \quad \frac{\partial H}{\partial x_2} = \frac{\partial G}{\partial x_3} \tag{27}$$

When written in terms of the pressure, we readily recover,

$$\frac{\partial p}{\partial x_1 \partial x_2} = \frac{\partial p}{\partial x_2 \partial x_1}; \quad \frac{\partial p}{\partial x_1 \partial x_3} = \frac{\partial p}{\partial x_3 \partial x_1}; \quad \frac{\partial p}{\partial x_2 \partial x_3} = \frac{\partial p}{\partial x_3 \partial x_2} \tag{28}$$

The outcome is fully consistent with Theorem 1. □

Corollary 2. *The pressure is integrable if, and only if, the velocity field satisfies the vorticity transport equation.*

Proof. The conditions given by Eq. (14) are equivalent to the vector identity

$$\nabla \times \nabla p = \mathbf{0} \tag{29}$$

This condition will be true if, and only if, the pressure belongs to a differentiability class of order 2 or higher. Physically, although a velocity field may be conjectured in such a manner to satisfy mass conservation, it may not (always) give rise to a pressure function that is twice differentiable and continuous. To ensure that the velocity field can generate a pressure distribution that fulfills Eq. (29), one may take the curl of the momentum equation, herein given by Eq. (13), to obtain

$$\rho\frac{\partial \boldsymbol{\Omega}}{\partial t} - \rho\nabla \times \boldsymbol{u} \times \boldsymbol{\Omega} - \mu\nabla^2\boldsymbol{\Omega} = -\nabla \times \nabla p = \mathbf{0} \tag{30}$$

The emergence of Eq. (30) is therefore contingent on the pressure being a continuous and twice differentiable scalar, $p(x_1, x_2, x_3, t) : \mathbb{R}^4 \longmapsto \mathbb{R}$. One concludes that the velocity field will satisfy the vorticity transport equation, if and only if, $\nabla \times \nabla p = \mathbf{0}$, a condition that arises when $p \in C^2$.

□

Corollary 3. *In a regular perturbation expansion, every order of the pressure must independently satisfy the integrability constraints applied at that level.*

Proof. Without loss of generality, we consider a regular perturbation series that employs integer powers of the gauge parameter ε. Forthwith, the perturbed pressure variable may be expanded as

$$p = \sum_{n=0}^{\infty} \varepsilon^n p_n \tag{31}$$

where $|\varepsilon| << 1$. Due to the linearity of the constraints in Eq. (14), backward substitution of Eq. (31) leads to an identical set of relations written for each p_n, $n \in \mathbb{Z}^+$, to the extent that every individual order of the pressure p_n must be independently made to satisfy the assortment of conditions set forth in Theorem 1. The proof equally applies to a perturbation series in non-integer powers of the gauge parameters. Such a series may be written as

$$p = \sum_{n=0}^{\infty} \delta_n (\varepsilon)\, p_n \tag{32}$$

where $\delta_n (\varepsilon)$ can be an arbitrary sequence of diminishing terms satisfying $\delta_{n+1} (\varepsilon) = o\, [\delta_n (\varepsilon)]$. □

IV. Implications and Examples

The ideas presented heretofore confirm that the vorticity transport equation plays a significant role in determining the characteristics of a flow field. Much to our chagrin, fulfillment of the mass conservation requirement by the velocity proves to be a minimal condition that does not in itself warrant the integrability of the pressure field. The latter can be secured only when all components of the velocity vector exactly satisfy the vorticity transport equation. We further realize that observance of the vorticity transport equation is tantamount to securing the symmetry of the second derivatives for the pressure and thereby its continuity on the domain of interest. From a kinematic standpoint, the equivalent conditions of validity that must be congruent with the velocity distribution are given in Theorem 1.

To illustrate the process of applying the integrability constraints, several known cases will be considered as testbeds for pressure evaluation. In this effort, it will be shown that the integrability constraints can be substantially simplified to the extent of providing valuable and compact relations that can be quickly verified, when needed, before pursuing the pressure field.

A. Potential Flows

For potential flows, the condition leading to irrotationality can be shown to be a special case of the integrability constraints. In fact, setting $\boldsymbol{\Omega} = \mathbf{0}$ in Eq. (30) leads to the immediate self-satisfaction of the vorticity transport equation. Meanwhile, the conditions of integrability given by Eq. (29) are identically met. It may thus be inferred that the pressure distribution can be straightforwardly determined, within a constant, in the treatment of irrotational fluid motions.

