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Short communication

Direct calculation of the average local Mach number in converging–diverging nozzles

Joseph Majdalani *,[1], Brian A. Maicke [2]

University of Tennessee Space Institute, Tullahoma, TN 37388, United States

A R T I C L E I N F O

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A B S T R A C T

Foremost amongst rocket nozzle relations is the area–Mach number expression linking the local velocity normalized by the speed of sound to the area ratio A_t/A, and the ratio of specific heats. Known as Stodola's equation, the attendant expression is transcendental and requires iteration or numerical root finding in extracting the solution under subsonic or supersonic nozzle operation. In this work, a novel analytical inversion of the problem is pursued to the extent of providing the local Mach number directly at any given cross-section. The inversion process is carried out using two unique approaches. In the first, Bürmann's theorem is employed to undertake a functional reversion from which the subsonic solution may be retrieved. In the second, the Successive Approximation Approach is repeatedly applied to arrive at a closed-form representation of the supersonic root. Both methods give rise to unique recursive approximations that permit the selective extraction of the desired solution to an arbitrary level of accuracy. Results are verified numerically and the precision associated with the supersonic solution is shown to improve with successive increases in the ratio of specific heats.

1. Introduction

For isentropic flow through a de Laval nozzle with throat area A_t (see Fig. 1), a transcendental equation named after Stodola [11] relates the area ratio A_t/A and the Mach number M at any cross-section of surface area A:

$$\left(\frac{A}{A_t}\right)^2 = \frac{1}{M^2}\left\{\frac{2}{\gamma+1}\left[1+\frac{1}{2}(\gamma-1)M^2\right]\right\}^{\frac{\gamma+1}{\gamma-1}} \tag{1}$$

where γ denotes the ratio of specific heats. Due to the nature of this expression, a perturbation parameter may be defined as $\varepsilon \equiv (A_t/A)^2$, where $\varepsilon \ll 1$ except for flow in the direct vicinity of the throat section. In practice, ε will reach its lowest value in the nozzle exit plane where $\varepsilon_e = (A_t/A_e)^2$ represents the square of the so-called *area expansion ratio*. Naturally, the calculation of the corresponding local or exit Mach number M_e is of interest to the propulsion and power generation subdisciplines with particular areas of concentration in nozzle design and optimization. This can be attributed to the connection between exit Mach numbers and ideal thrust coefficients, rocket specific impulses, characteristic exhaust velocities, gas turbine efficiencies, and so on, as well as the relevance of Stodola's relation to the problem of sizing and shape selection in a variety of combustion devices in which gases are expanded, such as rockets, ramjets, scramjets, and afterburners [1,2,9].

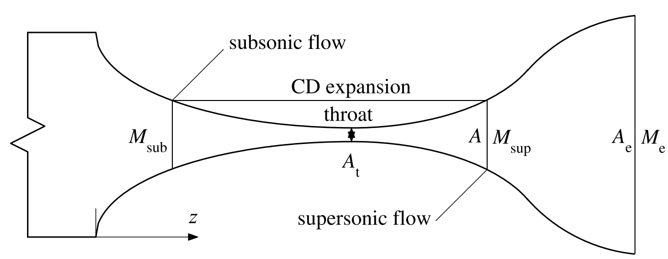

Fig. 1. Converging–diverging nozzle schematic showing relevant properties and Mach numbers.

Despite its simplicity and one-dimensional form, Eq. (1) continues to find uses in modern propulsion-related studies. In this context, one may cite Najjar et al. [10] who employ this relation to set up the initial conditions in their compressible flow solver Rocflo [6]. Along similar lines, Haselbacher et al. [7] use this expression to establish a test case for their slow-time acceleration problem in which time scales are estimated. Thakre and Yang [12] and Zhang et al. [14] also make use of the relation in question for the purpose of verifying their codes on nozzle erosion.

