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On the Compressible Bidirectional Vortex. Part 1: A Bragg-Hawthorne Stream Function Formulation

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The Bragg-Hawthorne equation, also named Squire-Long, takes advantage of a requirement that the stagnation pressure head, H, and angular momentum, B, may be generally related to the stream function, ψ. This reduces the Navier-Stokes equations to a single stream function representation, wherein H and B may be specified based on the application of interest. In this paper, the Bragg-Hawthorne equation (BHE) is extended in the context of a steady, inviscid, and compressible fluid. Then given an assortment of suitable assumptions, our approach leads to a pair of partial differential equations that must be treated simultaneously. Rather than solve the ensuing coupled equations numerically, we opt for a reduced-order model by implementing the Rayleigh-Janzen expansion method in which the square of the reference Mach number is employed as a perturbation parameter. The resulting linearized equations are retrievable to an arbitrary level of precision, thus leading to the establishment of a reliable, compressible BHE framework. From a practical standpoint, we find that a first-order correction is sufficient to capture the bulk compressible contribution in most physical settings. In the second part of this paper series, the procedure is tested to derive a compressible solution for the linear Beltramian model of the bidirectional vortex; this motion corresponds to a swirl-driven cyclonic flowfield with an axially reversing character that proves to be of particular interest to the development of an innovative, self-cooled, liquid rocket engine concept. It can therefore be seen that, although the application of interest remains focused on the bidirectional vortex motion, the framework that we offer retains sufficient generality to be useful in handling a wide range of axisymmetric problems, especially those that may be conveniently expressed in polar-cylindrical coordinates.

Nomenclature

a	chamber radius
A_i	inlet area
B	tangential angular momentum, rv
b	open radius
c_0	reference speed of sound, $\sqrt{\gamma R_0 T_0}$
H	stagnation enthalpy or total pressure head
L	chamber length
l	aspect ratio, L/a
M_0	reference Mach number, U/c_0
M_W, R_u	molecular weight and universal gas constant
$\dot{m}_i$	inlet mass flow rate
p	pressure
$\bar{Q}_i$	inlet volumetric flow rate, UA_i
R_0	reference gas constant, R_u/M_W
r	radial coordinate
T_0	reference temperature
t	time
$\boldsymbol{u}$	velocity vector, (u, v, w)
U	mean injection velocity
z	axial coordinate
Greek	
β	open radius fraction, b/a
γ	ratio of specific heats

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ν	kinematic viscosity
ρ	density
$\mathbf{\Omega}$	vorticity vector
ψ	stream function

Subscripts

0, i, o	reference, inlet, or outlet property
0, 1, 2, ...	perturbation order

I. Introduction

Despite the significance of compressible flow modeling as a vital area of research in fluid dynamics, analytical approximations for compressible flows have traditionally lagged behind other areas of investigation.[1–5] In retrospect, several factors may be responsible for this gap in the open literature, starting with the suitability of incompressible models in handling a large number of phenomenological problems. Undoubtedly, the complexities entailed in the resolution of compressible flow equations along with the attrition in available expertise may also stand as major contributors to the apparent neglect or abandonment of this vital area of investigation. When studying compressible motions, the resulting equations are often coupled in such a way that even approximate solutions become difficult to obtain.

One avenue to reduce the complexity of the Navier-Stokes equations rests on a stream function formulation. In aerospace applications, an often cited approach refers to the vorticity-stream function method used by Culick[6] in his mean flow modeling of a solid rocket motor. Along similar lines, Vyas and Majdalani[7] have employed a variant of this approach in deriving a complex-lamellar helical solution for the bidirectional vortex engine. These approaches have also been extended by Maicke and Majdalani[5] and Majdalani[8] to account for compressibility effects in planar and axisymmetric flow configurations, respectively.

Given the existence of several variants of the vorticity-stream function technique, the present study seeks to extend the incompressible framework employed by Bragg and Hawthorne[10] in their modeling of annular cascade actuators. In some research circles, this particular simplification of the axisymmetric Navier-Stokes equations is called the Squire-Long equation, after the classic book on the mechanics of swirling fluids[11] and a seminal paper on vortex dynamics.[12] Both of these studies, however, limit their scope to incompressible fluids. Later, in a paper on cyclone separators, Bloor and Ingham[9] extend the Bragg-Hawthorne approach to spherical coordinates (see Fig. 1). Within the context of aerospace applications, the same approach is revisited and refined by Barber and Majdalani[13] in their treatment of the conical bidirectional vortex. In addition to an improved formulation for a conical cyclone, a companion study by Majdalani[14] introduces a new class of exact Beltramian solutions to the analogous configuration associated with cylindrical cyclones. The latter are derived from the equivalent Bragg-Hawthorne equation (BHE) expressed in cylindrical coordinates. This particular framework leads to different types of confined, axially-reversing vortex motions that can be used to model the cyclonic flowfield associated with the Vortex Combustion Cold Wall Chamber attributed to Chiaverini *et al.*[15,16] and Knuth *et al.*[17]

In their setup (see Fig. 2), a fluid (e.g. propellant) is injected tangentially to the wall circumference in the vicinity of the VCCWC nozzle section. The incoming stream then spirals around while filling the outer, annular portion of the vortex chamber. After reaching the headwall, the flow reverses axial direction, swirls back through the inner region, and is finally discharged to the outer atmosphere through a partially open base. The use of such a cyclonic streaming pattern serves to protect the VCCWC walls from the combustion products by confining the high temperature gases to the core, outflow region.

