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Inner and Outer Solutions for the Injection Driven Channel Flow with Retractable Walls

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This work extends a previous study of the laminar flow in porous channels with retracting walls. After identifying an endpoint singularity that affects the former solution in its third derivative, a slow variable is introduced to capture the rapid variations near the channel core. The core refers to the midsection plane where the shear layer is relocated due to hard blowing at the walls. Using matched-asymptotic expansions with logarithmic corrections, a composite solution is developed following successive upward integrations that start with the third derivative. In the process, the inner solution is unraveled from the fourth-order equation governing the symmetric injection-driven flow near the core. The resulting approximation is expressed in terms of generalized hypergeometric functions and is verified using numerics and limiting process validations. The composite solution is shown to outperform the former, outer solution, as the core is approached or as the injection Reynolds number is increased. Without undermining the simplicity and practicality of the former solution outside the (thin) core region, the construction of a matched-asymptotic solution enables us to suppress singular terms and insure a uniformly valid outcome down to the fourth derivative.

I. Introduction

THIS article is concerned with providing a complete analytical solution for the steady two-dimensional flow of a viscous fluid in a porous channel with expanding or contracting walls. The channel is taken to be semi-infinite and uniform. An incompressible fluid is injected uniformly at the walls as shown in Fig. 1. Such an idealization serves to model a range of physical phenomena including transpiration cooling, boundary layer control, jet mixing, surface ablation, propellant burning, and membrane separation.

In transpiration cooling applications, the injection of a lower temperature fluid across the walls creates a thermal barrier that protects the walls of the channel carrying a hot fluid.[1] Boundary layer control, on the other hand, can be accomplished by injecting, redirecting, or pulsating streams of fluid which, on an aircraft wing or upstream of an ammunition bay, can serve to reduce drag or acoustic resonance.[2-5] In jet mixing processes, pulsating the jet in a controllable manner can improve jet turbulence and penetration while reducing the length to achieve a given mixing state.[6,7] In combustion chambers and nozzles, injecting a cool layer of oxidizer can help maintain tolerable wall temperatures.[8] In chemical propulsion, the ejection of hot gases inside a thrust chamber can be simulated by the uniform injection of a fluid across porous and regressing walls.[9-15] Another application that has given the original motivation for this class of studies is the separation of uranium isotopes U_{235} and U_{238} by differential gaseous diffusion.[16]

In a previous article,[17] both injection and suction-driven viscous flowfields were characterized inside a uniformly porous channel with expanding or contracting walls. By introducing a similarity solution for the streamfunction of the form $\psi = xF$, the Navier–Stokes equations were reduced to a single ODE

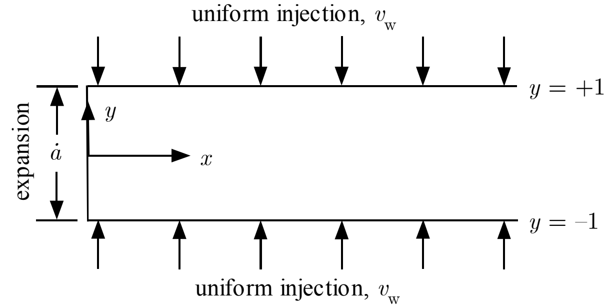

Fig. 1 Flow geometry.

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for the characteristic function F. An asymptotic solution could then be obtained, valid for large crossflow Reynolds number $R = v_w a / \nu$, where R was based on the wall injection speed v_w and the channel half-spacing a. The outer solution presented by Majdalani and Zhou[17] exhibited an endpoint singularity due to a logarithmic term that appeared in its third derivative. This term controlled the axial pressure gradient in the porous chamber and became suddenly unbounded at the core. Physically this singular behavior signals the presence of a viscous layer along the channel's midsection plane. In this study, we not only uncover the size and form of the viscous layer, but also show how this singularity can be removed by applying the principle of matched-asymptotic expansions. To complete our previous investigation, we hereby construct a uniformly valid composite solution that leads to holomorphic vorticity, shear, and pressure fields across the entire domain of interest.

II. Outer Solution

Consider the injection-driven viscous flow inside a uniformly porous channel with expanding or contracting walls. As shown previously,[17] one can apply exact similarity transformations in space and time to convert the Navier-Stokes equations into a well-posed fourth-order boundary value problem of the form

$$R^{-1}F'''' + \alpha R^{-1}\left(yF''' + 2F''\right) + FF''' - \left(F'\right)^2 = \lambda \quad (1)$$

where λ is a subsidary constant; boundary conditions are specified at the wall and midsection plane viz.