In order to determine the local and/or exit Mach numbers at a given cross-section, one can solve for M straightforwardly using either numerical root finding or compressible flow tables. On this

* Corresponding author. Tel.: +1 (931) 393 7280; fax: +1 (931) 393 7444.
E-mail address: maji@utsi.edu (J. Majdalani).
[1] H.H. Arnold Chair of Excellence in Advanced Propulsion, Department of Mechanical, Aerospace and Biomedical Engineering; Associate Fellow AIAA; Fellow ASME.
[2] Graduate Research Assistant, Department of Mechanical, Aerospace and Biomedical Engineering.

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Nomenclature

A	local cross-sectional area
A_t	nozzle throat area
M	local Mach number
α	constant exponent in Eq. (4), $\frac{1}{2}(\gamma+1)/(\gamma-1)$
ε	perturbation parameter, $(A_t/A)^2$
γ	ratio of specific heats, c_p/c_v
κ	constant exponent, $(\gamma-1)/(\gamma+1)$
ζ	constant related to γ in Eq. (4), $[\frac{1}{2}(\gamma+1)]^\alpha$

Subscripts and symbols

$0, 1, 2$	leading, first and second order
c	chamber stagnation condition
e	exit plane condition
n	asymptotic level
$\mathcal{O}(\varphi)$	Landau's big-O symbol, of same order as φ
$o(\varphi)$	Landau's little-o symbol, of smaller order than φ
sub, sup	subsonic or supersonic
t	nozzle throat condition

subject, Gaggioli [5] illustrates an elegant framework for compressible flow modeling using the Engineering Equation Solver (EES) program. These tools can also be used to handle highly nonlinear problems such as those arising in the calculation of the friction factors in pipe flow [4]. However, to solve the problem analytically, it is helpful to rearrange Eq. (1) into

$$\varepsilon^{\frac{1}{2}}\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}} - M\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 0 \qquad (2)$$

In this study an explicit solution for the Mach number will be presented in terms of the area ratio A/A_t and the ratio of specific heats γ.

2. Analysis

For a fixed nozzle area ratio, two mathematically and physically possible Mach number roots exist: one subsonic and the other supersonic. In what follows, the techniques leading to each root will be separately described. Subsonic and supersonic solutions at order $n \in \mathbb{N}$ will be denoted by $M_{\text{sub}}^{(n)}$ and $M_{\text{sup}}^{(n)}$, respectively.

2.1. Subsonic treatment based on Bürmann's theorem

In previous work by the authors [8], the subsonic solution to Stodola's equation was pursued using regular perturbation theory. At present, we provide a simple alternative based on classical analysis, namely, by means of Bürmann's theorem [13]. This form will lead to the establishment of a general recursive formulation from which the solution may be generated to any order. Bürmann's theorem forms an extension of a Taylor series reversion for arbitrary functions. In layman's words, if a function can be expanded about a particular point in terms of a second function, the converse is true, and one may express the second function in terms of the first. This theorem enables us to write an expression for the Mach number in terms of the local area ratio to arbitrary precision.

To start the analysis we revisit Stodola's equation and simplify it such that

$$\varepsilon^{\frac{1}{2}} = \phi(M) = M\zeta\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{-\alpha} \qquad (3)$$

where

$$\alpha \equiv \frac{\gamma+1}{2(\gamma-1)} \quad \text{and} \quad \zeta \equiv \left(\frac{\gamma+1}{2}\right)^\alpha \qquad (4)$$

In the above, $\phi(M)$ constitutes an analytic function over the closed region associated with the subsonic branch of solution, $0 \leqslant M \leqslant 1$. To ensure that an expansion is valid in this region, we examine the behavior of the function and its first derivative as $\varepsilon \to 0$. We identify, in particular, the point $M_0 \equiv 0$ where

$$\phi(0) = 0 \quad \text{and} \quad \phi'(0) \neq 0 \qquad (5)$$

Since $\phi(M_0)$ is finite and the derivative is non-zero, Bürmann's theorem can indeed be applied in this situation [13]. We subsequently define $f(M) = M$ and use M_0 as the anchor point to construct

$$\psi(M) = \frac{f(M)-f(M_0)}{\phi(M)-\phi(M_0)} = \frac{M}{\phi(M)} \qquad (6)$$

