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Some thoughts on the pressure integration requirements of the Navier–Stokes equations

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Abstract

The Navier–Stokes formulation represents a uniquely challenging system of partial differential equations that continues to influence modern applied science and engineering. In its simplest form, the system can be used to prescribe the motion of a viscous incompressible fluid with constant properties. It consists of four equations in three-dimensional space that account for both the kinematic and dynamic conditions that a fluid element senses. In this work, we investigate the pressure integration rules and restrictions that affect the resolution of the scalar pressure field. We begin our analysis by exploring the integration properties of Euler's equations in two dimensions while making use of Clairaut's theorem on the commutativity of mixed partial derivatives. We then extend our findings to three-dimensional space. This process gives rise to a theorem and four corollaries that help to clarify the conditions needed to obtain exact or asymptotic solutions for the pressure distribution. Consequently, we identify the fundamental conditions under which the Navier–Stokes equations can be properly integrated to arrive at an analytic expression for the pressure field, namely, one that is continuous and twice differentiable. In closing, several configurations are used to test the theorem and showcase its connection with the pressure formulation. These include potential flows for which the pressure can be obtained unconditionally, and inviscid rotational motions of the Taylor–Culick type with and without headwall injection.

1. Introduction

The mathematical relations describing fluid motions form such fascinating systems of partial differential equations (PDEs) that, for decades, they have intrigued and challenged

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mathematicians and scientists alike. Yet the inherent nonlinearity and tight coupling embedded in these equations make it nearly impossible to obtain closed form solutions except within the edifices of simplified models. To partially alleviate the emerging difficulties in these strongly nonlinear PDEs, a multitude of flow regimes have been identified wherein the flow equations can be systematically reduced to more manageable levels. Instances include classifying the field as irrotational, isentropic, inviscid-rotational, periodic, creeping, viscous with a prescribed pressure gradient or of the boundary layer type. To be worthwhile, each approximation thus conceived has to capture the essential physics of the flow in question despite the set of deficiencies that it inevitably implies. At the outset, numerous solution methodologies and mathematical tools have been developed and these have entailed concepts such as velocity potentials and conformal mapping for irrotational flows, vorticity-streamfunction formulations for inviscid-rotational motions (Saad and Majdalani 2010), similarity transforms and triple-deck theory for boundary layers, hodograph methods, Prandtl–Glauert, and Rayleigh–Janzen expansions for compressible flows (Majdalani 2007, Maicke and Majdalani 2008), Adomian decomposition, homotopy analysis methods (Liao 2003, Xu *et al* 2010), and others (see Batchelor 1967, Shivamoggi 1998).

Retrospectively, our present understanding of fluid mechanics eclipses quite a turbulent yet engaging history. On the upshot, most of the hydrostatic principles were already known by the early eighteenth century owing, in large part, to the cumulative works of notable figures such as: Torricelli, Pascal, Huygens, Mariotte, Newton and Bernoulli. On the downside, theoretically enticing formulations were timidly pursued during that era until somewhere in the middle of the eighteenth century. At that point, four key figures could be cited as having played a central role in advancing the theory of fluid mechanics. They are: Clairaut, for his work on differential forms; d'Alembert, for his propositions on the differential treatment of fluids; Euler, for his synthesis, notation and rigorous generalizations; and Lagrange, for introducing useful techniques, such as small parameter perturbations, to tackle the equations of motion (Calero 2008).

We start with Clairaut (1739, 1740) who, in his attempts at predicting the curvature of the Earth and the flattening of the poles, formulated a theorem on exact differentials that proved to be quintessential to many subsequent studies connected to his work. In short, he derived the constraint needed for a total differential to be exact, a development that is still viewed today as a milestone achievement in differential calculus. About a decade later, circa 1749, d'Alembert presented his findings to the Academy of Sciences of Berlin in a manuscript that would officially appear later in 1755. In the interim, Euler as usual continued to generate a string of thought-provoking and technically engaging inquiries, and may have been truly the first to approach the discipline of fluid mechanics from a purely centralistic and problem independent viewpoint. In this regard, he is most famous for his four manuscripts (Euler 1752, 1755a, 1755b, 1755c) published in the middle of the eighteenth century, which lay down the foundations of fluid dynamics. It may be further argued that a substantial coverage of the fundamental theory taught today is somewhat influenced by Euler's pioneering work. Following no doubt a similar path of mathematical rigor, Lagrange presented in 1788 his groundbreaking treatise on analytical mechanics. This work was not only profound, it provided the most comprehensive coverage of classical mechanics (since Newton) to the extent of precipitating the development of mathematical physics in the nineteenth century. Lagrange is credited today for being the first to have used asymptotic theory in the treatment of fluids-related problems (Calero 2008).