$$F''(0) = 0\,,\; F(0) = 0\,,\; F'(1) = 0\,, F(1) = 1. \quad (2)$$

As before, x and y represent the dimensionless axial and normal coordinates measured from the head-end and the midsection plane, respectively (see Fig. 1). The wall expansion ratio $\alpha = \dot{a}a / \nu$ is the Reynolds number based on the speed of wall regression $\dot{a}$. For a large injection Reynolds number, the small parameter $\varepsilon = R^{-1}$ was used by Majdalani and Zhou[17] to solve Eq. (1) asymptotically. Following successive applications of the variation of parameters method, one formerly obtained

$$F(\theta) = \sin\theta + \varepsilon\left\{-(2\alpha/\pi)\theta + \left(\tfrac{1}{4}\pi - 4\alpha/\pi\right)\right.$$
$$\times\left[(\theta\cos\theta - \sin\theta)\ln\tan\tfrac{1}{2}\theta - \cos\theta S(\theta)\right] + \alpha\sin\theta$$
$$\left.+\left[\left(\tfrac{1}{2} - 8\alpha\pi^{-2}\right)S\left(\tfrac{1}{2}\pi\right) + 4\alpha\pi^{-2} - \tfrac{1}{2}\right]\theta\cos\theta\right\};$$

$$\theta = \tfrac{1}{2}\pi y\,;\; S(\theta) \equiv \int_0^\theta \phi\csc\phi\,\mathrm{d}\phi \quad (3)$$

where the dimensionless axial and normal velocity components could be deduced using $u = xF'$, and $v = -F$. In like fashion, other flow attributes could be extrapolated from F and its derivatives. The axial pressure gradient, for example, could be evaluated from F''' via

$$\partial p / \partial x = x\left[\varepsilon F''' + FF'' + \left(F'\right)^2 + \alpha\varepsilon\left(2F' + yF''\right)\right]; \quad (4)$$

$$F''' = \tfrac{1}{8}\pi^3\left(-\cos\theta + \varepsilon\left\{-\alpha\cos\theta + \left(\tfrac{1}{4}\pi - 4\alpha/\pi\right)\right.\right.$$
$$\times\left[-\sin\theta S(\theta) - (2\cos\theta - \theta\sin\theta)\ln\tan\tfrac{1}{2}\theta - 1\right]$$
$$\left.\left.-\left[\left(\tfrac{1}{2} - 8\alpha\pi^{-2}\right)S\left(\tfrac{1}{2}\pi\right) + 4\alpha\pi^{-2} - \tfrac{1}{2}\right](3\cos\theta - \theta\sin\theta)\right\}\right)$$

(5)

The source of singularity is due to the ($\cos\theta\ln\tan\frac{1}{2}\theta$) term that appears for the first time in F'''. Since this term becomes suddenly unbounded as $\theta \to 0$, it marks the presence of a core boundary layer that has not been considered in previous work. This is due to large blowing at each wall which pushes the shear layer to the core region. Inside this layer the viscous term $R^{-1}F''''$ in Eq. (1) becomes comparable to the inertial term $FF''' - (F')^2$. Consequently, the full equation will have to be considered and rescaled.

By retaining the highest derivative, however, the order of the equation is increased by one. Previously, only three of the four boundary conditions needed to be satisfied by the outer solution. Under the premise of a large injection Reynolds number, it was not necessary to impose the first condition $F''(0) = 0$ in Eq. (2).[17] At present, the two core boundary conditions will have to be met by the inner solution whose two remaining integration constants have to be determined by matching.

It may be interesting to note that the case of large blowing at two opposing but fixed walls exhibits similar features to ours. This case has been previously investigated by Yuan[18] whose series solution also appeared to agree well with numerical solutions until it was differentiated three times. Much like ours, Yuan's third derivative was found to be infinite at the core. Since unboundedness could not occur in practice, Terrill[19] would later point out the existence of a viscous shear layer that was neglected in Yuan's analysis.

III. Inner Expansion of the Outer Solution

To eliminate the singularity in F''', an inner solution must be attempted in the thin core region. This can be accomplished by applying the concept of matched asymptotic expansions. Starting with

$$F = \sin\theta - \varepsilon\left\{(2\alpha/\pi)\theta - B\left[(\theta\cos\theta - \sin\theta)\ln\tan\tfrac{1}{2}\theta\right.\right.$$
$$\left.\left.-\cos\theta S(\theta)\right] - \alpha\sin\theta - A\theta\cos\theta\right\} \quad (6)$$

$$\begin{cases} A \equiv \left(\frac{1}{2} - 8\alpha/\pi^2\right) S\left(\frac{1}{2}\pi\right) + 4\alpha/\pi^2 - \frac{1}{2} \\ B \equiv \frac{1}{4}\pi - 4\alpha/\pi \end{cases} \tag{7}$$

a term-by-term expansion yields

$$\begin{aligned} F(\theta) &= \left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) + \varepsilon\left\{-2(\alpha/\pi)\theta\right. \\ &\quad + B\left[\theta\left(1 - \tfrac{1}{2!}\theta^2\right) - \left(\theta - \tfrac{1}{3!}\theta^3\right)\right]\ln\tfrac{1}{2}\theta \\ &\quad - B\left(1 - \tfrac{1}{2!}\theta^2\right)\left(\theta + \tfrac{1}{18}\theta^3\right) \\ &\quad \left. + \alpha\left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) + A\theta\left(1 - \tfrac{1}{2!}\theta^2\right)\right\} \\ &= \left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) + \varepsilon\left[\theta\left(\alpha + A - 2\alpha/\pi - B\right)\right. \\ &\quad \left. - \tfrac{1}{3}B\theta^3\ln\tfrac{1}{2}\theta + \theta^3\left(\tfrac{1}{3}B\ln 2 + \tfrac{4}{9}B - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\right]. \end{aligned} \tag{8}$$