Given $\psi(M)$, a straightforward application of the theorem leads to

$$f(M) = f(M_0) + \sum_{m=0}^{n}\frac{\phi^{m+1}(M)}{(m+1)!}\left[\frac{\mathrm{d}F}{\mathrm{d}M}\frac{\mathrm{d}^m\psi^{m+1}(M)}{\mathrm{d}M^m}\right]_{M=M_0} + R_{n+1} \qquad (7)$$

where R_{n+1} represents the truncation error remaining at that order. Substituting Eq. (6) into Eq. (7) yields

$$f(M) = \sum_{m=0}^{n}\frac{\varepsilon^{\frac{m+1}{2}}}{(m+1)!}\left[\frac{\mathrm{d}^m\psi^{m+1}(M)}{\mathrm{d}M^m}\right]_{M=0} + \mathcal{O}\left(\varepsilon^{\frac{n}{2}+1}\right)$$
$$= \sum_{m=0}^{n-1}\frac{\varepsilon^{\frac{m+1}{2}}}{(m+1)!}\left\{\frac{\mathrm{d}^m}{\mathrm{d}M^m}\left[\frac{M^{m+1}}{\phi^{m+1}(M)}\right]\right\}_{M=0} + \mathcal{O}\left(\varepsilon^{\frac{n}{2}+1}\right) \qquad (8)$$

whence

$$f(M) = \sum_{m=0}^{n}\frac{\varepsilon^{\frac{m+1}{2}}}{(m+1)!\zeta^{m+1}} \times \left\{\frac{\mathrm{d}^m}{\mathrm{d}M^m}\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{(m+1)\alpha}\right\}_{M=0} + \mathcal{O}\left(\varepsilon^{\frac{n}{2}+1}\right) \qquad (9)$$

Evaluating the first three terms of the summation renders

$$f = \frac{\varepsilon^{\frac{1}{2}}}{1!\zeta}\left\{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^\alpha\right\}_{M=0} + \frac{\varepsilon^{\frac{2}{2}}}{2!\zeta^2}\left\{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{2\alpha}\right\}'_{M=0} + \frac{\varepsilon^{\frac{3}{2}}}{3!\zeta^3}\left\{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{3\alpha}\right\}''_{M=0} + \cdots \qquad (10)$$

Each member of this series may be separately determined from the limit of the corresponding derivative at M_0. We get, for the first, zeroth-order member,

$$\lim_{M\to 0}\frac{\varepsilon^{\frac{1}{2}}}{\zeta}\left[1+\frac{1}{2}(\gamma-1)M^2\right]^\alpha = \frac{\varepsilon^{\frac{1}{2}}}{\zeta} = \varepsilon^{\frac{1}{2}}\left(\frac{2}{\gamma+1}\right)^\alpha \qquad (11)$$

Similarly, the second member gives

$$\lim_{M\to 0}\frac{\varepsilon^{\frac{2}{2}}}{2!\zeta^2}\left\{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{2\alpha}\right\}'$$
$$=\lim_{M\to 0}\frac{\varepsilon}{2\zeta^2}\left\{2\alpha\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{2\alpha-1}(\gamma-1)M\right\}=0 \quad (12)$$

It may be easily proven that all terms containing whole powers of ε, $\{1,\varepsilon,\varepsilon^2,\varepsilon^3,\ldots\}$ vanish identically. Finally, the cubic term may be calculated from