In their most general form, the equations of fluid motion capture the effects of inertial accumulation, convection, diffusion, compressibility, gravity etc. They consist of four equations: the continuity relation that expresses the kinematic conditions on the fluid, and three momentum equations that capture the co-existing dynamical effects while asserting the balance of internal and external forces on the evolution of the flow field. The most remarkable feature of these equations stands, perhaps, in the wealth of physics that they embody. For example, they can be used for modeling both laminar and turbulent flows, although the physical processes that accompany these flow regimes are quite dissimilar.

In this study, we consider the incompressible Navier–Stokes equations and investigate a fundamental aspect of their solution. Our objective may be stated as follows: *Given a velocity field that satisfies the continuity equation, we seek to identify the conditions under which the Navier–Stokes equations can be integrated to the extent of returning an analytic expression for the pressure.* Our discussion will also address the resulting nonlinear constraints that will be shown to constitute necessary and sufficient conditions for obtaining the pressure distribution from the direct integration of the momentum equation.

The work is organized as follows. First, we illustrate the process of integrating the two-dimensional (2D) Euler equations from which we are able to develop the pressure integrability rules. Then guided by Clairaut's theorem, the pressure integrability conditions will be applied to the three-dimensional (3D) Navier–Stokes equations from which three constraints will be derived. A theorem and four corollaries will follow, and these will be extended to the treatment of small perturbation problems. Finally, a list of implications and pertinent examples will be presented and discussed.

2. Integration of Euler's equations

Using standard nomenclature and a Cartesian frame of reference, the normalized form of the 2D, steady, Euler equations may be presented as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial p}{\partial x}=-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}=-\boldsymbol{u}\cdot\nabla u, \tag{2}$$

$$\frac{\partial p}{\partial y}=-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}=-\boldsymbol{u}\cdot\nabla v. \tag{3}$$

Then given a velocity field $\boldsymbol{u}$ that satisfies continuity (1), our objective here is to determine the additional conditions that must be met by $\boldsymbol{u}$ to ensure that the corresponding pressure field is solvable using (2)–(3). To this end, we first integrate (2) in the axial direction and write

$$p(x,y)=\int\frac{\partial p}{\partial x}\,\mathrm{d}x=-\int u\frac{\partial u}{\partial x}\,\mathrm{d}x-\int v\frac{\partial u}{\partial y}\,\mathrm{d}x+\mathcal{G}(y), \tag{4}$$

where $\mathcal{G}(y)$ is an arbitrary function of y. Forthwith, the substitution of $p(x,y)$ into (3) yields the differential equation for $\mathcal{G}(y)$:

$$\frac{\partial\mathcal{G}(y)}{\partial y}=-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}+\frac{\partial}{\partial y}\int u\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial}{\partial y}\int v\frac{\partial u}{\partial y}\,\mathrm{d}x. \tag{5}$$

The same procedure may be repeated by integrating (3) and then substituting the outcome into (2). A symmetrical result is obtained, namely,

$$\frac{\partial\mathcal{F}(x)}{\partial x}=-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}+\frac{\partial}{\partial x}\int u\frac{\partial v}{\partial x}\,\mathrm{d}y+\frac{\partial}{\partial x}\int v\frac{\partial v}{\partial y}\,\mathrm{d}y. \tag{6}$$

Recognizing that (5) can only be a function of y, it is clear that its partial derivative with respect to x must vanish. This implies

$$\frac{\partial^2 \mathcal{G}}{\partial y \partial x} = -\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right) = 0. \tag{7}$$

In like fashion, (6) may be differentiated with respect to y and set equal to zero. One gets

$$\frac{\partial^2 \mathcal{F}}{\partial x \partial y} = -\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right) = 0. \tag{8}$$