Letting the slow (inner) variable have the form $\xi = \theta/\varepsilon^n$, substitution into Eq. (1) reveals that a balance between inertial and viscous forces will be established for $n = \frac{1}{2}$. Recognizing that the core layer has a thickness of $O(\varepsilon^{\frac{1}{2}})$, the appropriate coordinate transformation becomes $\xi = \theta/\varepsilon^{\frac{1}{2}}$; forthwith, the outer solution given by Eq. (8) is expanded in the inner variable using

$$\begin{aligned} (F^{(o)})^i &= F^{(o)}(\xi) = \varepsilon^{\frac{1}{2}} f_1 + \varepsilon^{\frac{3}{2}} f_2 + \varepsilon^{\frac{5}{2}}\ln\varepsilon f_3 + \varepsilon^{\frac{5}{2}} f_4 \\ &= \varepsilon^{\frac{1}{2}}\xi - \varepsilon^{\frac{3}{2}}\left[\tfrac{1}{6}\xi^3 + \left(2\alpha/\pi + B - \alpha - A\right)\xi\right] - \tfrac{1}{6}\varepsilon^{\frac{5}{2}}\ln\varepsilon B\xi^3 \\ &\quad + \varepsilon^{\frac{5}{2}}\left[\tfrac{1}{120}\xi^5 - \tfrac{1}{3}B\xi^3\ln\xi + \left(\tfrac{1}{3}B\ln 2 + \tfrac{4}{9}B - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\xi^3\right] \end{aligned} \tag{9}$$

IV. Inner Solution

Next, the independent variable in Eq. (1) is changed from y to θ. This operation begets

$$\tfrac{1}{4}\pi^2\varepsilon F_{\theta\theta\theta\theta} + \alpha\varepsilon\left(\theta F_{\theta\theta\theta} + 3F_{\theta\theta}\right) + \tfrac{1}{2}\pi F F_{\theta\theta\theta} - \tfrac{1}{2}\pi F_\theta F_{\theta\theta} = 0 \tag{10}$$

where the subscript denotes the derivative with respect to θ. Using the spatial distortion $\theta = \varepsilon^{\frac{1}{2}}\xi$, one obtains the inner equation that dominates near the core. This emerges as

$$\tfrac{1}{4}\pi^2\varepsilon^{\frac{1}{2}} F_{\xi\xi\xi\xi} + \alpha\varepsilon^{\frac{3}{2}}\left(\xi F_{\xi\xi\xi} + 3F_{\xi\xi}\right) + \tfrac{1}{2}\pi F F_{\xi\xi\xi} - \tfrac{1}{2}\pi F_\xi F_{\xi\xi} = 0 \tag{11}$$

with the two local boundary conditions

$$F_{\xi\xi}(0) = 0 \text{ and } F(0) = 0 \tag{12}$$

At this juncture, the inner solution can be written in a series of progressively diminishing terms, specifically,

$$F^{(i)}(\xi) = \varepsilon^{\frac{1}{2}} g_1(\xi) + \varepsilon^{\frac{3}{2}} g_2(\xi) + \varepsilon^{\frac{5}{2}}\ln\varepsilon\, g_3(\xi) + \varepsilon^{\frac{5}{2}} g_4(\xi) \tag{13}$$

When substituted into Eq. (11), terms of the same order can be gathered. One finds

$$O(\varepsilon): \quad \tfrac{1}{2}\pi g_{1,\xi\xi\xi\xi} + g_1 g_{1,\xi\xi\xi} - g_{1,\xi} g_{1,\xi\xi} = 0 \tag{14}$$

$$O\left(\varepsilon^2\right): \quad \tfrac{1}{2}\pi g_{2,\xi\xi\xi\xi} + \xi g_{2,\xi\xi\xi} - g_{2,\xi\xi} = 0 \tag{15}$$

$$O\left(\varepsilon^3\ln\varepsilon\right): \quad \tfrac{1}{2}\pi g_{3,\xi\xi\xi\xi} + \xi g_{3,\xi\xi\xi} - g_{3,\xi\xi} = 0 \tag{16}$$

$$O\left(\varepsilon^3\right): \quad \tfrac{1}{2}\pi g_{4,\xi\xi\xi\xi} + \xi g_{4,\xi\xi\xi} - g_{4,\xi\xi} = \tfrac{1}{3}\xi^3 + (2\alpha/\pi)\xi \tag{17}$$

According to Van Dyke's matching principle, Eqs. (14)–(17) must be integrated and then matched with Eq. (9). This operation yields

$$g_1 = \xi \tag{18}$$

$$g_2 = -\tfrac{1}{6}\xi^3 + \left(\alpha + A - 2\alpha/\pi - B\right)\xi \tag{19}$$

$$g_3 = -\tfrac{1}{6}B\xi^3 \tag{20}$$

However, using $F_{\xi\xi}(0) = 0$ from Eq. (12), one gets $g_{4,\xi\xi}(0) = 0$. Equation (17) can then be integrated twice and put in the form

$$\begin{aligned} g_{4,\xi\xi} &= Z\left(\pi^2 - 4\alpha\right)\xi + \tfrac{1}{6}\xi^3 - \xi\left(\tfrac{1}{2}\pi - 2\alpha/\pi\right) \\ &\quad \times\int_0^\xi e^{\frac{-\xi^2}{\pi}}\operatorname{erfi}\left(\xi/\sqrt{\pi}\right)\mathrm{d}\xi - \left(\tfrac{1}{4}\pi^2 - \alpha\right)e^{\frac{-\xi^2}{\pi}}\operatorname{erfi}\left(\xi/\sqrt{\pi}\right) \end{aligned} \tag{21}$$

where the error function $\operatorname{erfi}(x) = \frac{1}{2}\pi\int_0^x e^{t^3}\mathrm{d}t$ and Z is a constant that needs to be determined by matching with $(F^{(o)})^i$.