At the heart of the Bragg-Hawthorne formulation, one leverages the idea that the stagnation pressure head, H, along with the angular momentum, B, may be expressed in terms of the stream function, ψ. The precise forms of the pressure head and angular momentum may be hence chosen in such a way to best reproduce the flowfield under consideration. This results in a single equation for ψ that may be solved either analytically or numerically. The freedom to choose the forms of B and H opens up a wide range of possibilities and stands as a principal reason for the continued development of the Bragg-Hawthorne approach. Naturally, this aspect of analysis rises to a higher level of complexity when density variations are allowed. The development of a compressible approximation to the ensuing flowfield will therefore require an appropriate extension of the Bragg-Hawthorne approach in which density variations are permitted. This effort will constitute the main focus of the present investigation.

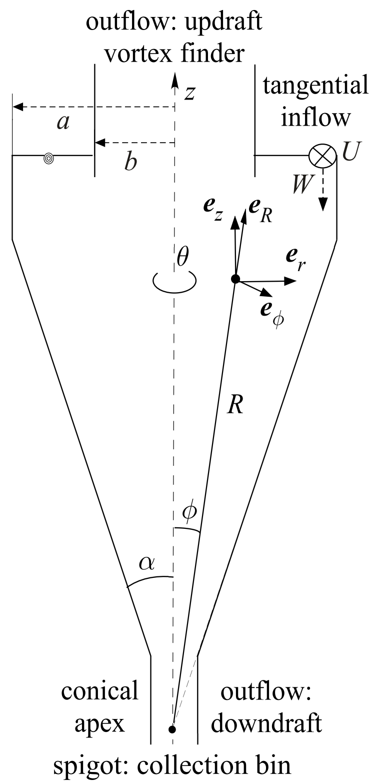

Figure 1. Sketch of Bloor and Ingham's conical cyclone separator depicting its general coordinate system and characteristic parameters. [9]

The remainder of this paper is structured as follows. First, the Bragg-Hawthorne equation is extended to account for variable density flows. In order to solve the ensuing equation, an additional relationship is required for the density. The stagnation enthalpy and ideal gas assumptions are therefore used in unison to develop an expression for the density in terms of the stream function. After some effort, one is left with two coupled partial differential equations (PDEs) that must be solved for the stream function and density. Our set is subsequently linearized and segregated by means of the Rayleigh-Janzen expansion method. This perturbation technique leads to the identification of a series of equations that may be solved sequentially to increasing orders of accuracy. This grants our approach some generality as the ensuing density-stream function formulation may be straightforwardly applied to a variety of problems that arise in the context of axisymmetric flows with constant stagnation enthalpy. In Part 2 of this two-paper series, [18] the analytical framework will be implemented and tested for the linear Beltramian model of the bidirectional vortex flowfield. [14]

II. Compressible Bragg-Hawthorne Formulation

A. Normalization

Before deriving the compressible relations, the governing equations may be conveniently converted to a dimensionless form. Using overbars to designate dimensional quantities, we take

$$z = \frac{\bar{z}}{a};\ r = \frac{\bar{r}}{a};\ \nabla = a\bar{\nabla};\ \beta = \frac{b}{a} \tag{1}$$

$$u = \frac{\bar{u}}{U};\ v = \frac{\bar{v}}{U};\ w = \frac{\bar{w}}{U};\ \boldsymbol{\Omega} = \frac{a\bar{\boldsymbol{\Omega}}}{U};\ \psi = \frac{\bar{\psi}}{\rho_0 U a^2};\ \mathrm{H} = \frac{\bar{\mathrm{H}}}{U^2} \tag{2}$$

$$p = \frac{\bar{p}}{p_0};\ \rho = \frac{\bar{\rho}}{\rho_0};\ Q_i = \frac{\bar{Q}_i}{Ua^2} = \frac{A_i}{a^2};\ Q_o = \frac{\bar{Q}_o}{Ua^2};\ \dot{m}_i = \frac{\bar{\dot{m}}_i}{\rho_0 U a^2};\ \dot{m}_o = \frac{\bar{\dot{m}}_o}{\rho_0 U a^2} \tag{3}$$

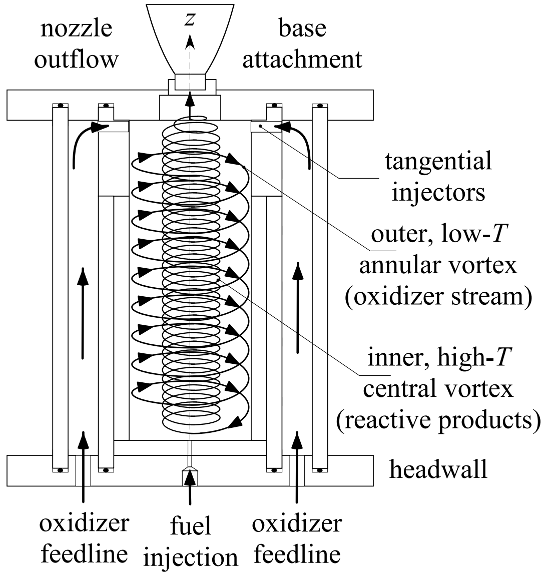

Figure 2. Conceptual sketch of the downward facing Vortex Combustion Cold-Wall Chamber (VCCWC) by Chiaverini *et al.*[15,16]