$$\lim_{M\to 0}\frac{\varepsilon^{\frac{3}{2}}}{3!\zeta^3}\left\{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{3\alpha}\right\}''$$
$$=\lim_{M\to 0}\frac{\varepsilon^{\frac{3}{2}}}{6\zeta^3}\left\{3\alpha\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{3\alpha-1}(\gamma-1)M\right\}'$$
$$=\lim_{M\to 0}\frac{\alpha(\gamma-1)\varepsilon^{\frac{3}{2}}}{2\zeta^3}\left\{(3\alpha-1)\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{3\alpha-2}(\gamma-1)M^2+\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{3\alpha-1}\right\}$$
$$=\frac{\alpha(\gamma-1)}{2\zeta^3}\varepsilon^{\frac{3}{2}}=\varepsilon^{\frac{3}{2}}2^{\frac{7-\gamma}{2(\gamma-1)}}(\gamma+1)^{-\frac{\gamma+5}{2(\gamma-1)}}$$
$$=\frac{\varepsilon^{\frac{3}{2}}}{4}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+5}{2(\gamma-1)}} \quad (13)$$

Higher-order terms may be readily extracted using symbolic programming of Eq. (9). At length, the subsonic solution to order $\mathcal{O}(\varepsilon^{\frac{7}{2}})$ may be arrived at, specifically

$$M_{\mathrm{sub}}^{(3)}=\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\varepsilon^{\frac{1}{2}}+\frac{1}{4}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+5}{2(\gamma-1)}}\varepsilon^{\frac{3}{2}}+\left[2^{\frac{15-5\gamma}{2(\gamma-1)}}(\gamma+1)^{\frac{3\gamma+7}{2-2\gamma}}(3\gamma+7)\right]\varepsilon^{\frac{5}{2}}+\mathcal{O}\left(\varepsilon^{\frac{7}{2}}\right) \quad (14)$$

Eq. (14) compares quite well with the numerical solution. For most propulsive applications, the three terms retained above are quite sufficient for engineering accuracy. The attending error remains smaller than 5% up to $\varepsilon=0.77$ or an area ratio of 0.88. Nonetheless, should further supplementary terms be required, Eq. (9) may be slightly modified to generate a solution to arbitrary order of precision viz.

$$M_{\mathrm{sub}}^{(n)}=\sum_{i=0}^{n}\frac{\varepsilon^{i+\frac{1}{2}}\zeta^{(2i+1)\alpha}}{(2i+1)!}\left\{\frac{\mathrm{d}^{2i}}{\mathrm{d}M^{2i}}\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{(2i+1)\alpha}\right\}_{M=0}+\mathcal{O}\left(\varepsilon^{n+\frac{3}{2}}\right) \quad (15)$$

After some effort, a recursive relation may be seen to exist, thus permitting the direct extraction of the subsonic solution to any desired order of accuracy:

$$M_{\mathrm{sub}}^{(n)}=\sum_{i=0}^{n}\frac{(\gamma-1)^i\varepsilon^{i+\frac{1}{2}}}{(2i+1)!\zeta^{2i+1}}\frac{(2i)!}{i!2^i}\prod_{j=0}^{i-1}\left[(2i+1)\alpha-j\right]+\mathcal{O}\left(\varepsilon^{n+\frac{3}{2}}\right) \quad (16)$$

As shown in Fig. 2, this expression is nearly identical to the numerical solution. A three-term approximation is more than adequate for the subsonic Mach number, even for large ε. For operational area ratios up to $A_t/A=0.47$ and an extreme case of $\gamma=1.7$, retaining *one term* leads to a viable approximation that accumulates an error of less than 5%. The range of ε increases as more terms are retained or as γ is lowered.

2.2. Supersonic treatment based on Successive Approximations

It seems that all regular and non-regular perturbation attempts fail in extracting the supersonic root directly from Eq. (2). Instead, we find it necessary to elevate Stodola's equation to the power of $\alpha^{-1}=2(\gamma-1)/(\gamma+1)$. The exponent-inverted form becomes

$$\varepsilon^{\frac{\gamma-1}{\gamma+1}}\left[1+\frac{1}{2}(\gamma-1)M^2\right]-M^{\frac{2(\gamma-1)}{\gamma+1}}\left[\frac{1}{2}(\gamma+1)\right]=0 \quad (17)$$