A. Outer Expansion of the Inner Solution

In order to evaluate the inner solution in the outer region, it is useful to introduce the large ξ expression

$$\operatorname{erfi}(\xi) \sim \frac{B}{\xi}\frac{4\sqrt{\pi}}{(\pi^2 - 4\alpha)}e^{\xi^2} \tag{22}$$

Based on Eq. (22), the third and fourth terms on the right hand side of Eq. (21) become

$$\begin{aligned} &\xi\left(\tfrac{1}{2}\pi - 2\alpha/\pi\right)\int e^{\frac{-\xi^2}{\pi}}\operatorname{erfi}\left(\xi/\sqrt{\pi}\right)\mathrm{d}\xi \\ &\sim \xi\left(\tfrac{1}{2}\pi - 2\alpha/\pi\right)\int\frac{B}{\xi\left(\frac{1}{4}\pi^2 - \alpha/\pi\right)}\mathrm{d}\xi \sim 2B\xi\ln\xi\,; \end{aligned} \tag{23}$$

$$\begin{aligned} &\left(\tfrac{1}{4}\pi^2 - \alpha\right)e^{\frac{-\xi^2}{\pi}}\operatorname{erfi}\left(\xi/\sqrt{\pi}\right) \\ &\quad \sim \left(\tfrac{1}{4}\pi^2 - \alpha\right)e^{\frac{-\xi^2}{\pi}}\frac{B}{\xi\left(\frac{1}{4}\pi - \alpha/\pi\right)}e^{\frac{\xi^2}{\pi}} \sim \frac{B\pi}{\xi} \end{aligned} \tag{24}$$

The $O\left(\xi^{-3}\right)$ outer expansion of Eq. (21) is hence

$$(g_{4,\xi\xi})^o = \tfrac{1}{6}\xi^3 + Z\left(\pi^2 - 4\alpha\right)\xi - 2B\xi\ln\xi - B\pi\xi^{-1}. \tag{25}$$

B. Matching with the Outer Solution

According to Eq. (13), $g_4(\xi)$ is the $O(\varepsilon^{\frac{5}{2}})$ correction to the inner solution $F^{(i)}$; $(g_{4,\xi\xi})^o$ is hence the second derivative of the $\varepsilon^{\frac{5}{2}}$ correction arising in $(F^{(i)})^o$. Pursuant to Van Dyke's matching principle, this term must be set equal to the second derivative $f_{4,\xi\xi}$ of the $\varepsilon^{\frac{5}{2}}$ correction arising in $(F^{(o)})^i$. From Eq. (9), it is clear that

$$f_4 = \tfrac{1}{120}\xi^5 - \tfrac{1}{3}B\xi^3\ln\xi + \left(\tfrac{1}{3}B\ln 2 + \tfrac{4}{9}B - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\xi^3 \quad (26)$$

and so

$$f_{4,\xi\xi} = \tfrac{1}{6}\xi^3 - 2B\xi\ln\xi + (2B\ln 2 - \alpha - 3A)\xi \quad (27)$$

By setting $(g_{4,\xi\xi})^o = f_{4,\xi\xi}$, Eqs. (25) and (27) can be equated in a manner to provide

$$Z = (2B\ln 2 - \alpha - 3A)/(\pi^2 - 4\alpha) \quad (28)$$

At the outset, Eq. (25) becomes

$$(g_{4,\xi\xi})^o = \tfrac{1}{6}\xi^3 - 2B\xi\ln\xi + (2B\ln 2 - \alpha - 3A)\xi - B\pi\xi^{-1} + O\left(\xi^{-3}\right) \quad (29)$$

hence

$$(g_4)^o = \tfrac{1}{120}\xi^5 + \left(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A + \tfrac{5}{18}B\right)\xi^3 - \tfrac{1}{3}B\xi^3\ln\xi + B\pi\xi - B\pi\xi\ln\xi \quad (30)$$

The outer expansion of the inner solution is finally at hand. Substituting g_i back into Eq. (13), one collects

$$(F^{(i)})^o = \varepsilon^{\frac{1}{2}}\xi + \varepsilon^{\frac{3}{2}}\left[-\tfrac{1}{6}\xi^3 + (\alpha + A - 2\alpha/\pi - B)\xi\right] - \tfrac{1}{6}\varepsilon^{\frac{5}{2}}\ln\varepsilon B\xi^3 + \varepsilon^{\frac{5}{2}}\left[-\tfrac{1}{3}B\xi^3\ln\xi + B\pi\xi - B\pi\xi\ln\xi + \tfrac{1}{120}\xi^5 + \left(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A + \tfrac{5}{18}B\right)\xi^3\right]. \quad (31)$$

This solution can be expressed in the original variable θ which puts it in the form

$$(F^{(i)})^o = \left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) + \varepsilon\left[\theta\left(\alpha + A - 2\alpha/\pi - B\right) + \left(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A + \tfrac{5}{18}B\right)\theta^3 - \tfrac{1}{3}B\theta^3\ln\theta\right] + \tfrac{1}{2}\varepsilon^2\ln\varepsilon B\pi\theta + O\left(\varepsilon^2\right). \quad (32)$$