where (z, r) represent the two primary spatial coordinates and b, the chamber exit radius. In Eq. (2), (u, v, w) denote the radial, tangential, and axial components of the velocity, whereas $\boldsymbol{\Omega}$, ψ, and H stand for the vorticity, stream function, and stagnation enthalpy, respectively. As usual, p, ρ, and Q refer to the pressure, density, and volumetric flow rate. In Eq. (1), all spatial parameters are normalized by the chamber radius, a. Similarly, the wall-tangential injection velocity, U, is used to non-dimensionalize the aforementioned velocities and other related variables as needed. Based on this choice of normalized quantities, the equations of motion become

$$\nabla \cdot (\rho \boldsymbol{u}) = 0 \qquad \text{conservation of mass} \tag{4}$$

$$\nabla \cdot (\rho \mathrm{H} \boldsymbol{u}) = 0 \qquad \text{conservation of energy} \tag{5}$$

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\gamma M_0^2 \rho} \qquad \text{conservation of momentum} \tag{6}$$

$$\mathrm{H} = \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{M_0^2 (\gamma - 1)} \frac{p}{\rho} \qquad \text{stagnation enthalpy} \tag{7}$$

where $M_0 = U/c_0$, with c_0 representing the reference speed of sound in the chamber. For convenience, we define the axisymmetric Stokes operator D^2 as

$$D^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{8}$$

and the compressible stream function viz.

$$u = -\frac{1}{\rho r}\frac{\partial \psi}{\partial z}; \quad w = \frac{1}{\rho r}\frac{\partial \psi}{\partial r} \tag{9}$$

B. Density-Stream Function Formulation

Before deriving the compressible Bragg-Hawthorne equation, it is useful to recast the stagnation enthalpy and tangential angular momentum in terms of ψ. Beginning with the energy equation, one may expand the dot product in Eq. (5) into

$$\nabla (\rho \mathrm{H}) \cdot \boldsymbol{u} + \rho \mathrm{H} (\nabla \cdot \boldsymbol{u}) = 0 \tag{10}$$

Conservation of mass may be expanded in a similar manner to provide

$$\nabla \cdot \boldsymbol{u} = -\frac{\boldsymbol{u} \cdot \nabla \rho}{\rho} \tag{11}$$

Substituting Eq. (11) back into Eq. (10) yields

$$\nabla(\rho \mathrm{H}) \cdot \boldsymbol{u} - \mathrm{H}(\boldsymbol{u} \cdot \nabla\rho) = 0 \tag{12}$$

Splitting the first member of Eq. (12) leads to a direct term-by-term cancelation that leaves us with

$$\nabla \mathrm{H} \cdot \boldsymbol{u} = 0 \tag{13}$$

Equation (13) proves that H can only vary along directions that are orthogonal to the velocity field. Hence, H remains constant along streamlines. This brings us to conclude that

$$\mathrm{H} = \mathrm{H}(\psi) \tag{14}$$

With the stagnation enthalpy in hand, the expanded tangential component of the momentum conservation expression in Eq. (6) may be revisited. Given the underlying assumption of axisymmetry, the θ-momentum equation may be reduced to

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = 0 \tag{15}$$

Equation (15) can be further multiplied by r to the extent of reproducing the material derivative of $\mathrm{B} \equiv rv$, where B represents the tangential angular momentum

$$ru\frac{\partial v}{\partial r} + rw\frac{\partial v}{\partial z} + uv = \frac{\mathrm{D}(rv)}{\mathrm{D}t} = \frac{\mathrm{DB}}{\mathrm{D}t} = 0 \tag{16}$$

A vanishing material derivative confirms that B must remain invariant along streamlines, and so $\mathrm{B} = \mathrm{B}(\psi)$.

With H and B in stream function form, they prove helpful in simplifying the momentum equation. In fact, this step is somewhat facilitated when paired with the isentropic flow relation, namely,

$$p = K\rho^{\gamma} \tag{17}$$

where K denotes a general constant. At this juncture, one may recognize that the momentum relation given by Eq. (6) incorporates a pressure gradient divided by the density, and so an equivalent form may be realized by manipulating Eq. (17) such that

$$\nabla\left(\frac{p}{\rho}\right) = K\nabla\rho^{\gamma-1} \tag{18}$$

To separate ∇p, we insert

$$\rho_t = \rho^{\gamma} \tag{19}$$

thus turning Eq. (18) into

$$\nabla\left(\frac{p}{\rho}\right) = K\nabla\rho_t^{1-\frac{1}{\gamma}} \tag{20}$$

Further chain rule differentiation enables us to isolate ρ_t and put

$$\nabla\left(\frac{p}{\rho}\right) = K\left(1 - \frac{1}{\gamma}\right)\rho_t^{-\frac{1}{\gamma}}\nabla\rho_t \tag{21}$$

Reverting back to the original density, we are left with

$$\nabla\left(\frac{p}{\rho}\right) = K\left(\frac{\gamma-1}{\gamma}\right)\rho^{-1}\nabla\rho^{\gamma} \tag{22}$$