The resulting expression may then be multiplied by 2 and rearranged into

$$(\gamma+1)M^{\frac{2(\gamma-1)}{\gamma+1}}-(\gamma-1)\varepsilon^{\frac{\gamma-1}{\gamma+1}}M^2-2\varepsilon^{\frac{\gamma-1}{\gamma+1}}=0 \quad (18)$$

or

$$(\gamma+1)M^{2\kappa}-(\gamma-1)\varepsilon^{\kappa}M^2-2\varepsilon^{\kappa}=0$$
$$\kappa\equiv(\gamma-1)/(\gamma+1) \quad (19)$$

Eq. (19) is a keystone relation that can be managed to produce the supersonic root. In what follows, we apply the Successive Approximation Method to obtain the first three terms of the solution from which a recursion formula may be deduced. We also recognize that $1\leqslant\gamma\leqslant\frac{5}{3}$ and so $0\leqslant\kappa\leqslant\frac{1}{4}$. Then for $M>1$, the members in Eq. (19) may be presented in descending order, with the largest and smallest terms corresponding to $(\gamma+1)M^{2\kappa}$ and $2\varepsilon^{\kappa}$, respectively. Assuming $M=M_0+o(M_0)$ one gets, at leading order:

$$(\gamma+1)M_0^{2\kappa}-(\gamma-1)\varepsilon^{\kappa}M_0^2-2\varepsilon^{\kappa}=0 \quad (20)$$

In view of $M_0>1$ and $\kappa<1$, the last term may be ignored, being of higher order. This enables us to achieve balance between the first two terms by setting

$$(\gamma+1)M_0^{2\kappa}=(\gamma-1)\varepsilon^{\kappa}M_0^2 \quad \text{or}$$
$$M_0=\left(\kappa\varepsilon^{\kappa}\right)^{\frac{1}{2(\kappa-1)}}=\left[\frac{\gamma+1}{(\gamma-1)\varepsilon^{\kappa}}\right]^{\frac{\gamma+1}{4}}$$
$$=\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma+1}{4}}\frac{1}{\varepsilon^{\frac{1}{4}(\gamma-1)}} \quad (21)$$

Next, we let $M=M_0+M_1+o(M_1)$ and expand Eq. (20) into

$$(\gamma+1)M_0^{2\kappa}\left(1+2\kappa\frac{M_1}{M_0}\right)-(\gamma-1)\varepsilon^{\kappa}M_0^2\left(1+2\frac{M_1}{M_0}\right)-2\varepsilon^{\kappa}=0 \quad (22)$$

where a binomial series is used. This allows the extraction of the first order correction from

$$2\kappa(\gamma+1)M_0^{2\kappa-1}M_1-2(\gamma-1)\varepsilon^{\kappa}M_0M_1-2\varepsilon^{\kappa}=0 \quad (23)$$

whence

$$M_1=\frac{\varepsilon^{\kappa}M_0}{\kappa(\gamma+1)M_0^{2\kappa}-(\gamma-1)\varepsilon^{\kappa}M_0^2}=\frac{1}{(\gamma-1)(\varepsilon^{-\kappa}M_0^{2\kappa-1}-M_0)} \quad (24)$$

To determine $\mathcal{O}(M_1)$ and, with it, the order of leading order truncation error, we evaluate