C. Inner Correction

Having determined the constant in Eq. (21), one can integrate twice and write

$$g_4 = \tfrac{1}{120}\xi^5 + \left(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\xi^3 - \tfrac{1}{120}\left(1 - 4\alpha/\pi^2\right) \times\xi^3\left\{10\pi + \xi^2\,{}_pF_q\left[(1,1);(3,\tfrac{7}{2});-\xi^2/\pi\right]\right\} + C_1\xi + C_0 \quad (33)$$

where C_1 and C_0 are pure constants and ${}_pF_q$ is the generalized hypergeometric function. Due to the boundary condition $F(0) = 0$, it can be readily seen that $g_4(0) = 0$ and $C_0 = 0$. The remaining constant C_1 becomes immaterial as it appears at $O(\varepsilon^2)$. This can be seen by first substituting Eq. (33) into Eq. (13) to render

$$F^{(i)}(\xi) = \varepsilon^{\frac{1}{2}}\xi + \varepsilon^{\frac{3}{2}}\left[-\tfrac{1}{6}\xi^3 + (\alpha + A - 2\alpha/\pi - B)\xi\right] - \tfrac{1}{6}\varepsilon^{\frac{5}{2}}\ln\varepsilon B\xi^3 + \varepsilon^{\frac{5}{2}}\left(\tfrac{1}{120}\xi^5 + \left(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\xi^3 - \tfrac{1}{120}\left(1 - 4\alpha/\pi^2\right)\xi^3\left\{10\pi + \xi^2\,{}_pF_q\left[(1,1);(3,\tfrac{7}{2});-\xi^2/\pi\right]\right\}\right) + C_1\xi \quad (34)$$

Since $\xi = \theta/\varepsilon^{\frac{1}{2}}$, the term involving $\varepsilon^{\frac{5}{2}}C_1\xi = \varepsilon^2 C_1\theta$ can be dismissed when returning to the original coordinate representation; the inner solution becomes

$$F^{(i)}(\theta) = \left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5\right) - \tfrac{1}{120}\left(1 - 4\alpha/\pi^2\right) \times\theta^5\,{}_pF_q\left[(1,1),(3,\tfrac{7}{2}),-\theta^2/(\varepsilon\pi)\right] - \tfrac{1}{6}\varepsilon\ln\varepsilon B\theta^3 + \varepsilon\left[(\alpha + A - 2\alpha/\pi - B)\theta + \left(\tfrac{1}{3}B\ln 2 - \tfrac{1}{6}\alpha - \tfrac{1}{2}A\right)\theta^3 - \tfrac{1}{12}(\pi - 4\alpha/\pi)\theta^3\right] + O\left(\varepsilon^2\right) \quad (35)$$

where terms at order ε^2 and higher have been ignored. Having determined $F^{(o)}(\theta)$, $F^{(i)}(\theta)$ and $(F^{(i)})^o$, a uniformly valid composite solution $F^{(c)}$ can be reached by combining

$$F^{(c)} = F^{(o)} + F^{(i)} - (F^{(i)})^o. \quad (36)$$

One can evaluate the net correction $\bar{F}^{(i)}$ that, when added to the formerly reported outer solution,[17] will make it singular-free down to the fourth derivative. Defining $\bar{F}^{(i)} \equiv F^{(i)} - (F^{(i)})^o$, one can put

$$\bar{F}^{(i)} = \tfrac{1}{120}\left(4\alpha/\pi^2 - 1\right)\theta^5\,{}_pF_q\left[(1,1),(3,\tfrac{7}{2}),-\theta^2/(\varepsilon\pi)\right] - \tfrac{1}{6}\varepsilon\ln\varepsilon B\theta^3 - \tfrac{1}{2}\pi\varepsilon^2\ln\varepsilon B\theta + \varepsilon\left[\tfrac{1}{3}B\theta^3\ln\theta - \tfrac{5}{18}B\theta^3 + \tfrac{1}{12}\left(4\alpha/\pi - \pi\right)\theta^3\right] \quad (37)$$

Based on this correction, the third derivative in $F^{(c)}$ is no longer unbounded at the core. The net corrections that must be added to the derivatives of the outer solution[17] are, therefore,

$$\bar{F}_\theta^{(i)} = \tfrac{1}{40}\left(4\alpha/\pi^2 - 1\right)\theta^4\,{}_pF_q\left[(1,1),(3,\tfrac{7}{2}),-\theta^2/(\varepsilon\pi)\right] - \tfrac{1}{2}\varepsilon\ln\varepsilon B\theta^2 + \varepsilon\theta^2\left(B\ln\theta - \tfrac{1}{2}B + \tfrac{1}{12}(4\alpha/\pi - \pi)\left\{\tfrac{13}{10} - {}_pF_q\left[(1,1),(2,\tfrac{5}{2}),-\theta^2/(\varepsilon\pi)\right]\right\}\right) - \tfrac{1}{2}\pi\varepsilon^2\ln\varepsilon B \quad (38)$$