Finally, Eq. (17) may be substituted to produce

$$\frac{\gamma}{\gamma-1}\nabla\left(\frac{p}{\rho}\right) = \frac{\nabla p}{\rho} \tag{23}$$

This expression may be inserted on the right-hand side of Eq. (6). Then using the vector identity, $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \frac{1}{2}\nabla\boldsymbol{u}^{\mathbf{2}} - \boldsymbol{u}\times\nabla\times\boldsymbol{u}$, we collect:

$$\frac{\nabla(\boldsymbol{u}\cdot\boldsymbol{u})}{2} + \frac{1}{M_0^2(\gamma-1)}\nabla\left(\frac{p}{\rho}\right) - \boldsymbol{u}\times\boldsymbol{\Omega} = 0 \tag{24}$$

By inspection of Eq. (7), it may be recognized that the first two terms in the above correspond to the spatial gradient of the total enthalpy. We hence arrive at

$$\nabla \mathrm{H} = \boldsymbol{u}\times\boldsymbol{\Omega} \tag{25}$$

To eliminate the vorticity, the right-hand side of Eq. (25) may be expanded in terms of the velocity viz.

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{\Omega} = &\left\{\frac{v}{r}\left[\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\theta}\right] + w\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right)\right\}\boldsymbol{e}_r \\ &- \left\{\frac{u}{r}\left[\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\theta}\right] - \frac{w}{r}\left[\frac{\partial w}{\partial\theta} - \frac{\partial(rv)}{\partial z}\right]\right\}\boldsymbol{e}_\theta + \left\{-u\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) - \frac{v}{r}\left[\frac{\partial w}{\partial\theta} - \frac{\partial(rv)}{\partial z}\right]\right\}\boldsymbol{e}_z\end{aligned} \tag{26}$$

As with the vorticity-stream function, the axial component of Eq. (25) may be segregated after imposing the axisymmetry condition. We get

$$\frac{\partial\mathrm{H}}{\partial z} = -u\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) + v\frac{\partial v}{\partial z} \tag{27}$$

Next, the velocities may be eliminated in favor of the stream function to produce

$$\frac{\partial\mathrm{H}}{\partial z} = \frac{1}{\rho r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial z}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial z}\right) + \frac{1}{\rho r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{1}{\rho r}\frac{\partial\psi}{\partial r}\right) + v\frac{\partial v}{\partial z} \tag{28}$$

At this stage, partial derivatives may be evaluated and the angular momentum factored out to obtain

$$\frac{\partial\mathrm{H}}{\partial z} = \frac{1}{\rho^2 r^2}\frac{\partial\psi}{\partial z}\left(\frac{\partial^2\psi}{\partial z^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial z}\frac{\partial\rho}{\partial z} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial r}\frac{\partial\rho}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\mathrm{B}}{r^2}\frac{\partial\mathrm{B}}{\partial z} \tag{29}$$

Further application of the chain rule to both H and B leads to

$$\frac{\mathrm{dH}}{\mathrm{d}\psi}\frac{\partial\psi}{\partial z} = \frac{1}{\rho^2 r^2}\frac{\partial\psi}{\partial z}\left(\frac{\partial^2\psi}{\partial z^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial z}\frac{\partial\rho}{\partial z} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial r}\frac{\partial\rho}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\mathrm{B}}{r^2}\frac{\mathrm{dB}}{\mathrm{d}\psi}\frac{\partial\psi}{\partial z} \tag{30}$$

and so

$$r^2\frac{\mathrm{dH}}{\mathrm{d}\psi} - \mathrm{B}\frac{\mathrm{dB}}{\mathrm{d}\psi} = \frac{1}{\rho^2}\left(\frac{\partial^2\psi}{\partial z^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial z}\frac{\partial\rho}{\partial z} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial r}\frac{\partial\rho}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right) \tag{31}$$

At length, taking advantage of vector notations and using the D^2 operator defined in Eq. (8), we arrive at the compact and convenient form

$$D^2\psi + \rho^2\left(\mathrm{B}\frac{\mathrm{dB}}{\mathrm{d}\psi} - r^2\frac{\mathrm{dH}}{\mathrm{d}\psi}\right) = \frac{1}{\rho}\nabla\rho\cdot\nabla\psi \tag{32}$$

It may be instructive to note that only B, H, and their derivatives with respect to ψ appear in Eq. (32). Therefore, given the general dependence of these quantities on the stream function, some freedom exists in the manner by which suitable forms of H and B may be specified. It is this flexibility that sets the Bragg-Hawthorne technique apart, particularly as a versatile and promising framework that can help to unravel multiple solutions for the same geometry and physical model. More detail on this point will be furnished in Part 2 of this series.[18]

C. Compressible Energy Relation

To achieve closure in the compressible Bragg-Hawthorne equation, a density relation is required. Hence, in the spirit of complementing Eq. (32), the expression for total enthalpy may be used. By rewriting Eq. (7) in terms of ψ, we find

$$\mathrm{H} - \frac{\mathrm{B}^2}{2r^2} = \frac{1}{2\rho^2 r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] + \frac{1}{M_0^2(\gamma-1)}\frac{p}{\rho} \tag{33}$$

Here, one may use $p = \rho^\gamma$ to eliminate p and produce another equation that links H, B, and ψ to ρ. We get

$$\mathrm{H} - \frac{\mathrm{B}^2}{2r^2} = \frac{1}{2\rho^2 r^2}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] + \frac{1}{M_0^2(\gamma-1)}\rho^{\gamma-1} \tag{34}$$

Equations (34) and (32) provide the basis for the compressible framework that we plan to establish for the steady-state analysis of axisymmetric gaseous motions in a friction-free environment.