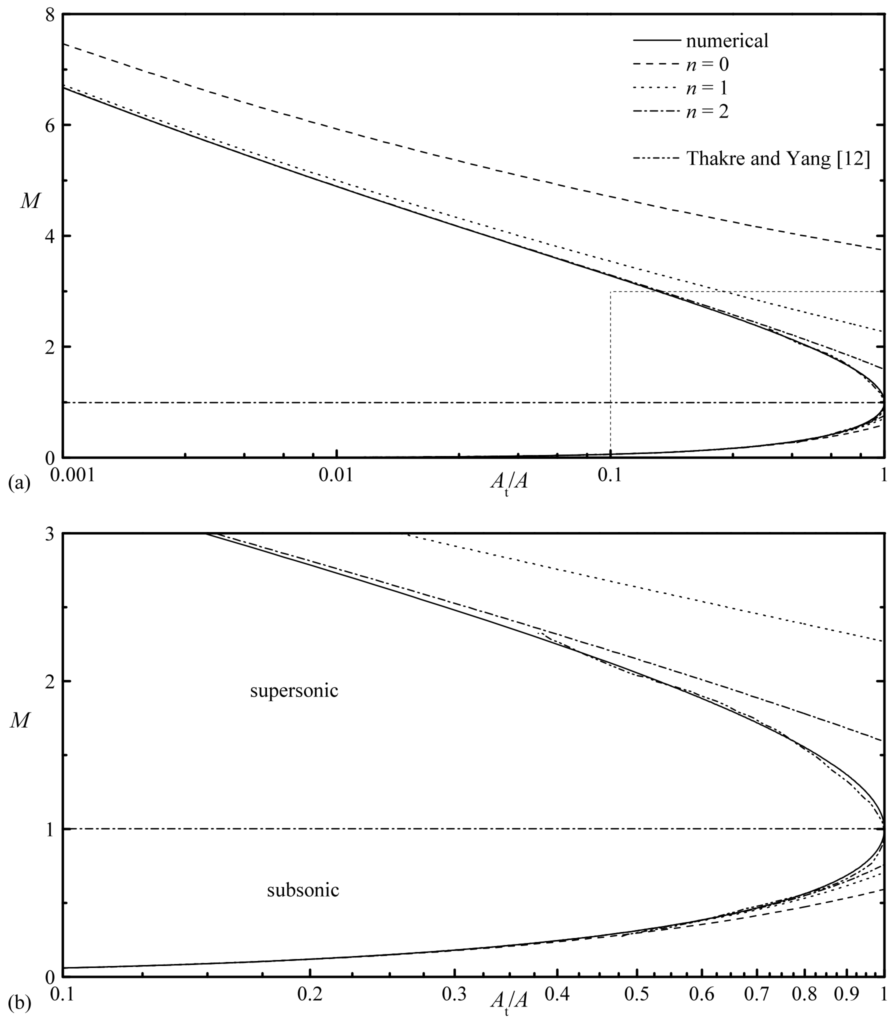

Fig. 2. Comparison between numerical and asymptotic solutions for $\gamma = 1.2$. The dashed line in (a) denotes the region enlarged in (b).

$$M_1 \approx \frac{1}{(\gamma-1)}\varepsilon^{\kappa} M_0^{1-2\kappa} = \frac{\varepsilon^{\frac{\gamma-1}{\gamma+1}}}{(\gamma-1)}\left[\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{\gamma+1}{4}}\varepsilon^{\frac{1}{4}(1-\gamma)}\right]^{\frac{3-\gamma}{\gamma+1}} = \frac{\varepsilon^{\frac{\gamma^2-1}{4(\gamma+1)}}}{(\gamma-1)}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{3-\gamma}{4}} = \mathcal{O}\left(\varepsilon^{\frac{1}{4}(\gamma-1)}\right) \tag{25}$$

Repeating once more, we substitute $M = M_0 + M_1 + M_2 + \mathcal{O}(M_2)$ into Eq. (20) and collect

$$(\gamma+1)(M_0+M_1)^{2\kappa}\left(1+2\kappa\frac{M_2}{M_0+M_1}\right) - (\gamma-1)\varepsilon^{\kappa}(M_0+M_1)^2\left(1+2\frac{M_2}{M_0+M_1}\right) - 2\varepsilon^{\kappa} = 0 \tag{26}$$

The second order correction may thus be retrieved. We obtain

$$M_2 = \frac{2\varepsilon^{\kappa} - (\gamma+1)(M_0+M_1)^{2\kappa} + (\gamma-1)\varepsilon^{\kappa}(M_0+M_1)^2}{2\kappa(\gamma+1)(M_0+M_1)^{2\kappa-1} - 2(\gamma-1)\varepsilon^{\kappa}(M_0+M_1)} \tag{27}$$