B. Inviscid Rotational Flows

For incompressible inviscid flows, the vorticity transport equation reduces to

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times \boldsymbol{u} \times \boldsymbol{\Omega} = \mathbf{0} \tag{33}$$

Despite their inviscid character, flowfields that fall under this category may be rotational because of their ability to advect vorticity.[1] Such situations may materialize when a source of vorticity appears at the boundaries or when the flow is initialized with a nonzero vorticity component. Among the myriad problems belonging to this classification, we are particularly interested in the Taylor–Culick profile that arises in the area of propulsion. First derived by Taylor[18] in 1956, the sinusoidal profile in question has been shown to adequately describe the flow bounded by porous surfaces.[19,20] In 1966, Culick[21] obtained an equivalent solution to model the bulk gaseous motion inside solid rocket motors (SRMs), particularly those that can be simulated as cylindrical tubes with porous and mass injecting sidewalls. In this context, Culick's model considered a semi-infinite pipe with a porous wall across which a uniform flux could be imposed. The attending profile has been extensively used in the propulsion community to model SRM mean flows in connection with stability and wave propagation studies.[22–25] For planar fluid motions in porous channels, the dimensionless streamfunction and velocity field are given by

$$\psi = x\sin(\tfrac{1}{2}\pi y) \tag{34}$$

$$\boldsymbol{u} = \tfrac{1}{2}\pi x\cos(\tfrac{1}{2}\pi y)\boldsymbol{e}_x - \sin(\tfrac{1}{2}\pi y)\boldsymbol{e}_y \tag{35}$$

where x and y stand for the axial and wall-normal coordinates, respectively. In this case, the integrability constraint takes the compact form

$$u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} = 0 \tag{36}$$

with u and v representing the axial and transverse velocities, respectively. It may be quickly verified that the solution given by Eq. (35) satisfies Eq. (36) identically. At the outset, the dimensionless pressure may be integrated and rearranged into

$$p(x,y) = p_0 - \tfrac{1}{8}\pi^2 x^2 + \tfrac{1}{4}\cos(\pi y) \tag{37}$$

where $p_0 = p(0,0)$ is the central pressure at the channel headwall. Similarly, Culick's solution[21] for the flow in a porous cylinder may be derived from

$$\psi = z\sin(\tfrac{1}{2}\pi r^2) \tag{38}$$

Then using the Stokes streamfunction, its corresponding velocity vector emerges as

$$\boldsymbol{u} = -r^{-1}\sin(\tfrac{1}{2}\pi r^2)\boldsymbol{e}_r + \pi z\cos(\tfrac{1}{2}\pi r^2)\boldsymbol{e}_z \tag{39}$$

where r and z are the radial and axial coordinates, respectively. For this case, the integrability constraint simplifies into

$$u\frac{\partial^2 w}{\partial r^2} + w\frac{\partial^2 w}{\partial r\partial z} - \frac{u}{r}\frac{\partial w}{\partial r} = 0 \tag{40}$$

where u and w stand for the radial and axial velocities, respectively. It may be readily shown that the velocity given by Eq. (39) is fully compliant with Eq. (40). Its subsequent integration renders

$$p(r,z) = p_0 - \tfrac{1}{2}\pi^2 z^2 - \tfrac{1}{2}r^{-2}\sin^2(\tfrac{1}{2}\pi r^2) \tag{41}$$

An interesting case of the Taylor–Culick problem arises when, in addition to the sidewall mass flux, one attempts to account for arbitrary injection at the headwall section of the fluid domain. The extended model can only be solved approximately[26] and, as such, gives rise to a velocity field that satisfies continuity but disagrees with the pressure integrability constraints. The treatment of the resulting motions may be found in previous work by the authors.[27,28] The solutions reported therein do not represent exact formulations. Instead, they offer solutions to an approximate form of the vorticity transport equation. To illustrate this paradigm, we select the profile corresponding to a parabolic injection pattern at the headwall of a planar channel.[28] The flow superimposed at the headwall is given by

$$u(0,y) = 1 - y^2 \tag{42}$$

to the extent that the induced steady-state velocity field becomes

$$\boldsymbol{u} = \left[\tfrac{1}{2}\pi x \cos(\tfrac{1}{2}\pi y) + f(y)\right]\boldsymbol{e}_x - \sin(\tfrac{1}{2}\pi y)\boldsymbol{e}_y \tag{43}$$

where

$$f(y) \equiv \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\left[(n+\tfrac{1}{2})\pi y\right] \tag{44}$$

In this case, Eq. (43) proves to be incongruent with Eq. (36) except at the center of the channel and the sidewall. To further illustrate the mathematical reason for the disagreement, we test the possibility of integrating the momentum equation. Starting with the y relation, we infer

$$p(x,y) = \int \frac{\partial p}{\partial y}\,\mathrm{d}y = -\int v\frac{\partial v}{\partial y}\,\mathrm{d}x + \mathcal{G}(x) = \tfrac{1}{4}\cos(\pi y) + \mathcal{G}(x) \tag{45}$$