III. Asymptotic Solution Strategy

In seeking analytical approximations to the two coupled density-stream function equations, the Rayleigh-Janzen expansion may be used to linearize the ensuing system of equations. The authors have used a similar technique in modeling the compressible Taylor flow in porous channels driven by wall-normal injection.[5] As done before, the principal variables of interest may be expanded in terms of M_0^2 using:

$$\begin{aligned}
&u = u_0 + M_0^2 u_1 + O(M_0^4) && \psi = \psi_0 + M_0^2\psi_1 + O(M_0^4) && \rho = 1 + M_0^2\rho_1 + M_0^4\rho_2 + O(M_0^6)\\
&v = v_0 + M_0^2 v_1 + O(M_0^4) && \mathrm{B} = B_0 + M_0^2 B_1 + O(M_0^4) && p = 1 + M_0^2 p_1 + M_0^4 p_2 + O(M_0^6)\\
&w = w_0 + M_0^2 w_1 + O(M_0^4) && \mathrm{H} = H_0 + M_0^2 H_1 + O(M_0^4) && T = 1 + M_0^2 T_1 + M_0^4 T_2 + O(M_0^6)
\end{aligned} \tag{35}$$

These expanded variables may be substituted back into the stream function and density expressions to produce a set of relations that may be solved sequentially.

A. Rayleigh-Janzen Expanded Equations

A Rayleigh-Janzen series expansion of the compressible Bragg-Hawthorne equation renders

$$\begin{aligned}
&(1 + M_0^2\rho_1 + M_0^4\rho_2)D^2(\psi_0 + M_0^2\psi_1 + M_0^4\psi_2)\\
&\quad + (1 + M_0^2\rho_1 + M_0^4\rho_2)^3\left[(B_0 + M_0^2 B_1 + M_0^4 B_2)\frac{\mathrm{d}}{\mathrm{d}\psi}(B_0 + M_0^2 B_1 + M_0^4 B_2) - r^2\frac{\mathrm{d}}{\mathrm{d}\psi}(H_0 + M_0^2 H_1 + M_0^4 H_2)\right]\\
&\qquad = \nabla(1 + M_0^2\rho_1 + M_0^4\rho_2)\cdot\nabla(\psi_0 + M_0^2\psi_1 + M_0^4\psi_2)
\end{aligned} \tag{36}$$

Collecting leading and first-order quantities in M_0^2 enables us to identify:

$$O(1): D^2\psi_0 + B_0\frac{\mathrm{d}B_0}{\mathrm{d}\psi} - r^2\frac{\mathrm{d}H_0}{\mathrm{d}\psi} = 0 \tag{37}$$

$$O(M_0^2): D^2\psi_1 + B_1\frac{\mathrm{d}B_1}{\mathrm{d}\psi} - r^2\frac{\mathrm{d}H_1}{\mathrm{d}\psi} = \frac{\partial\rho_1}{\partial z}\frac{\partial\psi_0}{\partial z} + \frac{\partial\rho_1}{\partial r}\frac{\partial\psi_0}{\partial r} - \rho_1\left[D^2\psi_0 + 3\left(B_0\frac{\mathrm{d}B_0}{\mathrm{d}\psi} - r^2\frac{\mathrm{d}H_0}{\mathrm{d}\psi}\right)\right] \tag{38}$$

Consistent with conventional perturbation theory, the leading-order equation reduces to the traditional incompressible Bragg-Hawthorne equation. The first-order correction, however, encapsulates the $O(M_0^2)$ compressible contribution. At first order, its left-hand side mirrors the leading-order operator while the terms on the right-hand side give rise to a non-homogenous PDE.

The same procedure may be straightforwardly applied to the stagnation enthalpy in Eq. (34). We find

$$\begin{aligned}
&(1 + M_0^2\rho_1 + M_0^4\rho_2)^2\left[(H_0 + M_0^2 H_1 + M_0^4 H_2) - \frac{(B_0 + M_0^2 B_1 + M_0^4 B_2)^2}{2r^2}\right] =\\
&\quad \frac{1}{2r^2}\left\{\left[\frac{\partial(\psi_0 + M_0^2\psi_1 + M_0^4\psi_2)}{\partial z}\right]^2 + \left[\frac{\partial(\psi_0 + M_0^2\psi_1 + M_0^4\psi_2)}{\partial r}\right]^2\right\} + \frac{1}{M_0^2(\gamma-1)}(1 + M_0^2\rho_1 + M_0^4\rho_2)^{\gamma+1}
\end{aligned} \tag{39}$$