$$\bar{F}^{(i)}_{\theta\theta} = -\tfrac{1}{20}\left(1+4\alpha/\pi^2\right)\theta^3\,{}_pF_q\left[(1,1),(3,\tfrac{7}{2}),-\theta^2/(\varepsilon\pi)\right]$$
$$-\varepsilon\ln\varepsilon B\theta\ +\varepsilon^2\left(\tfrac{1}{4}\pi^2-\alpha\right)/\theta\ +\varepsilon\theta\,(2B\ln\theta$$
$$+(4\alpha/\pi-\pi)\left\{\tfrac{7}{15}-\tfrac{1}{4}\,{}_pF_q\left[(1,1),(2,\tfrac{5}{2}),-\theta^2/(\varepsilon\pi)\right]\right\})$$
$$+\tfrac{1}{8}\pi\left(4\alpha-\pi^2\right)\varepsilon^{5/2}\,\mathrm{erfi}(\theta/\sqrt{\varepsilon\pi})\exp[-\theta^2/(\varepsilon\pi)]/\theta^2 \tag{39}$$

$$\bar{F}^{(i)}_{\theta\theta\theta} = -\tfrac{1}{20}\left(1+4\alpha/\pi^2\right)\theta^2\,{}_pF_q\left[(1,1),(3,\tfrac{7}{2}),-\theta^2/(\varepsilon\pi)\right]$$
$$-\varepsilon\ln\varepsilon B\ +\varepsilon\,(2B-\tfrac{29}{30}(\pi-4\alpha/\pi)$$
$$+\tfrac{1}{4}\left\{(\pi-4\alpha/\pi)\,{}_pF_q\left[(1,1),(2,\tfrac{5}{2}),-\theta^2/(\varepsilon\pi)\right]+8B\ln\theta\right\})$$
$$+\tfrac{1}{4}\left(\pi^2-4\alpha\right)\varepsilon^{3/2}\,\mathrm{erfi}(\theta/\sqrt{\varepsilon\pi})e^{-\theta^2/(\varepsilon\pi)}/\theta$$
$$+\tfrac{1}{4}\left(\pi^2-4\alpha\right)\varepsilon^{5/2}\left[1-\tfrac{1}{2}\pi\,\mathrm{erfi}(\theta/\sqrt{\varepsilon\pi})e^{-\theta^2/(\varepsilon\pi)}/\theta\right]/\theta^2 \tag{40}$$

and, finally,

$$\bar{F}^{(i)}_{\theta\theta\theta\theta} = \tfrac{1}{2}\left(4\alpha/\pi-\pi\right)\varepsilon^{1/2}\,\mathrm{erfi}(\theta/\sqrt{\varepsilon\pi})e^{-\theta^2/(\varepsilon\pi)}+2B\varepsilon/\theta \tag{41}$$

V. Corrected Core Values

Recalling that $\mathrm{d}^nF/\mathrm{d}y^n = (\tfrac{1}{2}\pi)^n\mathrm{d}^nF/\mathrm{d}\theta^n$, one can use either L'Hôpital's rule or Taylor series expansions to study the behavior of F and its derivatives as $y\to 0^+$. Without the inner correction, the formerly reported solution can be shown to exhibit

$$F(0) = F''(0) = 0 \tag{42}$$

$$F'(0) = \tfrac{1}{2}\pi\left\{1+\varepsilon\begin{bmatrix}\mathcal{C}-\tfrac{1}{2}-\tfrac{1}{4}\pi\\ +\left(4-16\mathcal{C}+2\pi+\pi^2\right)\alpha/\pi^2\end{bmatrix}\right\} \tag{43}$$

$$F'''(y) \sim \tfrac{1}{8}\pi^3\varepsilon\left(8\alpha/\pi-\tfrac{1}{2}\pi\right)\ln\left(\tfrac{1}{2}\pi y\right)$$
$$\xrightarrow[y\to 0^+]{}\left[\mathrm{sgn}(\pi^2-16\alpha)\right]\infty \tag{44}$$

Table 1 Comparison between outer (*o*), composite (*c*), and numeric (*n*) predictions for F, F', F'' and F''' at $\alpha=10$ and $\varepsilon=0.01$ ($R=100$)

	F			F'			F''			F'''		
$y/(2\varepsilon)$	$F^{(o)}$	$F^{(c)}$	$F^{(n)}$	$F^{(o)\prime}$	$F^{(c)\prime}$	$F^{(n)\prime}$	$F^{(o)\prime\prime}$	$F^{(c)\prime\prime}$	$F^{(n)\prime\prime}$	$F^{(o)\prime\prime\prime}$	$F^{(c)\prime\prime\prime}$	$F^{(n)\prime\prime\prime}$
0.0	0	0	0	1.6525	1.6389	1.6476	0	0	0	-∞	-6.2752	-6.0567
0.1	0.0033	0.0033	0.0033	1.6525	1.6389	1.6476	-0.0190	-0.0126	-0.0121	-8.5722	-6.2750	-6.0562
0.2	0.0066	0.0066	0.0066	1.6524	1.6389	1.6475	-0.0354	-0.0251	-0.0242	-7.9301	-6.2746	-6.0547
0.3	0.0099	0.0098	0.0099	1.6523	1.6388	1.6475	-0.0509	-0.0376	-0.0363	-7.5543	-6.2740	-6.0523
0.4	0.0132	0.0131	0.0132	1.6522	1.6387	1.6474	-0.0657	-0.0502	-0.0484	-7.2875	-6.2732	-6.0489
0.5	0.0165	0.0164	0.0165	1.6521	1.6386	1.6473	-0.0801	-0.0627	-0.0605	-7.0803	-6.2722	-6.0446
0.6	0.0198	0.0197	0.0198	1.6519	1.6385	1.6471	-0.0941	-0.0753	-0.0726	-6.9109	-6.2709	-6.0393
0.7	0.0231	0.0229	0.0231	1.6517	1.6383	1.647	-0.1077	-0.0878	-0.0847	-6.7674	-6.2695	-6.0331
0.8	0.0264	0.0262	0.0264	1.6515	1.6381	1.6468	-0.1211	-0.1004	-0.0967	-6.6430	-6.2678	-6.0259
0.9	0.0297	0.0295	0.0296	1.6512	1.6379	1.6466	-0.1343	-0.1129	-0.1088	-6.5330	-6.2660	-6.0178
1.0	0.0330	0.0328	0.0329	1.6509	1.6377	1.6464	-0.1473	-0.1254	-0.1208	-6.4345	-6.2639	-6.0088