Equations (7) and (8) are certainly identical and can be re-arranged into

$$\frac{\partial}{\partial x}\left(\boldsymbol{u}\cdot\nabla v\right) - \frac{\partial}{\partial y}\left(\boldsymbol{u}\cdot\nabla u\right) = 0. \tag{9}$$

When the above constraint is fulfilled, it guarantees an analytic expression for the pressure field, namely, one that is continuous, smooth-differentiable, and equal to its Taylor series expansion at any point in its domain. This will be the case granted that the velocity vector is observant of the continuity equation. Then recalling the right-hand side of (2)–(3), we may re-write (9) as

$$\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}. \tag{10}$$

As affirmed by Blank and Krantz (2006), this familiar expression represents, in actuality, a statement of Clairaut's theorem on the equality of mixed partials, also named after Schwarz (see Schwarz 1873, Higgins 1940, Doussineau and Levelut 2002). According to Rogawski (2008), *'the mixed second derivatives of a continuous function f on a domain $\mathcal{D}$ are equal if, and only if, its mixed derivatives f_{xy} and f_{yx} are continuous on $\mathcal{D}$.'* Thus, whenever the mixed derivatives p_{xy} and p_{yx} prove to be continuous on a domain of fluid $\mathcal{D}$, their equality is ascertained along with the existence of a continuous parent function p on the same domain of interest. The commutative character of continuous mixed partials and their symmetry is also confirmed through Young's theorem (Young 1909). For a review of the history of mixed derivatives, the reader is referred to Higgins (1940). These properties ensure the existence of an exact total differential for the pressure which, according to Price (1857), can be synthesized from

$$\mathrm{d}p = \frac{\partial p}{\partial x}\,\mathrm{d}x + \frac{\partial p}{\partial y}\,\mathrm{d}y. \tag{11}$$

This simple, yet powerful result may be extended to 3D viscous motions.

3. Integration of the Navier–Stokes equations

The analysis presented above may be generalized by applying it to the 3D Navier–Stokes equations with constant properties. These are routinely given by

$$\nabla\cdot\boldsymbol{u} = 0 \tag{12}$$

and

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{b}, \tag{13}$$

where ρ, ν and $\boldsymbol{b}$ denote the fluid density, kinematic viscosity and body force per unit mass (Hughes and Gaylord 1964). For the sake of generality, we write the viscous momentum equations (13) in orthogonal coordinates using

$$\frac{\partial p}{\partial x_i} = \mathcal{F}_i(x_1, x_2, x_3, t); \quad i = 1, 2, 3, \tag{14}$$

where (x_1, x_2, x_3) denote the generalized coordinates in an orthogonal reference frame in which $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ represent the base unit vectors, and where $\mathcal{F}_i$ is given by

$$\mathcal{F}_i = -\frac{\partial u_i}{\partial t} - u_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_i^2} + b_i. \tag{15}$$

In what follows, one theorem and four corollaries connected to Clairaut's fundamental theorem will be formulated and discussed.

Theorem 1. *Given a velocity field* $\boldsymbol{u}$ *of class* C^2 *in a fluid region* $\mathcal{V}$ *that satisfies the incompressible continuity equation,* $\nabla \cdot \boldsymbol{u} = 0$*, the corresponding viscous momentum equation can be integrated for the pressure to within a constant if the following constraints are satisfied:*

$$\frac{\partial \mathcal{F}_i}{\partial x_j} = \frac{\partial \mathcal{F}_j}{\partial x_i}; \quad i, j = 1, 2, 3; \quad i \neq j. \tag{16}$$

Proof. We start by integrating (14) in the x_1 direction for the pressure. This operation yields

$$p = \int \frac{\partial p}{\partial x_1}\,\mathrm{d}x_1 = \int \mathcal{F}_1\,\mathrm{d}x_1 + \mathcal{G}_1(x_2, x_3, t). \tag{17}$$

As before, this expression may be substituted into the x_2 momentum equation to produce

$$\frac{\partial p}{\partial x_2} = \mathcal{F}_2 = \frac{\partial}{\partial x_2} \int \mathcal{F}_1\,\mathrm{d}x_1 + \frac{\partial \mathcal{G}_1(x_2, x_3, t)}{\partial x_2}. \tag{18}$$