Next, by substituting Eq. (45) into the x-momentum equation, we collect

$$\frac{\partial p}{\partial x} = \frac{\partial \mathcal{G}(x)}{\partial x} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \tag{46}$$

According to Eq. (45), $\mathcal{G} = \mathcal{G}(x)$ and so should Eq. (46) behave, as a sole function of x. Its evaluation is simple to carry out. It yields

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \tfrac{1}{4}\pi^2 x\cos^2(\tfrac{1}{2}\pi y) + \tfrac{1}{2}\pi\cos(\tfrac{1}{2}\pi y)f(y) - \sin(\tfrac{1}{2}\pi y)f'(y) \tag{47}$$

Clearly, the right-hand-side (RHS) poses a problem as it contains occurrences of y as well. This test exemplifies how the velocity field defined through Eq. (43) can be divergence free and yet disallow the direct analytical integration of the pressure. Such behavior is due, of course, to the solution being approximate.

Before leaving this subject, it may be instructive to note that the constraints given by Eq. (14) are not restrictive on the domain of applicability. From this perspective, the pressure may be determined in any fluid subdomain on which Eq. (14) applies. Consequently, if one restricts the domain of interest to the midsection plane where $y = 0$, then Eq. (46) will reduce to a function of x only, a simplification that permits the extraction of the pressure viz.

$$p(x,0) = p_0 - \tfrac{1}{8}\pi^2 x^2 + \tfrac{1}{4} \tag{48}$$

where $p_0 = p(0,0)$ is the pressure at the headwall center. The evolution of p along the channel's midsection plane is hence possible, albeit approximate.

V. Conclusions

In this work, we derive a set of constraints that, when satisfied, establish the necessary and sufficient conditions for obtaining an analytic expression for the pressure through direct integration of the momentum equation. Our pressure integrability rules and restrictions are provided in a general curvilinear coordinate system to the extent of making them applicable to both two-dimensional and three-dimensional flow configurations. From a mathematical perspective, our constraints are directly connected to the Clairaut-Schwarz theorem on the symmetry and commutativity of mixed derivatives. For a viscous incompressible fluid, we show that the velocity field must not only satisfy mass conservation, but also an assortment of relations that prove to be equivalent to the vorticity transport equation. We establish the attendant equalities rigorously by conceiving a theorem and three corollaries. These extend to the treatment of small perturbation problems. The consequences of the theorem are subsequently tested on select problems that range from purely potential to two-dimensional and axisymmetric flows with vorticity. Some of the examples are chosen from recent work by the authors as they help to explain the inability of approximate solutions to generate a pressure distribution over the entire fluid domain. The restrictive expressions that we identify are found to be equally important to mass conservation as a requisite for the establishment of a meaningful fluid motion. From a historical perspective, our findings appear to be consistent with those of Euler,[6] especially vis-à-vis his statement qualifying continuity as a necessary but not sufficient condition for the establishment of a flow field. The symmetry of the momentum equation, which balances the forces that sustain fluid motion, must also be secured. It is also our conclusion that the equality of the cross derivatives of the momentum equation is vitally important and quintessential to the extraction of the pressure field.

Acknowledgments

This material is based on work supported by the National Science Foundation through Grant No. CMMI-0928762, Dr. Eduardo A. Misawa, Program Director. The authors thank Dr. Christian G. Parigger for his valuable comments regarding Clairaut's theorem and Dr. Roy J. Schulz for his insightful remarks concerning the analytic nature of the pressure field.

References

[1]Karamcheti, K., *Principles of Ideal-Fluid Aerodynamics*, John Wiley, New York, 1966.

[2]Shivamoggi, B. K., *Theoretical Fluid Dynamics*, John Wiley, New York, 2nd ed., 1998.

[3]Calero, J. S., *The Genesis of Fluid Mechanics, 16401780*, Springer Verlag, The Netherlands, 2008. doi:10.1007/978-1-4020-6414-2.

[4]Clairaut, A.-C., "Recherches Générales sur le Calcul Intégral," *Histoire de l'Académie Royale des Sciences avec les Mémoires de Mathématique et de Physique*, Vol. 1, 1739, pp. 425–436.

[5]Clairaut, A.-C., "Sur l'Intégration ou la Construction des Équations Différentielles du Premier Ordre," *Histoire de l'Académie Royale des Sciences avec les Mémoires de Mathématique et de Physique*, Vol. 1, 1740, pp. 293–323.

[6]Euler, L., "Principia Motus Fluidorum," *Novi Commentarii Academiae Scientiarum Petropolitanae*, 1752.

[7]Euler, L., "Continuation des Rechèrches sur la Théorie du Mouvement des Fluides," *Mémoires de l'Académie des Sciences de Berlin*, Vol. 11, 1755, pp. 316–361.

[8]Euler, L., "Principes Généraux de l'État d'Équilibre des Fluides," *Mémoires de l'Académie des Sciences de Berlin*, Vol. 11, 1755, pp. 217–273.