Table 2 Outer (*o*), composite (*c*), and numeric (*n*) predictions for $F'(0)$ and $F'''(0)$ at $\alpha=10$ and an increasing range of R. The composite solutions at the core are given using the first term, the first two terms, and the first three terms that constitute Eqs. (47) and (48)

	F'					F'''				
R	$F^{(o)\prime}$	$(F^{(c)\prime})_1$	$(F^{(c)\prime})_2$	$(F^{(c)\prime})_3$	$F^{(n)\prime}$	$F^{(o)\prime\prime\prime}$	$(F^{(c)\prime\prime\prime})_1$	$(F^{(c)\prime\prime\prime})_2$	$(F^{(c)\prime\prime\prime})_3$	$F^{(n)\prime\prime\prime}$
10	2.3877	1.5708	2.3877	1.7089	2.1281	-∞	-3.8758	-14.538	-17.208	-18.165
20	1.9793	1.5708	1.9793	1.7585	1.8963	-∞	-3.8758	-10.812	-12.147	-12.294
50	1.7342	1.5708	1.7342	1.6881	1.7173	-∞	-3.8758	-7.4986	-8.0326	-7.8156
100	1.6525	1.5708	1.6525	1.6389	1.6476	-∞	-3.8758	-6.0082	-6.2752	-6.0567
200	1.6116	1.5708	1.6116	1.6077	1.6102	-∞	-3.8758	-5.1025	-5.2360	-5.0798
500	1.5871	1.5708	1.5871	1.5864	1.5869	-∞	-3.8758	-4.4513	-4.5047	-4.4242
1,000	1.5790	1.5708	1.5790	1.5788	1.5789	-∞	-3.8758	-4.1956	-4.2223	-4.1776
10,000	1.5716	1.5708	1.5716	1.5716	1.5716	-∞	-3.8758	-3.9184	-3.9211	-3.9160
100,0000	1.5709	1.5708	1.5709	1.5709	1.5709	-∞	-3.8758	-3.8811	-3.8814	-3.8809

and

$$F''''(y) \sim \tfrac{1}{16}\pi^4\varepsilon\left(16\alpha/\pi^2-1\right)/y \xrightarrow[y\to 0^+]{} -\left[\operatorname{sgn}(\pi^2-16\alpha)\right]\infty \tag{45}$$

where $\mathcal{C}=0.9159656$ is Catalan's constant. When the composite solution is constructed, singular terms are no longer present; one finds

$$F^{(c)}(0)=F^{(c)''}(0)=0 \tag{46}$$

$$F^{(c)'}(0)=\tfrac{1}{2}\pi\left\{1+\varepsilon\left[\mathcal{C}-\tfrac{1}{2}-\tfrac{1}{4}\pi+\left(4-16\mathcal{C}+2\pi+\pi^2\right)\alpha/\pi^2\right]+\left(2\alpha-\tfrac{1}{8}\pi^2\right)\varepsilon^2\ln\varepsilon\right\} \tag{47}$$

$$F^{(c)'''}(0)=\tfrac{1}{8}\pi^3\left(-1+\left(4\alpha-\tfrac{1}{4}\pi^2\right)\varepsilon\ln\varepsilon/\pi+\varepsilon\left\{3\left(\tfrac{1}{2}-\mathcal{C}\right)+\pi\left(\tfrac{1}{5}+\tfrac{1}{2}\ln 2\right)-\left[\pi^2+\left(\tfrac{19}{5}+8\ln 2\right)\pi+12-48\mathcal{C}\right]\alpha/\pi^2\right\}\right) \tag{48}$$

$$F^{(c)''''}(0)=0 \tag{49}$$

Clearly, a distinct improvement can be observed near the core in the first, third, and fourth derivatives. The difference between $F'(0)$ and $F^{(c)'}(0)$ appears at a small logarithmic order in view of

$$\overline{F}^{(i)'}(0)=\tfrac{1}{2}\pi\left(2\alpha-\tfrac{1}{8}\pi^2\right)\varepsilon^2\ln\varepsilon \tag{50}$$