At this point, we recall that $\mathcal{G}_1$ depends solely on x_2, x_3 and t; we may therefore impose

$$\frac{\partial^2 \mathcal{G}_1}{\partial x_2 \partial x_1} = 0. \tag{19}$$

Then, through the use of (18), we retrieve,

$$\frac{\partial^2 \mathcal{G}_1}{\partial x_2 \partial x_1} = \frac{\partial \mathcal{F}_2}{\partial x_1} - \frac{\partial \mathcal{F}_1}{\partial x_2} = 0 \tag{20}$$

or

$$\frac{\partial \mathcal{F}_1}{\partial x_2} = \frac{\partial \mathcal{F}_2}{\partial x_1}. \tag{21}$$

In similar fashion, substitution of (17) into the x_3 momentum equation renders

$$\frac{\partial \mathcal{F}_1}{\partial x_3} = \frac{\partial \mathcal{F}_3}{\partial x_1}. \tag{22}$$

To retrieve the third constraint, (14) may be first integrated in the x_2-direction

$$p = \int \frac{\partial p}{\partial x_2}\,\mathrm{d}x_2 = \int \mathcal{F}_2\,\mathrm{d}x_2 + \mathcal{G}_2(x_1, x_3, t) \tag{23}$$

and then inserted into the x_3 momentum equation to obtain

$$\frac{\partial p}{\partial x_3} = \mathcal{F}_3 = \frac{\partial}{\partial x_3}\int \mathcal{F}_2\,\mathrm{d}x_2 + \frac{\partial \mathcal{G}_2(x_1, x_3, t)}{\partial x_3}. \tag{24}$$

Since $\mathcal{G}_2 = \mathcal{G}_2(x_1, x_3, t)$, differentiation with respect to x_2 gives

$$\frac{\partial^2 \mathcal{G}_2}{\partial x_3 \partial x_2} = 0. \tag{25}$$

Finally, by taking the partial of (24) with respect to x_2, accounting for (25), and substituting the relations between $\mathcal{F}_2$, $\mathcal{F}_3$ and the pressure gradients that they represent, we get

$$\frac{\partial \mathcal{F}_2}{\partial x_3} = \frac{\partial \mathcal{F}_3}{\partial x_2}. \tag{26}$$

It is hence established that the three constraints given by (21), (22) and (26) provide necessary and sufficient conditions for deriving an analytic expression for the pressure field directly from the momentum equations. □

Corollary 1. *The pressure is integrable if its mixed partial derivatives are equal.*

Proof. This follows directly from theorem 1 where $\mathcal{F}_i = \frac{\partial p}{\partial x_i}$ may be substituted such that

$$\frac{\partial^2 p}{\partial x_1 \partial x_2} = \frac{\partial^2 p}{\partial x_2 \partial x_1}; \quad \frac{\partial^2 p}{\partial x_2 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_2}; \quad \frac{\partial^2 p}{\partial x_1 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_1}. \tag{27}$$

The resulting form is identical to the one presented in theorem 1 and will be used interchangeably in subsequent sections. □

Corollary 2. *The pressure is integrable if its total differential is exact.*

Proof. In general orthogonal coordinates, the total differential for the pressure may be written as

$$\mathrm{d}p = \frac{\partial p}{\partial x_1}\mathrm{d}x_1 + \frac{\partial p}{\partial x_2}\mathrm{d}x_2 + \frac{\partial p}{\partial x_3}\mathrm{d}x_3 + \frac{\partial p}{\partial t}\mathrm{d}t \equiv F\,\mathrm{d}x_1 + G\,\mathrm{d}x_2 + H\,\mathrm{d}x_3 + E\,\mathrm{d}t. \tag{28}$$

Note that the time derivative of the pressure is excluded in (28) by virtue of its irrelevance to the steps that lie ahead. Then according to Price (1857), the total differential in (28) will be exact only if

$$\frac{\partial G}{\partial x_1} = \frac{\partial F}{\partial x_2}; \quad \frac{\partial F}{\partial x_3} = \frac{\partial H}{\partial x_1}; \quad \frac{\partial H}{\partial x_2} = \frac{\partial G}{\partial x_3}. \tag{29}$$