An asymptotically expanded form of the above may be expressed as

$$M_2 \simeq -\frac{M_1^3}{2}\frac{(\gamma-1)\varepsilon^{\kappa} - (\gamma+1)\kappa(2\kappa-1)M_0^{2\kappa-2}}{[(\gamma-1)\varepsilon^{\kappa} - (\gamma+1)\kappa(2\kappa-1)M_0^{2\kappa-2}]M_1^2 - \varepsilon^{\kappa}} \tag{28}$$

The corresponding order may be deduced using binomial expansions and simplifications leading to the identification of

$$M_2 \approx \frac{1}{2}(\gamma-1)M_1^3 = \mathcal{O}\left(\varepsilon^{\frac{3}{4}(\gamma-1)}\right) \tag{29}$$

It is clear that for $i \geqslant 1$, a recursive relation exists for M_i in terms of M_{i-1} from which all terms beyond M_0 may be extrapolated. Higher order approximations may be realized by adding each successive correction to the sum:

$$M_{\text{sup}}^{(n)} = M_0 + \sum_{i=1}^{n} M_i$$
$$M_i = \frac{2\varepsilon^{\kappa} - (\gamma+1)[M_{\text{sup}}^{(i-1)}]^{2\kappa} + (\gamma-1)\varepsilon^{\kappa}[M_{\text{sup}}^{(i-1)}]^2}{2\kappa(\gamma+1)[M_{\text{sup}}^{(i-1)}]^{2\kappa-1} - 2(\gamma-1)\varepsilon^{\kappa}M_{\text{sup}}^{(i-1)}}$$
$$M_{\text{sup}}^{(i-1)} = \sum_{j=0}^{i-1} M_j \tag{30}$$

Results displayed in Fig. 2 are taken with expansions up to $n = 2$ so both the supersonic and subsonic solutions will contain three terms. We note a substantial agreement with the use of three terms, even at large ε. In addition to the present work, we feature in Fig. 2 the solution predicted by Thakre and Yang [12] in their numerical investigation of nozzle erosion. Their treatment employs a density-based finite volume solver that accounts for chemical reactions. Their results are taken at the centerline of the nozzle and agree quite favorably with the one-dimensional model employed here.

Before leaving this subject, it may be useful to remark that one potential advantage of the analytical form over compressible flow tables or basic numerical routines (unlike EES) lies in the handling of the specific heat ratio. Compressible flow tables only catalogue values for fixed specific heat ratios. In many situations, solving Eq. (1) numerically becomes limited to a single value of γ, though more sophisticated solvers can incorporate variable specific heat ratios [5]. The present formulation can readily accommodate temperature variations using empirically determined equations. For example, a proposed linear function [3] may be written as

$$\gamma = \gamma_0 - K_1(T - T_{\mathrm{ref}})/1000 \tag{31}$$

where γ_0 and K_1 denote pure constants while T_{ref} represents a reference temperature. For cases with known temperature profiles, Eq. (31) can be directly substituted into the subsonic or supersonic expressions to the extent of providing a solution with the requisite temperature-dependent specific heat ratio. Other empirically developed relations with polynomial or exponential dependence could just as easily be incorporated into the present work.

3. Concluding remarks

In this study, two asymptotic formulations are presented as numerical equivalents to the traditional area–Mach number relationship that is ubiquitously used in rocket nozzle analysis. The first is derived from Bürmann's theorem and the second, using the Method of Successive Approximations. Both techniques unravel the distinct dependence of the Mach number on the nozzle area ratio and the ratio of specific heats. The tacit relations that we arrive at allow for swift and robust computation of essential flowfield properties in a de Laval nozzle under either subsonic or supersonic operation. This is accomplished by granting the use of selectively controlled recursive formulations that produce the desired branch of solution to any degree of accuracy. The present work leads to simple and novel recursions with well prescribed truncation orders. These increase our repertoire of engineering approximations for compressible flows and enable us to compute the subsonic and supersonic Mach numbers at any cross-section and to an arbitrary degree of precision. In practice, a maximum of three non-zero terms in each approximation may be sufficient to yield a satisfactory level of precision.

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