As usual, by segregating terms of the same order, we recover

$$O(1): H_0 - \frac{B_0^2}{2r^2} = \frac{1}{2r^2}\left[\left(\frac{\partial\psi_0}{\partial r}\right)^2 + \left(\frac{\partial\psi_0}{\partial z}\right)^2\right] + \frac{\gamma+1}{\gamma-1}\rho_1 \tag{40}$$

$$O(M_0^2): 2\rho_1\left(H_0 - \frac{B_0}{2r^2}\right) + H_1 - \frac{B_0 B_1}{2r^2} = \frac{1}{2r^2}\left(\frac{\partial\psi_0}{\partial r}\frac{\partial\psi_1}{\partial r} + \frac{\partial\psi_0}{\partial z}\frac{\partial\psi_1}{\partial z}\right) + \frac{\gamma+1}{\gamma-1}\left(\rho_2 + \gamma\rho_1^2\right) \tag{41}$$

When Eq. (37) is used to solve for ψ_0, substitution into Eq. (40) will directly unravel the density correction, ρ_1. With the density in hand, the right-hand side of Eq. (38) will be fully determined and the resulting non-homogeneous PDE may be solved for the first compressible stream function correction. In principle, this sequence may be repeated until a satisfactory truncation error is reached. Presently, the procedure will enable us to extract closed-form expressions for the leading and first-order corrections. Generally speaking, the complexity of the particular solutions can grow rapidly to the extent that a compressible approximation at the second order or beyond may require considerable effort. In most problems, the first-order compressible correction will be sufficiently accurate to convey the bulk compressibility effects, and this may be largely attributed to the typical values of M_0^2. This provision is especially true for swirl-dominated flows such as those arising in the context of a bidirectional vortex engine in which the reference Mach number remains smaller than unity.

B. Specification of B and H

Modeling the bidirectional vortex, or any other motion for that matter, begins with the selection of suitable forms for B and H in the compressible Bragg-Hawthorne equations. To facilitate analytical closure, several test functions may be considered, specifically

$$\mathrm{B}\frac{\mathrm{dB}}{\mathrm{d}\psi} = \text{constant} \qquad \mathrm{B} = \sqrt{B_0\psi + B_1} \tag{42}$$

$$\mathrm{B}\frac{\mathrm{dB}}{\mathrm{d}\psi} = \psi \qquad \mathrm{B} = \sqrt{B_0\psi^2 + B_1} \tag{43}$$

$$\frac{\mathrm{dH}}{\mathrm{d}\psi} = \text{constant} \qquad \mathrm{H} = H_0\psi + H_1 \tag{44}$$

$$\frac{\mathrm{dH}}{\mathrm{d}\psi} = \psi \qquad \mathrm{H} = H_0\psi^2 + H_1 \tag{45}$$

Although the number of candidate functions may be limitless, the selections above lead to linear relations that increase the likelihood of producing explicit analytical formulations. Higher-order polynomial relations may require a numerical treatment of the density-stream function equations. For example, the (original) incompressible model of the bidirectional vortex by Vyas and Majdalani[7] may be recovered by setting $\mathrm{B} = 1$ and $\mathrm{dH}/\mathrm{d}\psi = -C_n^2\psi$, where C_n is a constant. To make further headway in illustrating this procedure, one may attempt to follow Bloor and Ingham[9] or Majdalani[14] by specifying B and H such that

$$\frac{\mathrm{dH}}{\mathrm{d}\psi} = 0; \quad \mathrm{B} = \sqrt{B_0^2\psi^2 + B_1^2}; \quad \frac{\mathrm{dB}}{\mathrm{d}\psi} = \frac{B_0^2\psi}{\sqrt{B_0^2\psi^2 + B_1^2}} \tag{46}$$

Interestingly, it turns out that, in the compressible case, these declarations prove insufficient in reproducing a congruent first-order system of equations. The source of this disparity may be traced back to the right-hand side of Eq. (38) where third-order multiples of the stream function emerge. As per Eq. (40), the density correction contains ψ_0^2 terms and these are multiplied by another ψ_0 during final book-keeping. To compensate for these additional powers of ψ, a modification of Eq. (46) is warranted. This may be accomplished by taking

$$\frac{\mathrm{dH}}{\mathrm{d}\psi} = 0; \quad \mathrm{B} = \sqrt{B_0^2\psi^2 + B_1^2 + M_0^2\psi^2\left(B_2^2 + \tfrac{1}{2}B_3^2\psi^2\right)}; \quad \frac{\mathrm{dB}}{\mathrm{d}\psi} = \frac{B_0^2\psi + M_0^2\psi\left(B_2^2 + B_3^2\psi^2\right)}{\sqrt{B_0^2\psi^2 + B_1^2 + M_0^2\psi^2\left(B_2^2 + \tfrac{1}{2}B_3^2\psi^2\right)}} \tag{47}$$

It may be instructive to remark that the reference Mach number, M_0, remains invariant under steady-state flow conditions. At the outset, its inclusion in the fundamental definition of B does not violate in any way the stream function constraint. Furthermore, realizing that B and $\mathrm{dB}/\mathrm{d}\psi$ appear only as a product in the Bragg-Hawthorne equation, their combination may be expanded as:

$$\mathrm{B}\frac{\mathrm{dB}}{\mathrm{d}\psi} = B_0^2\psi + M_0^2\psi\left(B_2^2 + B_3^2\psi^2\right) \tag{48}$$