When written in terms of the pressure, we readily recover,

$$\frac{\partial p}{\partial x_1 \partial x_2} = \frac{\partial p}{\partial x_2 \partial x_1}; \quad \frac{\partial p}{\partial x_1 \partial x_3} = \frac{\partial p}{\partial x_3 \partial x_1}; \quad \frac{\partial p}{\partial x_2 \partial x_3} = \frac{\partial p}{\partial x_3 \partial x_2}. \tag{30}$$

This outcome is fully consistent with theorem 1 and corollary 1. □

Corollary 3. *The pressure is integrable if, and only if, the velocity field satisfies the vorticity transport equation.*

Proof. The conditions given by (27) are equivalent to the vector identity

$$\nabla \times \nabla p = \mathbf{0}. \tag{31}$$

This condition will be true if, and only if, the pressure belongs to a differentiability class of order 2 or higher. Physically, although a velocity field may be conjectured in such a manner to satisfy mass conservation, it may not (always) give rise to a pressure function that is twice differentiable and continuous. To ensure that the velocity field can generate a pressure distribution that fulfills (31), one may take the curl of the momentum equation, herein given by (13), to obtain

$$\rho \frac{\partial \mathbf{\Omega}}{\partial t} - \rho \nabla \times \boldsymbol{u} \times \mathbf{\Omega} - \mu \nabla^2 \mathbf{\Omega} = -\nabla \times \nabla p = \mathbf{0}. \tag{32}$$

The emergence of (32) is therefore contingent on the pressure being a continuous and twice differentiable scalar, $p(x_1, x_2, x_3, t) : \mathbb{R}^4 \longmapsto \mathbb{R}$. One concludes that the velocity field will satisfy the vorticity transport equation, if and only if, $\nabla \times \nabla p = \mathbf{0}$, a condition that arises when $p \in C^2$. □

Corollary 4. *In a regular perturbation expansion, every order of the pressure must independently satisfy the integrability constraints applied at that level.*

Proof. Without loss of generality, we consider a regular perturbation series that employs integer powers of the gauge parameter ε. Forthwith, the perturbed pressure variable may be expanded as

$$p = \sum_{n=0}^{\infty} \varepsilon^n p_n, \tag{33}$$

where $|\varepsilon| \ll 1$. Due to the linearity of the constraints in (27), backward substitution of (33) leads to an identical set of relations written for each p_n, $n \in \mathbb{Z}^+$, to the extent that every individual order of the pressure p_n must be independently made to satisfy the assortment of conditions set forth in theorem 1. The proof equally applies to a perturbation series in non-integer powers of the gauge parameters. Such a series may be written as

$$p = \sum_{n=0}^{\infty} \delta_n(\varepsilon)\, p_n, \tag{34}$$

where $\delta_n(\varepsilon)$ can be an arbitrary sequence of diminishing terms satisfying $\delta_{n+1}(\varepsilon) = o[\delta_n(\varepsilon)]$. □

4. Implications and examples

The ideas presented heretofore confirm that the vorticity transport equation plays a significant role in determining the characteristics of a flow field. Much to our chagrin, fulfillment of the mass conservation requirement by the velocity proves to be a minimal condition that does not in itself warrant the integrability of the pressure field. The latter can be secured only when all components of the velocity vector exactly satisfy the vorticity transport equation. We further realize that observance of the vorticity transport equation is tantamount to securing the symmetry of the second derivatives for the pressure and thereby its continuity on the domain of interest. From a kinematic standpoint, the equivalent conditions of validity that must be congruent with the velocity distribution are encapsulated in theorem 1.

To illustrate the process of applying the integrability constraints, several known cases will be considered as testbeds for pressure evaluation. In this effort, it will be shown that the integrability constraints can be substantially simplified to the extent of providing valuable and compact relations that can be quickly verified, when needed, before seeking to determine the pressure field.

4.1. Potential flows

For potential flows, the condition leading to irrotationality can be shown to be a special case of the integrability constraints. In fact, setting $\mathbf{\Omega} = \mathbf{0}$ in (32) leads to the immediate self-satisfaction of the vorticity transport equation. Meanwhile, the conditions of integrability given by (31) are identically met. It may be hence inferred that the pressure distribution can be straightforwardly determined, within a constant, in the treatment of irrotational fluid motions.