Insofar as Eq. (50) enhances the accuracy of the solution in the vicinity of the core, it leads to local ameliorations in axial velocity and both axial and normal pressure gradients whose dependence on F' has been established in previous work.[17] In fact, when inner corrections are accounted for, a better agreement is achieved at all levels with the numerical solution of the problem. This can be seen in Table 1 where the outer, composite, and numerical predictions for F and its derivatives are catalogued. This comparison is focused on the limited core region associated with $0\le y\le 2\varepsilon$ and for representative values of $R=100$ and $\alpha=10$. In addition to the gradual refinement in the composite solution over the former expressions for F, F', and F'', a significant improvement in the performance of the composite solution can be noted for F'''. Overall, the ability of the matched-asymptotic approximation to outperform the outer solution is consistent as $\varepsilon\to 0$. This can be inferred from Table 2 where estimates for both $F'(0)$ and $F'''(0)$ are

Table 3 Outer (*o*), composite (*c*), and numeric (*n*) predictions for $F'(0)$ at $\alpha=\pm 5$ and an increasing range of R. The composite solution at the core is obtained using the first term, the first two terms, and the first three terms that constitute Eq. (47)

	$\alpha=5$ (expanding walls)					$\alpha=-5$ (contracting walls)				
R	$F^{(o)'}$	$(F^{(c)'})_1$	$(F^{(c)'})_2$	$(F^{(c)'})_3$	$F^{(n)'}$	$F^{(o)'}$	$(F^{(c)'})_1$	$(F^{(c)'})_2$	$(F^{(c)'})_3$	$F^{(n)'}$
10	1.9502	1.5708	1.9502	1.6332	1.8116	1.0753	1.5708	1.0753	1.4816	1.5659
20	1.7605	1.5708	1.7605	1.6574	1.7152	1.3230	1.5708	1.3230	1.4552	1.4061
50	1.6467	1.5708	1.6467	1.6251	1.6372	1.4717	1.5708	1.4717	1.4993	1.4888
100	1.6087	1.5708	1.6087	1.6024	1.6059	1.5212	1.5708	1.5212	1.5294	1.5263
200	1.5898	1.5708	1.5898	1.5879	1.5890	1.5460	1.5708	1.5460	1.5484	1.5475
500	1.5784	1.5708	1.5784	1.5780	1.5782	1.5609	1.5708	1.5609	1.5613	1.5612
1,000	1.5746	1.5708	1.5746	1.5745	1.5745	1.5658	1.5708	1.5658	1.5660	1.5659

Table 4 Outer (*o*), composite (*c*), and numeric (*n*) predictions for $F'(0)$ and $F'''(0)$ at $R=1000$ and a range of α

R	$F^{(o)'}$	$(F^{(c)'})_1$	$(F^{(c)'})_2$	$(F^{(c)'})_3$	$F^{(n)'}$	$F^{(o)'''}$	$(F^{(c)'''})_1$	$(F^{(c)'''})_2$	$(F^{(c)'''})_3$	$F^{(n)'''}$
-100	1.4827	1.5708	1.4827	1.4849	1.4872	-∞	-3.8758	-0.4459	-0.1589	-1.5951
-10	1.5615	1.5708	1.5615	1.5617	1.5616	-∞	-3.8758	-3.5139	-3.4835	-3.5544
-1	1.5693	1.5708	1.5693	1.5694	1.5694	-∞	-3.8758	-3.8207	-3.8160	-3.8250
-0.1	1.5701	1.5708	1.5701	1.5701	1.5701	-∞	-3.8758	-3.8513	-3.8492	-3.8529
0	1.5702	1.5708	1.5702	1.5702	1.5702	-∞	-3.8758	-3.8548	-3.8529	-3.8560
0.1	1.5703	1.5708	1.5703	1.5703	1.5703	-∞	-3.8758	-3.8582	-3.8566	-3.8591
1	1.5711	1.5708	1.5711	1.5711	1.5711	-∞	-3.8758	-3.8888	-3.8899	-3.8873
10	1.5790	1.5708	1.5790	1.5788	1.5789	-∞	-3.8758	-4.1956	-4.2223	-4.1776
100	1.6577	1.5708	1.6577	1.6556	1.6598	-∞	-3.8758	-7.2636	-7.5470	-8.1171

produced at $\alpha = 10$ and a successively increasing Reynolds number ranging from 10 to 10^5. These estimates are compared to the numerical values with and without inner corrections. In order to depict the improvement with each successive asymptotic correction, the composite solutions at the core are given using the first term, the first two terms, and the first three terms that appear in Eqs. (47) and (48). Note that, in the case of $F'(0)$, a two-term composite solution is identical to the outer approximation. It is the third correction of order $\varepsilon^2 \ln \varepsilon$ that sets the difference in Table 2. This result is further confirmed in Table 3 where the behavior of $F'(0)$ is illustrated for expanding and contracting wall expansions with $\alpha = \pm 5$. In both situations, the asymptotic behavior is improved as $\varepsilon \to 0$.

The dependence of the core values on α is also captured in Table 4 where both $F'(0)$ and $F'''(0)$ are calculated over a range of expansion ratios ranging from -100 to 100 at fixed $R = 1000$. It is interesting to note that the error in the asymptotic predictions becomes more appreciable as $|\alpha|$ is increased at constant R. This can be attributed to our approximation being contingent on $|\alpha R^{-1}| \ll 1$ in Eq. (1). In practice, this constraint does not pose any physical limitations in view of the small reported $|\alpha|$ in most applications. The present development of a composite solution with critical core corrections completes our laminar flow treatment of the porous channel with retractable walls.

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