From an asymptotic standpoint, Eq. (48) does not entail a loss of generality. It is obtained by expanding the angular momentum and its derivative to the appropriate truncation order before substituting the outcome into the stream function relation. The next step is to insert the perturbed form of ψ and write:

$$\mathrm{B}\frac{\mathrm{dB}}{\mathrm{d}\psi} = B_0^2\left(\psi_0 + M_0^2\psi_1 + M_0^4\psi^2\right) + M_0^2\left[B_2^2\left(\psi_0 + M_0^2\psi_1 + M_0^4\psi^2\right) + B_3^2\left(\psi_0 + M_0^2\psi_1 + M_0^4\psi^2\right)^3\right] \tag{49}$$

C. Compressible Bragg-Hawthorne Solution Strategy

By gathering $O(M_0^2)$ in B dB/dψ, we retrieve

$$\mathrm{B}\frac{\mathrm{dB}}{\mathrm{d}\psi} = B_0^2\psi_0 + M_0^2\left(B_0^2\psi_1 + B_2^2\psi_0 + B_3^2\psi_0^3\right) + O(M_0^4) \tag{50}$$

Inserting these contributions back into Eq. (37) and Eq. (38) gives rise to a congruent set of linearized Bragg-Hawthorne equations at the first two successive perturbation orders, namely,

$$O(1): D^2\psi_0 + B_0^2\psi_0 = 0 \tag{51}$$

$$O(M_0^2): D^2\psi_1 + B_0\psi_1 = \frac{\partial\rho_1}{\partial z}\frac{\partial\psi_0}{\partial z} + \frac{\partial\rho_1}{\partial r}\frac{\partial\psi_0}{\partial r} - \rho_1\left(D^2\psi_0 + 3B_0^2\psi_0\right) - B_2^2\psi_0 - B_3^2\psi_0^3 \tag{52}$$

Equation (52) can be further simplified by realizing that the left-hand side of Eq. (51) partially appears on its right-hand side. This permits reducing Eq. (52) into

$$D^2\psi_1 + B_0\psi_1 = \frac{\partial\rho_1}{\partial z}\frac{\partial\psi_0}{\partial z} + \frac{\partial\rho_1}{\partial r}\frac{\partial\psi_0}{\partial r} - 2\rho_1 B_0^2\psi_0 - B_2^2\psi_0 - B_3^2\psi_0^3 \tag{53}$$

Similar substitutions may be implemented in the density relation to unravel

$$O(1): \frac{\gamma+1}{\gamma-1}\rho_1 = -\frac{1}{2r^2}\left[\left(\frac{\partial\psi_0}{\partial r}\right)^2 + \left(\frac{\partial\psi_0}{\partial z}\right)^2 + B_0^2\psi_0^2\right] \tag{54}$$

$$O(M_0^2): \frac{\gamma+1}{\gamma-1}\left(\rho_2 + \gamma\rho_1^2\right) = -\frac{B_0^2\psi_0}{r^2}\left(\psi_0\rho_1 + \psi_1\right) - \frac{1}{2r^2}\left[\frac{\partial\psi_0}{\partial r}\frac{\partial\psi_1}{\partial r} + \frac{\partial\psi_0}{\partial z}\frac{\partial\psi_1}{\partial z} + B_2^2\psi_0^2 + B_3^2\psi_0^4\right] \tag{55}$$

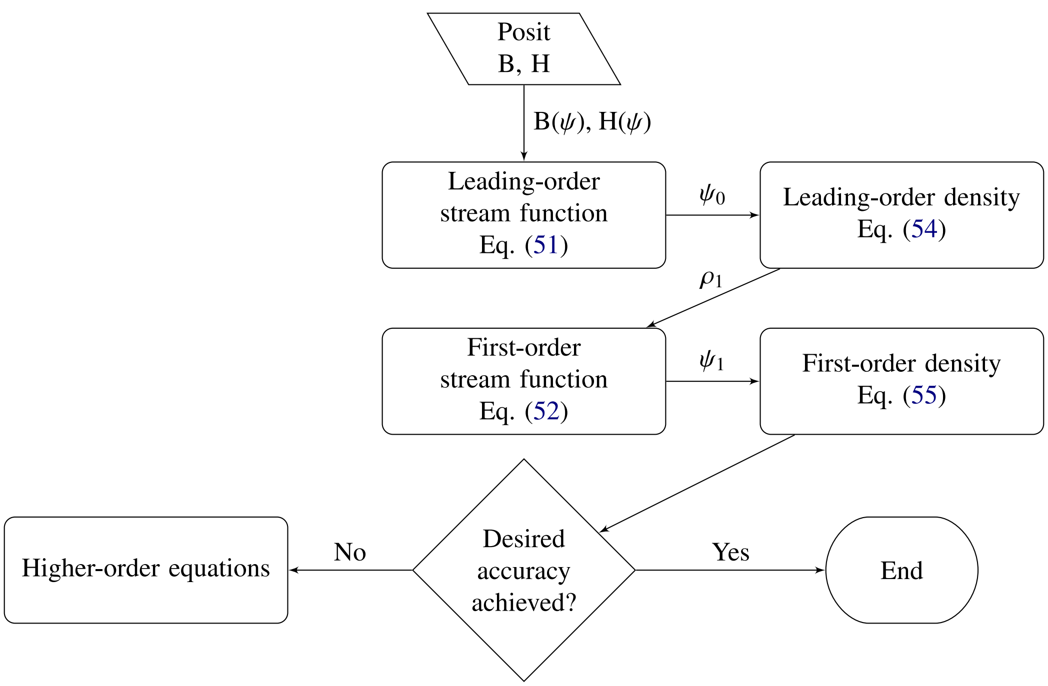

Figure 3. Flowchart for the density-stream function formulation needed to obtain a compressible Bragg-Hawthorne solution.