[9]Euler, L., "Principes Généraux du Mouvement des Fluides," *Mémoires de l'Académie des Sciences de Berlin*, Vol. 11, 1755, pp. 274–315.

[10]Blank, B. E., and Krantz, S. G., *Calculus: Multivariable*, Key College Publishing, Emeryville, CA, 2006.

[11]Higgins, T. J., "A Note on the History of Mixed Partial Derivatives," *Scripta Mathematica*, 1940, pp. 59–62.

[12]Schwarz, H. A., "Communication: Ciquante-Sixième Session de la Société Helvétique des Sciences Naturelles," *Archives des Sciences Physiques et Naturelles*, Vol. 48, 1873, pp. 38–44.

[13]Doussineau, P., and Levelut, A. L., "Differentiability Breaking and Schwarz Theorem Violation in an Aging Material," *Physical Review B*, Vol. 66, No. 2, Jul 2002, pp. 024105–9. doi:10.1103/PhysRevB.66.024105.

[14]Rogawski, J. D., *Calculus: Early Transcendentals*, W. H. Freeman, New York, 2008.

[15]Young, W. H., "On Differentials," *Proceedings of the London Mathematical Society*, Vol. s2-7, No. 1, Jan. 1909, pp. 157–180. doi:10.1112/plms/s2-7.1.157.

[16]Price, B., *A Treatise on Infinitesimal Calculus, Containing Differential and Integral Calculus, Calculus of Variations, Applications to Algebra and Geometry, and Analytical Mechanics*, Vol. 2, Oxford University Press, 1857.

[17]Hughes, W., and Gaylord, E., *Basic Equations of Engineering Science*, McGraw-Hill, New York, 1964.

[18]Taylor, G. I., "Fluid Flow in Regions Bounded by Porous Surfaces," *Proceedings of the Royal Society of London, Series A*, Vol. 234, No. 1199, March 1956, pp. 456–475. doi:10.1098/rspa.1956.0050.

[19]Chaouat, B., and Schiestel, R., "Reynolds Stress Transport Modelling for Steady and Unsteady Channel Flows with Wall Injection," *Journal of Turbulence*, Vol. 3, No. 1, 2002, pp. 021–15. doi:10.1088/1468-5248/3/1/021.

[20]Venugopal, P., Moser, R. D., and Najjar, F. M., "Direct Numerical Simulation of Turbulence in Injection-Driven Plane Channel Flows," *Physics of Fluids*, Vol. 20, No. 10, 2008, pp. 105103–22. doi:10.1063/1.2963137.

[21]Culick, F. E. C., "Rotational Axisymmetric Mean Flow and Damping of Acoustic Waves in a Solid Propellant Rocket," *AIAA Journal*, Vol. 4, No. 8, 1966, pp. 1462–1464. doi:10.2514/3.3709.

[22]Majdalani, J., "Multiple Asymptotic Solutions for Axially Travelling Waves in Porous Channels," *Journal of Fluid Mechanics*, Vol. 636, No. 1, 2009, pp. 59–89. doi:10.1017/S0022112009007939.

[23]Chedevergne, F., Casalis, G., and Féraille, T., "Biglobal Linear Stability Analysis of the Flow Induced by Wall Injection," *Physics of Fluids*, Vol. 18, No. 1, 2006, pp. 014103–14. doi:10.1063/1.216052.

[24]Abu-Irshaid, E. M., Majdalani, J., and Casalis, G., "Hydrodynamic Stability of Rockets with Headwall Injection," *Physics of Fluids*, Vol. 19, No. 2, 2007, pp. 024101–11. doi:10.1063/1.2434797.

[25]Dunlap, R., Blackner, A. M., Waugh, R. C., Brown, R. S., and Willoughby, P. G., "Internal Flow Field Studies in a Simulated Cylindrical Port Rocket Chamber," *Journal of Propulsion and Power*, Vol. 6, No. 6, 1990, pp. 690–704. doi:10.2514/3.23274.

[26]Terrill, R. M., and Colgan, T., "Some Simple Analytic Solutions of the Navier-Stokes Equations," *International Journal of Engineering Science*, Vol. 29, No. 1, 1991, pp. 55–68. doi:10.1016/0020-7225(91)90076-F.

[27]Majdalani, J., and Saad, T., "The Taylor–Culick Profile with Arbitrary Headwall Injection," *Physics of Fluids*, Vol. 19, No. 9, 2007, pp. 093601–10. doi:10.1063/1.2746003.

[28]Saad, T., and Majdalani, J., "Rotational Flowfields in Porous Channels with Arbitrary Headwall Injection," *Journal of Propulsion and Power*, Vol. 25, No. 4, 2009, pp. 921–929. doi:10.2514/1.41926.