4.2. Inviscid rotational flows

For incompressible inviscid flows, the vorticity transport equation reduces to

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times \boldsymbol{u} \times \mathbf{\Omega} = \mathbf{0}. \tag{35}$$

Despite their inviscid character, flow fields that fall under this category may be rotational because of their ability to advect vorticity (Batchelor 1967). Such situations may materialize when a source of vorticity appears at the boundaries or when the flow is initialized with a nonzero vorticity component. Among the myriad problems belonging to this classification, we are particularly interested in the Taylor–Culick profile that arises in the area of propulsion. First derived by Taylor (1956), the sinusoidal profile in question has been shown by several investigators such as Chaouat and Schiestel (2002) and Venugopal *et al* (2008) to adequately describe the flow bounded by porous surfaces. Culick (1966) obtained an equivalent solution to model the bulk gaseous motion inside solid rocket motors (SRMs), particularly those that can be simulated as cylindrical tubes with porous and mass injecting sidewalls. In this context, Culick's model considers a semi-infinite pipe with a porous wall across which a uniform flux can be imposed. The attending profile has been extensively used in the propulsion community to model SRM mean flows in connection with stability and wave propagation studies (e.g. Dunlap *et al* 1990, Chedevergne *et al* 2006, Abu-Irshaid *et al* 2007, Majdalani 2009). For planar fluid motions in porous channels, the dimensionless streamfunction and velocity field are given by

$$\psi = x \sin\left(\tfrac{1}{2}\pi y\right), \tag{36}$$

$$\boldsymbol{u} = \tfrac{1}{2}\pi x \cos\left(\tfrac{1}{2}\pi y\right) \boldsymbol{e}_x - \sin\left(\tfrac{1}{2}\pi y\right) \boldsymbol{e}_y, \tag{37}$$

where x and y stand for the axial and wall-normal coordinates, respectively. In this case, the integrability constraint takes the compact form

$$u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} = 0, \tag{38}$$

with u and v representing the axial and transverse velocities, respectively. It may be quickly verified that the solution given by (37) satisfies (38) identically. At the outset, the dimensionless pressure may be integrated and rearranged into

$$p(x, y) = p_0 - \tfrac{1}{8}\pi^2 x^2 + \tfrac{1}{4}\cos(\pi y), \tag{39}$$

where $p_0 = p(0, 0)$ is the central pressure at the channel headwall. Similarly, the solution by Culick (1966) for the flow in a porous cylinder may be derived from

$$\psi = z \sin\left(\tfrac{1}{2}\pi r^2\right). \tag{40}$$

Then using the Stokes streamfunction, its corresponding velocity vector emerges as

$$\boldsymbol{u} = -r^{-1} \sin\left(\tfrac{1}{2}\pi r^2\right) \boldsymbol{e}_r + \pi z \cos\left(\tfrac{1}{2}\pi r^2\right) \boldsymbol{e}_z, \tag{41}$$

where r and z are the radial and axial coordinates, respectively. For this case, the integrability constraint simplifies into

$$u\frac{\partial^2 w}{\partial r^2} + w\frac{\partial^2 w}{\partial r \partial z} - \frac{u}{r}\frac{\partial w}{\partial r} = 0, \tag{42}$$

where u and w stand for the radial and axial velocities, respectively. It may be readily shown that the velocity given by (41) is fully compliant with (42). Its subsequent integration renders

$$p(r, z) = p_0 - \tfrac{1}{2}\pi^2 z^2 - \tfrac{1}{2}r^{-2} \sin^2\left(\tfrac{1}{2}\pi r^2\right). \tag{43}$$

An interesting case of the Taylor–Culick problem arises when, in addition to the sidewall mass flux, one attempts to account for arbitrary injection at the headwall section of the fluid domain. As confirmed by Terrill and Colgan (1991), the extended model can only be solved approximately and, as such, gives rise to a velocity field that satisfies continuity but disagrees with the pressure integrability constraints. The treatment of the resulting motions may be found in previous work by Majdalani and Saad (2007) and Saad and Majdalani (2009) for the axisymmetric and planar problems, respectively. The solutions reported therein do not represent exact formulations. Instead, they offer solutions to an approximate form of the vorticity transport equation in view of its pending linearity with successive increases in the distance from the headwall (Kurdyumov 2008). To illustrate this paradigm, we select the profile corresponding to a parabolic injection pattern at the headwall of a planar channel as described by Saad and Majdalani (2009). The flow superimposed at the headwall corresponds to