In seeking a compressible mean flow approximation, our procedure consists of solving Eqs. (51), (54), (52), and (55) in this staggered sequence. A flowchart describing this process is posted as Fig. 3. In Part 2 of this series, the bidirectional vortex flowfield will be used as a test case.[18]

The bidirectional vortex engine may be represented as a right-cylindrical chamber with radius a and height L, as shown in Fig. 4. The coordinate system may be fixed at the center of the inert headwall and the chamber may be taken as partially open at the base where the outflow cross-section has radius b. The dimensional radial and axial coordinates may be denoted by $\bar{r}$ and $\bar{z}$ consistently with the present model. Swirl is generated by injecting a fluid with a wall-tangential speed U in the aft end section. Naturally, the injectant develops an axial velocity while spiraling towards the headwall. When the flow reaches the headwall, it reverses axial polarity while maintaining the same swirling direction. It returns along the central part of the chamber, coursing its way towards and through the partially open base. The inner and outer vortex regions that characterize this cyclonic flowfield are separated by a spinning, non-translating layer termed the mantle. It may hence be seen that the geometric parameters of interest are the aspect ratio $l = L/a$ and the open radial fraction $\beta = b/a$. The bidirectional vortex flowfield differs from that associated with industrial cyclones in view of its single flow outlet (here, a nozzle) in lieu of two distinct openings (e.g., a vortex finder and a spigot) through which the injected fluid may exit the chamber (cf. Fig. 1 versus Fig. 2). The ensuing motion exhibits strong axisymmetry with respect to the chamber axis, thus making it ideally suited for treatment using the BHE approach.

IV. Concluding Remarks

In this paper, the compressible Bragg-Hawthorne equation and its accompanying density relation are transformed into a general, application-free solution method. In addition to the framework constructed here, a handful of candidate functions for the stagnation enthalpy and angular momentum are identified as likely relations that can lead to meaningful flowfield descriptions. The resulting equations are expanded using the Rayleigh-Janzen perturbation technique to unravel a series of equations that can be solved sequentially to the extent of providing approximate compressible solutions. To this end, revised forms of the traditional definitions of H and B are introduced in a manner to achieve balance with the non-homogenous terms that appear in the first-order compressible correction.

The compressible framework presented here is not limited to confined vortices. Our approach may be applied with equal merit to either confined or unconfined motions. With the proper selection of B and H, non-swirling flows that exhibit strong axisymmetries may be explored equally successfully in the context of the Bragg-Hawthorne formulation. Then given this crucial compressible flow extension, many previously untreated problems involving incompressible motions may be revisited and resolved.

In the forthcoming sequel,[18] the BHE framework will be used to obtain a compressible solution for the bidirectional vortex. The perturbed Bragg-Hawthorne equations will be systematically tackled using a conglomeration of analytical and numerical tools to the extent of producing an approximation that suitably captures the effects of compressibility on the ensuing swirling motion. In particular, the velocity behavior and pressure distributions will be examined to assess the sensitivity of profile shape and mantle location on density variations. Naturally, the accurate determination of the mantle location can have a profound effect on the performance and characterization of the bidirectional vortex. Such advancements, as well as others, would not be realizable in the absence of a well-posed compressible BHE procedure.

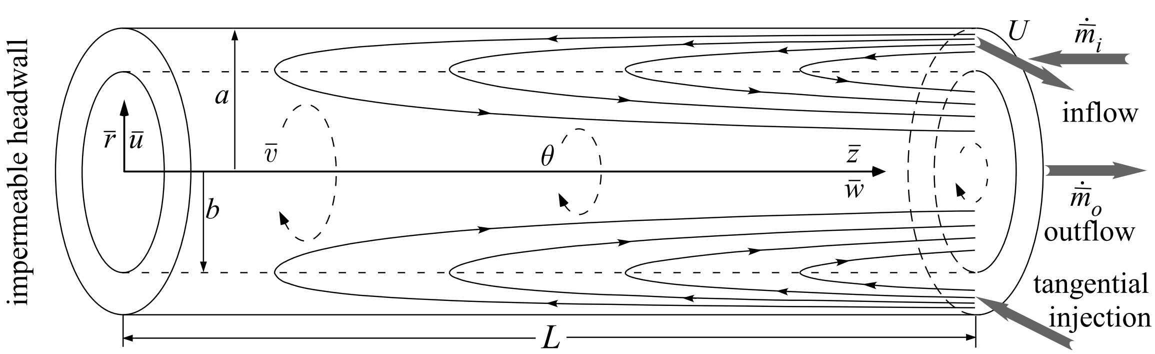

Figure 4. Idealization of the bidirectional vortex chamber depicting its key physical characteristics, coordinate system, and dimensions.

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