$$u(0, y) = 1 - y^2, \tag{44}$$

and the induced steady-state velocity field is given by

$$\boldsymbol{u} = \left[\tfrac{1}{2}\pi x \cos(\tfrac{1}{2}\pi y) + f(y)\right] \boldsymbol{e}_x - \sin(\tfrac{1}{2}\pi y)\boldsymbol{e}_y, \tag{45}$$

where

$$f(y) \equiv \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos\left[\left(n + \tfrac{1}{2}\right)\pi y\right]. \tag{46}$$

In this case, (45) proves to be incongruent with (38) except at the center of the channel and the sidewall. To further illustrate the mathematical reason for this disagreement, we test the possibility of integrating the momentum equation. Starting with the y relation, we have

$$p(x, y) = \int \frac{\partial p}{\partial y}\,\mathrm{d}y = -\int v\frac{\partial v}{\partial y}\,\mathrm{d}x + \mathcal{G}(x) = \tfrac{1}{4}\cos(\pi y) + \mathcal{G}(x). \tag{47}$$

Next, by substituting (47) into the x-momentum equation, we collect

$$\frac{\partial p}{\partial x} = \frac{\partial \mathcal{G}(x)}{\partial x} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}. \tag{48}$$

According to (47), $\mathcal{G} = \mathcal{G}(x)$ and so should (48) behave, as a sole function of x. Its evaluation is simple to carry out. It yields

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \tfrac{1}{4}\pi^2 x \cos^2(\tfrac{1}{2}\pi y) + \tfrac{1}{2}\pi \cos(\tfrac{1}{2}\pi y) f(y) - \sin(\tfrac{1}{2}\pi y) f'(y). \quad (49)$$

Clearly, the right-hand side poses a problem as it contains occurrences of y as well. This test exemplifies how the velocity field defined through (45) can be divergence free and yet disallow the direct analytical integration of the pressure. Such behavior is due, of course, to the solution being approximate.

Before leaving this subject, it may be instructive to note that the constraints given by (27) are not restrictive on the domain of applicability. From this perspective, the pressure may be determined in any fluid subdomain on which (27) applies. Consequently, if one restricts the domain of interest to the midsection plane where $y = 0$, or any $y =$ constant line, then (48) will reduce to a function of x only, a simplification that permits the extraction of the pressure viz.

$$p(x, 0) = p_0 - \tfrac{1}{8}\pi^2 x^2 + \tfrac{1}{4}, \quad (50)$$

where $p_0 = p(0, 0)$ is the pressure at the headwall center. The evolution of p along the channel's midsection plane is hence possible, albeit approximate.

5. Conclusions

In this work, we derive a set of constraints that, when satisfied, establish the necessary and sufficient conditions for obtaining an analytic expression for the pressure through direct integration of the momentum equations. Our pressure integrability rules and restrictions are provided in cartesian tensor form that can be easily extended to curvilinear coordinates. From a mathematical perspective, our constraints are directly connected to the Clairaut–Schwarz theorem on the symmetry and commutativity of mixed derivatives. For a viscous incompressible fluid, we show that the velocity field must not only satisfy mass conservation, but also an assortment of relations that prove to be equivalent to the vorticity transport equation. We establish the attendant equalities rigorously by conceiving a theorem and four corollaries. These extend to the treatment of small perturbation problems. The consequences of the theorem are subsequently tested on select problems that range from purely potential to 2D and axisymmetric flows with vorticity. Some of the examples are chosen from recent work by the authors as they help to explain the inability of approximate solutions to generate a pressure distribution over the entire fluid domain. The restrictive expressions that we identify are found to be equally important to mass conservation as a requisite for the establishment of a meaningful fluid motion. From a historical perspective, our findings appear to be consistent with those of Euler (1752), especially vis-à-vis his statement qualifying continuity as a necessary but not sufficient condition for the establishment of a flow field. The symmetry of the momentum equation, which balances the forces that sustain fluid motion, must also be secured. We also deduce that the equality of the cross derivatives of the momentum equation is vitally important to the extraction of the pressure field.

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